A supplement to the paper "Differentiable roads for real functions" by J. G. Ceder (*)

by

P. Holický (Praha)

In the paper Differentiable roads for real functions by J. G. Ceder [1] the following theorem is proved.

**Theorem 1.** Let \( f \) be any real-valued function defined on an uncountable subset \( A \) of reals. Then, there exists a countable set \( C \) such that for each \( x \in A - C \) there exists a bilaterally dense-in-itself set \( B \) containing \( x \) such that \( f|B \) is monotonic and differentiable.

In that paper the following theorem and lemmas were proved.

**Lemma 1.** Let \( f \) have a domain \( A \) where \( A \) is uncountable. Let \( B \) be the domain of the bilateral condensation points of \( f \). Then \( A - B \) is countable and, for each \( x \in B \), \( [x, f(x)] \) is a bilateral condensation point for \( f|B \).

**Definition.** Let \( f \) be any real-valued function defined on a subset of reals. Let \( x \) be any point of the domain of \( f \). The left-derived set \( D_L(f, x) \) and the right-derived set \( D_R(f, x) \) of function \( f \) at the point \( x \) are defined to be the sets of all possible sequential limits of the difference quotient \( \frac{f(y) - f(x)}{y - x} \) as \( y \) approaches \( x \) from the left and from the right, respectively.

**Theorem 2.** Suppose \( f \) has an uncountable domain \( A \). Then, there exists a countable subset \( C \) of \( A \) such that for each \( x \in A - C \)

\[
D_L(f)(A - C), x) \cap D_R(f)(A - C), x) \neq \emptyset.
\]

**Lemma 2.** Let \( f \) be any real-valued function defined on an uncountable subset \( A \) of reals. Then, there exists a countable set \( C \) such that for each \( x \in A - C \) there exists a bilaterally dense-in-itself set \( B \subset A - C \) containing \( x \) such that \( f|B \) is differentiable.

To complete the proof of Theorem 1 in the paper Differentiable roads for real functions the following lemma is proved:


13 — Fundamenta Mathematicae, T. LXXVII
Suppose $B$ is bilaterally dense-in-itself and $f|B$ is differentiable. Then, there exists an $A \subseteq B$ such that $A$ is bilaterally dense-in-itself and $f|A$ is differentiable and monotonic.

But the proof of this lemma is incorrect. The following example shows that this lemma does not hold.

**Example.** There exists a real-valued function $f$ defined on a bilaterally dense-in-itself subset of reals $H$ such that $f'(x) = 0$ for each $x \in H$ and for no bilaterally dense-in-itself subset $K \subseteq H$ the function $f|K$ is monotonic.

1. We shall construct a sequence of functions $(f_n)_{n=0}^\infty$ and a sequence of sets $(A_n)_{n=0}^\infty$ where $A$ is a set of sequences of reals such that $f_n(a) = f_n(b)$ for $a, b \in A_n$ and for no bilaterally dense-in-itself subset $K \subseteq H$ the function $f|K$ is monotonic.

2. Let the sets $A_1, ..., A_m (m \geq 0)$ and the function $f_m$ defined on $H$ be defined such that the following assertions hold.

(i) If $x \in H_i (0 \leq i \leq m - 1)$, then there exist in $A_{i+1}$ exactly two sequences $(a_{i+1})_{i=0}^{m-1}, (b_{i+1})_{i=0}^{m-1}$ such that $a_{i+1} = x, b_{i+1} = x$. We put $Q_n(x) = (x_{i+1})_{i=0}^{m-1}, (b_{i+1})_{i=0}^{m-1}, ...angle.$

(ii) If $a \in A_m$ then $f_m(a) = (x - a)^2 + f_m(a)$ for $x \in \phi_m(a)$.

(iii) There exists a positive real-valued function $a_m$ defined on $\bigcup_{i=1}^m H_i$ and such that

a) $(x - a_m(x), x + a_m(x)) \cap (y - a_m(y), y + a_m(y)) = \emptyset$ for each real number $x, y \in H_i, x \neq y$, where $i = 0, 1, ..., m - 1$.

b) If $x \in H_i (i = 0, 1, ..., m - 1)$, then $\phi_m(x) \subseteq [x - a_m(x), x + a_m(x)]$.

c) If $a \in \phi_m(a), a \in H_i (0 \leq i \leq m - 1)$, then $(x - a)^2 + f_m(a)$ for every $a \in [x - a_m(x), x + a_m(x)]$.

d) If $y \in \phi_m(a), x \in A_m$, then $(y - a_m(y), y + a_m(y)) \subseteq [x - a_m(x), x + a_m(x)]$.

We shall define $A_{m+1}$ (therefore $H_{m+1}$) as the assertions (i) (ii), (iii) hold, where we write $m+1$ instead of $m$. For every $x \in H_m$ there exist an $a_m$ and a $\beta(x) > 0$ such that $x \in \phi_m(a_m)$ and for $y \in (x - \beta(x), x + \beta(x))$

$$(y - x)^2 + f_m(a_m) \leq \frac{1}{2}(y - x)^2 + f_m(a_m) \quad (m \geq 1).$$

The existence of $\beta(x)$ follows from the facts that functions $\frac{1}{2}(y - x)^2 + f_m(a_m)$ and $(y - x)^2 + f_m(a_m)$ are continuous and that $f_m(a_m) = \frac{1}{2}(x - a_m(x))^2 + f_m(a_m)$.

Further, for $x \in H_m$ there exists a $\gamma(x) > 0$ such that $(x - \gamma(x), x + \gamma(x)) \cap H_m = \emptyset$ because every point of $H$ is an isolated point. We write $\delta(x) = \min(\beta(x), \gamma(x))$. As $x \in H_m (m \geq 1)$, there exists an $a_m$ such that $x \in \phi_m(a_m)$. Then, there exists an $a_{m+1}(x) > 0$ such that

$$(x - a_{m+1}(x), x + a_{m+1}(x)) \subseteq \{x - \delta(x), x + \delta(x)\} \cap [x - a_m(x), x + a_m(x)](x).$$

We define $a_0(x) = 1$. There exist different points $a_m, b_m \in [x - a_{m+1}(x), x + a_{m+1}(x)]$ such that $a_m > x$, $b_m < x$. We put $\phi_m(x) = B_m, a_m \subseteq \bigcup_{i=1}^m H_i = \emptyset$, $f_m \bigcup_{i=1}^m H_i = f_m$, and define $A_{m+1}, H_{m+1}$.

and define $A_{m+1}, H_{m+1}$. Therefore also $H_{m+1}$ is defined. Now we define $f_{m+1}$ such that (ii) holds.

The assertions (i), (ii), (iii) hold for the sequences $(A_n)_{n=0}^\infty, (f_n)_{n=0}^\infty$.

We have defined sequences of sets $(A_n)_{n=0}^\infty, (H_n)_{n=0}^\infty$. We put $H = \bigcup_{i=0}^\infty H_i$ and define $f(x) = f_n(x)$ for $x \in H_n$. If $n$ is a natural number, $n > 1$, then (i), (ii), (iii) hold for $H_1, ..., H_n, A_1, ..., A_n; f_1 \bigcup_{i=1}^n H_i = f_n$.

III. Obviously $H$ is bilaterally dense-in-itself.

IV. Now we show that $f|K$ is not monotonic for any bilaterally dense-in-itself set $K \subseteq H$, where $K$ is non-empty. We shall contradict the supposition that there exist such a set $K$. The interval $(x - a_m(x), x + a_m(x))$ contains no point of $H_n$ for $k < m$, because such a point would be a limit point of $H_n$ and this contradicts (iii) a). As $f(a) > (a) > f(a)$ for $x \in \bigcap_{i=0}^{m-1} (x - a_{m+1}(x), x + a_{m+1}(x)) \cap K \neq \emptyset$ and for $m < p$ it is $f(a) > f(a)$ for $x \in \bigcap_{i=0}^{m-1} (x - a_{m+1}(x), x + a_{m+1}(x)) \cap K \neq \emptyset$. Therefore $K$ is not monotonic.

V. It remains to show that $f'(x) = 0$ for each $x \in H$. There exist an $a > 0$ such that $x \in H_n$. According to IV it is $f(y) = f(x)$ for $y \in (x - a_m(x), x + a_m(x)) \setminus (x)$. Therefore $f(y) = f(x) > 0$ for $y > x$.

Hence $D_x f(x) = 0$. Similarly we prove that $D_y f(x) = 0$. We shall show that for the other Dini derivatives it is $D_y f(x) \leq 0$, $D_x f(x) \geq 0$. We define $\phi(x) = \phi_m(x)$ and $a_m(x) = a_m(x)$ for $x \in H_{m+1}$ where $m \geq 1$. At first we prove the following inequality.
Let \( y' \in \{ x - a_{n_0}(x), x + a_{n_0}(x) \} \cap H \) where \( n_0 \gg 1 \); let \( y_1, \ldots, y_{n-1} \) be a finite sequence of points of \( H \) such that \( y', y_1, \ldots, y_{n-1} \in \phi(a(x)) \). Then

\[
(y - y_{n-1})^2 + f(y_{n-1}) < (y - a)^2 + \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} f(x)
\]

for \( y \in \{ y' - a(y'), y' + a(y') \} \cup \{ y_n = a \} \).

We prove this inequality by induction.

1. If \( n = 1 \), then the assertion easily follows from the definition of \( f \).
2. Suppose that the inequality holds for some \( n > 1 \). Let \( y' \in \phi(a(x), \ldots, y_n \in \phi(a(x)) \). At first we prove this inequality:

\[
(y - y_{n-1})^2 \leq \frac{1}{2^{n-1}} (y - a)^2.
\]

For \( n = 1 \) the inequality is clearly satisfied. We suppose \( n > 1 \).

If \( (y_{n-1} - y_{n-2})^2 \geq (y - y_{n-1})^2 \), then \( (y - y_{n-1})^2 < \frac{1}{2} (y - y_{n-2})^2 \) because \( (y - y_{n-1})^2 + f(y_{n-1}) < \frac{1}{2} (y - y_{n-2})^2 + f(y_{n-2}) \) and \( f(y_{n-1}) = (y_{n-1} - y_{n-1})^2 + f(y_{n-1}) \).

If \( (y_{n-1} - y_{n-2})^2 < (y - y_{n-1})^2 \), then \( (y - y_{n-1})^2 < (x - y_{n-1})^2 \) and \( (y - y_{n-1})^2 < (y_{n-1} - y_{n-2})^2 < \frac{1}{2} (y_{n-1} - y_{n-2})^2 < \frac{1}{2} (y - y_{n-1})^2 \).

Hence

\[
(y - y_{n-1})^2 \leq \frac{1}{2^{n-1}} (y - a)^2.
\]

According to this inequality it is

\[
(y - y_{n-1})^2 + f(y_{n-1}) < \frac{3}{2} (y - y_{n-1})^2 + f(y_{n-1})
\]

\[
\leq \frac{1}{2} (y - y_{n-1})^2 + \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} f(x)
\]

\[
\leq \frac{1}{2} (y - a)^2 + \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} (y - a)^2 + f(x)
\]

\[
= \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} (y - a)^2 + f(x).
\]

This last inequality implies that \( f(x) < 2(y - a)^2 + f(x) \) for each \( x \in \{ x - a(x), x + a(x) \} \cap H \). We put \( g(x) = 2(y - a)^2 + f(x) \). Then \( g'(x) = 0 \) and \( f(x) = g(x) \).

Therefore

\[
\frac{f(x) - f(a)}{x - a} < \frac{g(x) - g(a)}{x - a} \quad \text{for} \quad x > a \quad \text{and} \quad D^4 f(x) \leq 0.
\]

Similarly we prove \( D^{-1} f(x) \to 0 \) as \( f' = 0 \) on \( H \).

Proof of Theorem 1.

Notation. Let \( f \) be any real-valued function defined on an uncountable subset \( D(f) \) of reals.

a) We denote by \( M_f \) the set of all \( a \in D(f) \) for that the following conditions hold:

(i) \( a \) is the point of bilateral condensation of \( f \),

(ii) there exist a \( \delta > 0 \) and an \( \epsilon > 0 \) such that

\[
\left( y \in D(f) \mid 0 < |y - a| < \delta, f(y) \leq f(a), \frac{|f(y) - f(a)|}{y - a} < \epsilon \right)
\]

is countable,

(iii) \( D_1(f, a) \cap D_{-1}(f, a) = \{ 0 \} \).

b) We denote by \( N_f \) the set of all \( a \in D(f) \) such that the following conditions hold:

(i) \( a \) is the point of bilateral condensation of \( f \),

(ii) there exist a \( \delta > 0 \) and an \( \epsilon > 0 \) such that

\[
\left( y \in D(f) \mid 0 < |y - a| < \delta, f(y) \geq f(a), \frac{|f(y) - f(a)|}{y - a} < \epsilon \right)
\]

is countable,

(iii) \( D_1(f, a) \cap D_{-1}(f, a) = \{ 0 \} \).

Let \( f \) be a real-valued function defined on an uncountable set \( D(f) \) of reals.

We prove that \( M_f \cap N_f \) is countable.

Proof. We prove that the set \( M_f \) is countable. The proof for \( N_f \) is similar. It is \( M_f = \bigcup_{m \in P} \mathcal{M}_f(m, n) \), where \( P \) is the set of all natural numbers, \( \mathcal{M}_f(m, n) \) is the set of all points \( x \in M_f \) for which

\[
\{ (y, f(y)) \mid y \in D(f), 0 < |y - a| < \frac{1}{m}, f(y) \leq f(x), \frac{|f(y) - f(a)|}{y - a} < \frac{1}{n} \}
\]

is countable. Suppose that \( M \) is uncountable. Then there exist natural numbers \( m, n \) such that \( M_f(m, n) \) is uncountable. Lemma 1 implies that there exists a countable set \( C \subseteq M_f(m, n) \) such that every point \( a \in M_f(m, n) \setminus C \) is a point of bilateral condensation of \( f/M \). We can suppose that \( C = \emptyset \). Theorem 2 implies that there exists a point \( y \in M_f(m, n) \) such that

\[
D_1(f, M_f(m, n), y) \cap D_{-1}(f, M_f(m, n), y) \neq \emptyset.
\]
There exist sequences \( \{x_k\}_{n=1}^{\infty} \) and \( \{y_k\}_{n=1}^{\infty} \) such that \( x_k \uparrow y_k \), \( y_k \downarrow y \), where \( x_k \in M_f(m, n), y_k \in M_f(m, n) \) for each \( k \geq 1 \). We can choose the sequences \( \{x_k\}_{n=1}^{\infty}, \{y_k\}_{n=1}^{\infty} \) such that \( f(x_k) \uparrow f(y) \), \( f(y_k) \downarrow f(y) \). As \( D_x f(y) \cap D_y f(y) = \{0\} \), it is \( D_{x_k} f(x_{n+1}, m, n), y \cap D_{x_k} f(x_{n+1}, m, n), y = \{0\} \).

We write \( x_k = x_k \) and \( x_{k+1} = y_k \). Then \( x_k \to y \) and \( f(x_k) - f(y) \to 0 \). Therefore there exists a \( \delta_k \) such that
\[
|f(y) - f(x_k)| < \frac{1}{m} (x_k - y) \quad \text{and} \quad \frac{|x_k - y|}{m} < \frac{1}{m}.
\]
As \( x_k \in M_f(m, n) \), the set \( M = \{(x, f(x)) | x \in M_f(m, n), 0 < |x_k - x| < \frac{1}{m}, (f(x_k) - f(x)) - \frac{|x_k - x|}{m} < \frac{1}{m}, f(x) < f(x_k)\} \) is empty.

The point \( y \) is a point of \( D(f) \) and \( 0 < |y - x_k| < \frac{1}{m}, f(y) < f(x_k) \), \( f(x_k) - f(y) \frac{1}{m} < \frac{1}{m} \); therefore \( y, f(y) \in M \). But \( M = \emptyset \) and this is a contradiction. Hence the set \( M_f \) is countable.

The main part of proof of Theorem 1. We can suppose that for all points \( x \) of \( D(f) \) the following assertions hold:

a) \( x \) is a point of bilateral condensation,

b) \( x \) is neither a point of the set \( M_f \) nor a point of the set \( N_f \),

c) \( D_x f(x) \cap D_y f(x) \neq \emptyset \).

We first prove Theorem 1 in an easy case.

A) Suppose that \( f \) is constant in an uncountable subset of \( D(f) \). Then the assertion of Theorem 1 is obvious.

B) Suppose that \( f \) is not constant in any uncountable subset of \( D(f) \).

We shall choose a sequence of sets \( \{A_n\}_{n=1}^{\infty} \) such that \( A_n \) is a set of sequences of \( D(f) \) for all \( n \geq 0 \).

1. Let \( a_0 \) be a point of \( D(f) \). We put \( a_0 = a_0, A_0 = \{a_m\}_{m=0}^{\infty}, \) and \( H_1 = A_0 \).

2. Suppose that the sets \( A_1, \ldots, A_n \) are defined for \( n \geq 0 \). \( H_n \) is always the set of all points of the sequences from \( A_n \) \( (k \geq 0) \), where \( A_n \) is defined. For \( A_n, \ldots, A_{n+1} \) the following assertions hold:

(i) For all \( x \in H_k \) \( (0 \leq k \leq n-1) \) there exist exactly two sequences \( \{x_m\}_{m=1}^{\infty}, \{x_m\}_{m=1}^{\infty} \) of points of \( D(f) \) in \( A_{k+1} \) such that \( x_m \to x, a_m \to x \). The restriction \( f(\{x_m\}, a_m, \{x_m\}, a_m, \ldots) \) is strictly monotonic. We put \( p(x) = \{x_m, a_m, \ldots\} \).

(ii) If \( x \in \bigcup_{n=0}^{\infty} H_n \) then \( \bigcup_{n=0}^{\infty} H_n \) exists. If \( D_{x_k} f(x) \cap D_{y_k} f(x) \neq \emptyset \) then \( \bigcup_{n=0}^{\infty} H_n \neq 0 \).

(iii) If \( 0 \leq k \leq n-1 \), then there exists a positive real function \( \varphi \) defined on \( H_k \) and such that for \( x \not= y \), \( x, y \in H_k \) \( (x - \varphi(x), x + \varphi(x)) \cap (y - \varphi(y), y + \varphi(y)) = \emptyset \).

If \( \{a_m\}_{m=0}^{\infty} \) is a sequence of \( A_{k+1} \) then \( x_k \to x \), where \( x \in H_k \) and \( a_0 \in (x - \varphi(x), x + \varphi(x)) \) for all natural \( i \). Further, in the case where \( k < n-2 \) it is

\[
(a_i - \varphi(a_i), a_i + \varphi(a_i)) \subseteq (x - \varphi(x), x + \varphi(x)).
\]

(iv) There exists a real positive function \( \varphi \) defined on \( \bigcup_{n=0}^{\infty} H_n \) and such that \( (f(x) - \varphi(a_m), f(x) + \varphi(a_m)) \subseteq (f(y) - \varphi(a_m), f(y) + \varphi(a_m)) \).

for \( x \in p(y) \) and that \( f(y) \neq f(x) \).

(v) If \( 0 \leq k \leq n-1 \) and \( x \in H_k \) then there exist real functions \( \psi, \varphi \) such that \( \psi \) is continuous on \( (x - \varphi(x), x + \varphi(x)) \) and \( \varphi \) is defined and continuous on \( (x - \varphi(x), x + \varphi(x)) \) and such that \( \psi(x) \) is \( \subseteq \) \( \{\varphi(x)\} = \{\psi(x)\} \).

We choose the set \( A_{k+1} \) such that for the sets \( A_0, \ldots, A_{k+1} \) the assertions (i)-(v) for \( n+1 \) hold. As every point of \( H_k \) is an isolated point, there exists a real function \( \varphi_{n+1} \) defined on \( \bigcup_{n=0}^{\infty} H_n \) and such that (iii) holds for \( n+1 \).

The function \( \varphi_{n+1} \) can be chosen such that \( \psi(x) = \varphi_{n+1}(x) \) for \( x \in p(y) \), \( x \in H_k \) and \( x \in (x - \varphi_{n+1}(x), x + \varphi_{n+1}(x)) \).

Then there exists obviously an \( \varepsilon(x) > 0 \) such that
\[
f(x) + \varepsilon(x) < \varphi_{n+1}(x) \quad \text{and} \quad \psi(x) < f(x) - \varepsilon(x) \quad \text{for} \quad x \in (x - \varphi_{n+1}(x), x + \varphi_{n+1}(x)).
\]

It is easy to prove that there exists a real function \( \varphi_{n+1} \) such that condition (iv) for \( n+1 \) holds.

Let \( x \) be any point of \( H_n \) and suppose that \( \lambda \in D_x f(x) \cap D_y f(x) \), where \( \lambda \neq 0 \). There exists a \( y \in H_{n-1} \) such that \( x \in p(y) \). We choose the
sequences \( \{a_m\}_{m=1}^{\infty}, \{b_m\}_{m=1}^{\infty} \) such that \( a_m \uparrow x, b_m \downarrow x, f(p(x)) \text{ is strictly monotonic, where} \),

\[
p(x) = (a_1, b_1, a_2, b_2, \ldots), \quad (f(p(x))'(x) = \lambda,
\]

\[
\max(f(x) - \varepsilon(x), f(x) - a_{m+1}(x)) < f(a_m) < \min(f(x) + \varepsilon(x), f(x) + a_{m+1}(x)),
\]

\[
\max(f(x) - \varepsilon(x), f(x) - a_{m+1}(x)) < f(b_m) < \min(f(x) + \varepsilon(x), f(x) + b_{m+1}(x)).
\]

We put \( \psi(x) = \{a_m\}_{m=1}^{\infty}, \{b_m\}_{m=1}^{\infty} \). Let \( D_{f, x} = D_p(f, x) \cap D_q(f, x) = \emptyset \); let there exist no two sequences for which (i)--(iv) hold true. As there exist two sequences \( \{a_m\}_{m=1}^{\infty}, \{b_m\}_{m=1}^{\infty}, a_m \uparrow x, b_m \downarrow x \) such that \( f(p(x))'(x) = 0 \), there is a point of \( M_f \cup N_f \). This contradicts our propositions on the \( D(f) \). Hence there exist sequences with conditions as in the case of \( D_p(f, x) \cap D_q(f, x) \subset \emptyset \). Therefore these sequences satisfy (i)--(iv).

We denote the set of these two sequences by \( \psi(x) \).

We put \( A_{n+1} = \bigcup_{\psi(x)} \psi(x) \) with \( \psi \) for every point of \( H_n \). Let \( x \) be any point of \( H_n \). For \( n = 1 \) the construction of \( \psi(x) \) is easy. Suppose that \( n > 1 \), \( y \in H_{n+1} \) such that \( y \in \psi(x) \). Therefore there exist a sequence \( \{a_m\}_{m=1}^{\infty} \in A_{n+1} \) and a natural number \( n \) such that \( y = a_n \) and \( a_m \to x \). There exist real numbers \( \gamma, \delta \) such that

\[a) f(x) - \varepsilon(x) < \delta_m < f(a_m) < \gamma_m < f(x) + \varepsilon(x),\]

\[b) \quad \frac{|f(a_m) - \gamma_m|}{a_m - x} < \frac{1}{m^2} \quad \frac{|f(a_m) - \delta_m|}{a_m - x} < \frac{1}{m^2}.\]

We put \( \psi(x)(a_m) = \gamma_m, \psi(x)(a_m) = \delta_m \). Between \( a_k \) and \( a_{k+1} \) we define \( \psi(x) \) and \( \psi(x) \) continuous and monotonic. Then \( \psi(x)'(x) = (\psi(x))'(x) = (f(p(x))'(x).\)

We have chosen the set \( A_{n+1} \) and the functions \( \psi_{n+1} : a_{n+1} \to \psi_{n+1} \) from the conditions (i)--(iv) for \( n+1 \).

We put \( H = \bigcup_{\psi(x)} H \), \( H \) is obviously bilaterally dense-in-itself.

\( H = H' \cup H'' \), where \( H' \) is the set of points \( x \) such that \( f(p(x)) \) is increasing and \( H'' = H - H' \). It is easy to prove that each \( H' \) or \( H'' \) contains a non-empty bilaterally dense-in-itself subset \( B \). Then \( fB \) is monotonic and differentiable; this completes the proof of Theorem 1.

References


Reçu par la Rédaction le 17. 8. 1971.