A supplement to the paper "Differentiable roads for real functions" by J. G. Ceder (*)

by

P. Holický (Praha)

In the paper *Differentiable roads for real functions* by J. G. Ceder [1] the following theorem is proved.

THEOREM 1. *Let f be any real-valued function defined on an uncountable subset A of reals. Then, there exists a countable set C such that for each $x \in A-C$ there exists a bilaterally dense-in-itself set B containing x such that f/B is monotonic and differentiable.*

In that paper the following theorem and lemmas were proved.

LEMMA 1. *Let f have a domain A where A is uncountable. Let B be the domain of the bilateral condensation points of f. Then $A-B$ is countable and, for each $x \in B$, $(x, f(x))$ is a bilateral condensation point for f/B.*

DEFINITION. *Let f be any real-valued function defined on a subset of reals. Let x be any point of the domain of f. The left-derived set $D_L(f, x)$ and the right-derived set $D_R(f, x)$ of function f at the point x are defined to be the sets of all possible sequential limits of the difference quotient $\frac{f(y)-f(x)}{y-x}$ as y approaches x from the left and from the right, respectively.*

THEOREM 2. *Suppose f has an uncountable domain A. Then, there exists a countable subset C of A such that for each $x \in A-C$*

$$D_L\big(f/(A-C), x\big) \cap D_R\big(f/(A-C), x\big) \neq \emptyset .$$

LEMMA 2. *Let f be any real-valued function defined on an uncountable subset A of reals. Then, there exists a countable set C such that for each $x \in A-C$ there exists a bilaterally dense-in-itself set $B \subset A-C$ containing x such that f/B is differentiable.*

To complete the proof of Theorem 1 in the paper *Differentiable roads for real functions* the following lemma is proved:

(*) Fund Math. 65 (1969), pp. 351–358.

Suppose B is bilaterally dense-in-itself and f/B is differentiable. Then, there exists an $A \subseteq B$ such that A is bilaterally dense-in-itself and f/A is differentiable and monotonic.

But the proof of this lemma is incorrect. The following example shows that this lemma does not hold.

Example. There exists a real-valued function f defined on a bilaterally dense-in-itself subset of reals H such that $f'(x) = 0$ for each $x \in H$ and for no bilaterally dense-in-itself subset $K \subseteq H$ the function f/K is monotonic.

I. We shall construct a sequence of functions $\{f_n\}_{n=0}^{\infty}$ and a sequence of sets $\{A_n\}_{n=0}^{\infty}$ where A are sets of sequences of reals such that $f_{n+1}/\bigcup_{i=1}^{n} H_i = f_n$ and f_n is defined on $\bigcup_{i=0}^{n} H_i$, where H_i is the set of all points of all sequences from A_i.

1. We put $A_0 = \{\{a_n\}_{n=1}^{\infty}\}$, where $a_n = 0$ for $n \geqslant 1$. Therefore $H_0 = \{0\}$. We defined $f_0(0) = 0$.

2. Let the sets $A_0, \ldots, A_m$ $(m \geqslant 0)$ and the function f_m defined on $\bigcup_{i=0}^{m} H_i$ be defined such that the following assertions hold.

(i) If $x \in H_i$ $(0 \leqslant i \leqslant m-1)$, then there exist in A_{i+1} exactly two sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ such that $a_n \uparrow x$, $b_n \downarrow x$. We put $\varphi_m(x) = \{a_1, b_1, a_2, b_2, \ldots\}$.

(ii) If $a \in \bigcup_{i=1}^{m-1} H_i$ then $f_m(x) = (x-a)^2 + f_{m-1}(a)$ for $x \in \varphi_m(a)$.

(iii) There exists a positive real-valued function a_m defined on $\bigcup_{i=1}^{m-1} H_i$ and such that

a) $(x - a_m(x), x + a_m(x)) \cap (y - a_m(y), y + a_m(y)) = \emptyset$ for each real number $x, y \in H_i$, $x \neq y$, where $i = 0, 1, \ldots, m-1$,

b) if $x \in H_i$ $(i = 0, 1, \ldots, m-1)$, then $\varphi_m(x) \subseteq (x - a_m(x), x + a_m(x))$,

c) if $a_1 \in \varphi_m(a_0)$, $a_0 \in H_i$ $(0 \leqslant i \leqslant m-2)$, then $(a-a_1)^2 + f_m(a_1) \leqslant \frac{3}{2}(a-a_0)^2 + f_m(a_0)$ for every $a \in (a_1 - a_m(a_1), a_1 + a_m(a_1))$,

d) if $y \in \varphi_m(x)$, $x \in \bigcup_{i=0}^{m-1} H_i$, then $(y - a_m(y), y + a_m(y)) \subseteq (x - a_m(x), x + a_m(x))$.

We shall define A_{m+1} (therefore H_{m+1}) such that the assertions (i (ii), (iii) hold, where we write $m+1$ instead of m. For every $x \in H_m$ ther exist an x_0 and a $\beta(x) > 0$ such that $x \in \varphi_m(x_0)$ and for $y \in (x - \beta(x), x + \beta(x))$

$$(y - x^2) + f_m(x) \leqslant \tfrac{3}{2}(y - x_0)^2 + f_m(x_0) \qquad (m \geqslant 1).$$

The existence of $\beta(x)$ follows from the facts that functions $\frac{3}{2}(y-x_0)^2 + f_m(x_0)$ and $(y-x)^2 + f_m(x)$ are continuous and that $f_m(x) < \frac{3}{2}(x-x_0)^2 + f_m(x_0)$. Further, for $x \in H_m$ there exists a $\gamma(x) > 0$ such that $(x - \gamma(x), x + \gamma(x)) \cap H_m = \{x\}$ because every point of H is an isolated point. We write $\delta(x) = \min(\beta(x), \frac{1}{2}\gamma(x))$. As $x \in H_m$ $(m \geqslant 1)$, there exists an x_0 such that $x \in \varphi_m(x_0)$. Then, there exists an $a_{m+1}(x) > 0$ such that

$$(x - a_{m+1}(x), x + a_{m+1}(x)) \subseteq (x - \delta(x), x + \delta(x)) \cap (x_0 - a_m(x_0), x_0 + a_m(x_0)).$$

We define $a_0(0) = 1$. There exist different points $a_n, b_n \in (x - a_{m+1}(x), x + a_{m+1}(x))$ such that $a_n \uparrow x$, $b_n \downarrow x$. We put

$$\varphi_{m+1}(x) = \{a_1, b_1, a_2, b_2, \ldots\},$$

$$f_{m+1}/\bigcup_{i=0}^{m-1} H_i = \varphi_m, \qquad a_{m+1}/\bigcup_{i=0}^{m-1} H_i = a_m,$$

$$\psi(x) = \{\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}\}$$

and define $A_{m+1} = \bigcup_{x \in H_m} \psi(x)$. Therefore also H_{m+1} is defined. Now we define f_{m+1} such that (ii) holds.

The assertions (i), (ii), (iii) hold for the sequences $\{A_n\}_{n=0}^{\infty}$, $\{f_n\}_{n=0}^{\infty}$.

II. We have defined sequences of sets $\{A_n\}_{n=0}^{\infty}$, $\{H_n\}_{n=0}^{\infty}$. We put $H = \bigcup_{i=0}^{\infty} H_i$ and define $f(x) = f_n(x)$ for $x \in H_n$. If n is a natural number, $n \geqslant 1$, then (i), (ii), (iii) hold for $H_0, \ldots, H_n$; $A_0, \ldots, A_n$; $f/\bigcup_{i=0}^{\infty} H_i = f_n$.

III. Obviously H is bilaterally dense-in-itself.

IV. Now we show that f/K is not monotonic for any bilaterally dense-in-itself set $K \subseteq H$, where K is non-empty. We shall contradict the supposition that there exist such a set K. The interval $(x - a_{m+1}(x), x + a_{m+1}(x))$ contains no point of H_k for $k < m$, because such a point would be a limit point of H_m and this contradicts (iii) a). As $f_p(z) > f(x)$ for $z \in \{(x - a_{m+1}(x), x + a_{m+1}(x)) - \{x\}\} \cap K \neq \emptyset$ and for $m < p$ it is $f(z) > f(x)$ for $z \in \{(x - a_{m+1}(x), x + a_{m+1}(x)) - \{x\}\} \cap K$. Therefore f/K is not monotonic.

V. It remains to show that $f'(x) = 0$ for each $x \in H$. There exists an $n_0 \geqslant 0$ such that $x \in H_{n_0}$. According to IV it is $f(y) > f(x)$ for $y \in (x - a_{n_0+1}(x), x + a_{n_0+1}(x)) - \{x\}$. Therefore $\dfrac{f(y)-f(x)}{y-x} > 0$ for $y > x$. Hence $D_+f(x) \geqslant 0$. Similarly we prove that $D^-f(x) \leqslant 0$. We shall show that for the other Dini derivatives it is $D^+f(x) \leqslant 0$, $D_-f(x) \geqslant 0$. We define $\varphi(x) = \varphi_n(x)$ and $a_n(x) = a_n(x)$ for $x \in H_{n-1}$ where $n \geqslant 1$. At first we prove the following inequality.

Let $y' \in \big(x - a_{n_0+1}(x),\, x + a_{n_0+1}(x)\big) \cap H$ where $n_0 \geqslant 1$; let $y_1, \ldots, y_{n-1}$ be a finite sequence of points of H such that $y' \in \varphi(y_{n-1}), \ldots, y_1 \in \varphi(x)$. Then

$$(y - y_{n-1})^2 + f(y_{n-1}) \leqslant (y-x)^2 \sum_{i=0}^{n-1} \frac{1}{2^i} + f(x)$$

$$\text{for} \quad y \in \big(y' - a(y'),\, y' + a(y')\big) \cdot (y_0 = x)\,.$$

We prove this inequality by induction.

1. If $n = 1$, then the assertion easily follows from the definition of f.

2. Suppose that the inequality holds for some $n \geqslant 1$. Let $y' \in \varphi(y_n), \ldots$ $\ldots, y_1 \in \varphi(x)$. At first we prove this inequality:

$$(y - y_{n-1})^2 \leqslant \frac{1}{2^{n-1}} (y-x)^2\,.$$

For $n = 1$ the inequality is clearly satisfied. We suppose $n > 1$.

If $(y_{n-1} - y_{n-2})^2 \geqslant (y - y_{n-2})^2$, then $(y - y_{n-1})^2 \leqslant \frac{1}{2}(y - y_{n-2})^2$ because $(y - y_{n-1})^2 + f(y_{n-1}) \leqslant \frac{3}{2}(y - y_{n-2})^2 + f(y_{n-2})$ and $f(y_{n-1}) = (y_{n-1} - y_{n-2})^2 + f(y_{n-2})$.

If $(y_{n-1} - y_{n-2})^2 \leqslant (y - y_{n-2})^2$, then $(y - y_{n-1})^2 = (z - y_{n-1})^2$ and $(z - y_{n-2})^2 \leqslant (y_{n-1} - y_{n-2})^2$ for any z. Hence

$$(y - y_{n-1})^2 = (z - y_{n-1})^2 \leqslant \tfrac{1}{2}(z - y_{n-2})^2 \leqslant \tfrac{1}{2}(y_{n-1} - y_{n-2})^2 \leqslant \tfrac{1}{2}(y - y_{n-2})^2\,.$$

Hence

$$(y - y_{n-1})^2 \leqslant \frac{1}{2^{n-1}} (y-x)^2\,.$$

According to this inequality it is

$$\begin{aligned}(y - y_n)^2 + f(y_n) &\leqslant \frac{3}{2}(y - y_{n-1})^2 + f(y_{n-1}) \\ &\leqslant \frac{1}{2}(y - y_{n-1})^2 + \sum_{i=0}^{n-1} \frac{1}{2^i}(y-x)^2 + f(x) \\ &\leqslant \frac{1}{2^n}(y-x)^2 + \sum_{i=0}^{n-1} \frac{1}{2^i}(y-x)^2 + f(x) \\ &= \sum_{i=0}^{n} \frac{1}{2^i}(y-x)^2 + f(x)\,.\end{aligned}$$

This last inequality implies that $f(z) < 2(z-x)^2 + f(x)$ for each $z \in \big(x - a(x),\, x + a(x)\big) \cap H$. We put $g(z) = 2(z-x)^2 + f(x)$. Then $g'(x) = 0$

and $f(x) = g(x)$. Therefore

$$\frac{f(z) - f(x)}{z - x} < \frac{g(z) - g(x)}{z - x} \quad \text{for} \quad z > x \text{ and } D^+ f(x) \leqslant 0\,.$$

Similarly we prove $D_- f(x) \geqslant 0$; thus $f' = 0$ on H.

Proof of Theorem 1.

Notation. Let f be any real-valued function defined on an uncountable subset $D(f)$ of reals.

a) We denote by M_f the set of all $x \in D(f)$ for that the following conditions hold:

(i) x is the point of bilateral condensation of f,

(ii) there exist a $\delta > 0$ and an $\varepsilon > 0$ such that

$$\left\langle y \in D(f) \,|\, 0 < |y - x| < \delta,\, f(y) \leqslant f(x),\, \left|\frac{f(y) - f(x)}{y - x}\right| < \varepsilon \right\rangle$$

is countable,

(iii) $D_L(f, x) \cap D_R(f, x) = \{0\}$.

b) We denote by N_f the set of all $x \in D(f)$ such that the following conditions hold:

(i) x is the point of bilateral condensation of f,

(ii) there exist a $\delta > 0$ and an $\varepsilon > 0$ such that

$$\left\langle y \in D(f) \,|\, 0 < |y - x| < \delta,\, f(y) \geqslant f(x),\, \left|\frac{f(y) - f(x)}{y - x}\right| < \varepsilon \right\rangle$$

is countable,

(iii) $D_L(f, x) \cap D_R(f, x) = \{0\}$.

Lemma 3. *Suppose f is a real-valued function defined on an uncountable subset of reals. Then the set $M_f \cup N_f$ is countable.*

Proof. We prove that the set M_f is countable. The proof for N_f is similar. It is $M_f = \bigcup_{m \in P} \bigcup_{n \in P} M_f(m, n)$, where P is the set of all natural numbers, $M_f(m, n)$ is the set of all points $x \in M_f$ for which

$$\left\{(y, f(y)) \,|\, y \in D(f),\, 0 < |y - x| < \frac{1}{m},\, f(y) \leqslant f(x),\, \left|\frac{f(y) - f(x)}{y - x}\right| < \frac{1}{n}\right\}$$

is countable. Suppose that M is uncountable. Then there exist natural numbers m, n such that $M_f(m, n)$ is uncountable. Lemma 1 implies that there exists a countable set $C \subseteq M_f(m, n)$ such that every point $x \in M_f(m, n) - C$ is a point of bilateral condensation of f/M. We can suppose that $C = \emptyset$. Theorem 2 implies that there exists a point $y \in M_f(m, n)$ such that

$$D_L\big(f/M_f(m, n), y\big) \cap D_R\big(f/M_f(m, n), y\big) \neq \emptyset\,.$$

There exist sequences $\{x_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty}$ such that $x_k \uparrow y$, $y_k \downarrow y$, where $x_k \in M_f(m, n)$, $y_k \in M_f(m, n)$ for each $k \geqslant 1$. We can choose the sequences $\{x_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty}$ such that $f(x_k) \uparrow f(y)$, $f(y_k) \downarrow f(y)$. As $D_L(f, y) \cap D_R(f, y) = \{0\}$ it is

$$D_L(f/M_f(m, n), y) \cap D_R(f/M_f(m, n), y) = \{0\}.$$

We write $z_{2k} = x_k$ and $z_{2k+1} = y_k$. Then $z_k \to y$ and $\dfrac{f(z_k)-f(y)}{z-y} \to 0$. Therefore there exists a k_0 such that

$$f(y) < f(x_{k_0}) < \left|\frac{1}{n}(x_{k_0}-y)\right| + f(y) \quad \text{and} \quad |x_{k_0}-y| < \frac{1}{m}.$$

As $x_{k_0} \in M_f(m, n)$ the set

$$M = \left\{(x, f(x)) \,|\, x \in M_f(m, n),\ 0 < |x_{k_0}-x| < \frac{1}{m},\ \left(\frac{f(x_{k_0})-f(x)}{x_{k_0}-x}\right) < \frac{1}{n},\ f(x) < f(x_{k_0})\right\}$$

is empty.

The point y is a point of $D(f)$ and $0 < |y-x_{k_0}| < \dfrac{1}{m}$, $f(y) < f(x_{k_0})$, $\left|\dfrac{f(x_{k_0})-f(y)}{x_{k_0}-y}\right| < \dfrac{1}{n}$; therefore $(y, f(y)) \in M$. But $M = \emptyset$ and this is a contradiction. Hence the set M_f is countable.

The main part of proof of Theorem 1. We can suppose that for all points x of $D(f)$ the following assertions hold:

a) x is a point of bilateral condensation,

b) x is neither a point of the set M_f nor a point of the set N_f,

c) $D_L(f, x) \cap D_R(f, x) \neq \emptyset$.

We first prove Theorem 1 in an easy case.

A) Suppose that f is constant in an uncountable subset of $D(f)$. Then the assertion of Theorem 1 is obvious.

B) Suppose that f is not constant in any uncountable subset of $D(f)$. We shall choose a sequence of sets $\{A_n\}_{n=0}^{\infty}$ such that A_n is a set of sequences of points of $D(f)$ for all $n \geqslant 0$.

1. Let x_0 be a point of $D(f)$. We put $a_m = x_0$, $A_0 = \{a_m\}_{m=1}^{\infty}$ and $H_0 = \{x_0\}$.

2. Suppose that the sets $A_0, \ldots, A_n$ are defined for $n \geqslant 0$. H_k is always the set of all points of the sequences from A_k $(k \geqslant 0)$, where A_k is defined. For $A_0, \ldots, A_n$ the following assertions hold:

(i) For all $x \in H_k$ $(0 \leqslant k \leqslant n-1)$ there exist exactly two sequences $\{x_m^L\}_{m=1}^{\infty}$, $\{x_m^R\}_{m=1}^{\infty}$ of points of $D(f)$ in A_{k+1} such that $x_m^L \uparrow x$, $x_m^R \downarrow x$. The restriction $f/\{x_1^L, x_1^R, x_2^L, x_2^R, \ldots\}$ is strictly monotonic. We put $p(x) = \{x_1^L, x_2^R, \ldots\}$.

(ii) If $x \in \bigcup_{i=0}^{n-1} H_i$ then $(f/\bigcup_{i=0}^{n} H_i)'(x)$ exists. If $D_L(f, x) \cap D_R(f, x) \supsetneqq \{0\}$ then $(f/\bigcup_{i=0}^{n} H_i)'(x) \neq 0$.

(iii) If $0 \leqslant k \leqslant n-1$, then there exists a positive real function φ_n defined on H_k and such that for $x \neq y$, $x, y \in H_k$

$$(x-\varphi_n(x),\, x+\varphi_n(x)) \cap (y-\varphi_n(y),\, y+\varphi_n(y)) = \emptyset .$$

If $\{a_i\}_{i=1}^{\infty}$ is a sequence of A_{k+1}, then $a_i \to x$, where $x \in H_k$ and $a_i \in (x-\varphi_n(x),\, x+\varphi_n(x))$ for all natural i. Further, in the case where $k \leqslant n-2$ it is

$$(a_i-\varphi_n(a_i),\, a_i+\varphi_n(a_i)) \subseteq (x-\varphi_n(x),\, x+\varphi_n(x)) .$$

(iv) There exists a real positive function α_n defined on $\bigcup_{i=0}^{n-1} H_i$ and such that

$$(f(x)-\alpha_n(x),\, f(x)+\alpha_n(x)) \subseteq (f(y)-\alpha_n(y),\, f(y)+\alpha_n(y))$$

for $x \in p(y)$ and that $f(y) \notin (f(x)-\alpha_n(x),\, f(x)+\alpha_n(x))$.

(v) If $0 \leqslant k \leqslant n-1$ and $x \in H_k$ then there exist real functions $\underline{\psi}(x)$ and $\overline{\psi}(x)$, $\underline{\psi}(x) \leqslant \overline{\psi}(x)$, $\underline{\psi}(x)(x) = \overline{\psi}(x)(x)$ defined and continuous on $(x-\varphi_n(x),\, x+\varphi_n(x))$ and such that $p(x) \subseteq M(x)$, where

$$M(x) = \{y \,|\, y \in (x-\varphi_n(x),\, x+\varphi_n(x)) \cap D(f),\ \underline{\psi}(x)(y) < f(y) < \overline{\psi}(x)(y)\} .$$

If $x \in H_{k-1}$ and $y \in p(x)$, then $M(y) \subseteq M(x)$ and $(\underline{\psi}(x))'(x) = (\overline{\psi}(x))'(x)$.

We choose the set A_{k+1} such that for the sets $A_0, \ldots, A_{k+1}$ the assertions (i)–(v) for $n+1$ hold. As every point of H_k is an isolated point, there exists a real function φ_{n+1} defined of $\bigcup_{i=0}^{n} H_i$ and such that (iii) holds for $n+1$.

The function φ_{n+1} can be chosen such that

$$\underline{\psi}(x_0)(y) < \overline{\psi}(x_0)(y) \text{ for } x \in p(x_0),\ x \in H_n \text{ and } y \in \langle x-\varphi_{n+1}(x),\, x+\varphi_{n+1}(x)\rangle .$$

Then there exists obviously an $\varepsilon(x) > 0$ such that

$$f(x)+\varepsilon(x) < \overline{\psi}(x_0)(y) \quad \text{and} \quad \underline{\psi}(x_0)(y) < f(x)-\varepsilon(x)$$

for $y \in (x-\varphi_{n+1}(x),\, x+\varphi_{n+1}(x))$. It is easy to prove that there exists a real function α_{n+1} such that condition (iv) for $n+1$ holds.

Let x be any point of H_n and suppose that $\lambda \in D_L(f, x) \cap D_R(f, x)$, where $\lambda \neq 0$. There exists a $y \in H_{n-1}$ such that $x \in p(y)$. We choose the sequences $\{a_m\}_{m=1}^{\infty}$, $\{b_m\}_{m=1}^{\infty}$ such that $a_m \uparrow x$, $b_m \downarrow x$, $f/p(x)$ is strictly monotonic, where

$$p(x) = \{a_1, b_1, a_2, b_2, \ldots\}, \qquad \big(f/p(x)\big)'(x) = \lambda,$$

$$\max\big(f(x)-\varepsilon(x), f(x)-\alpha_{m+1}(x)\big) < f(a_m) < \min\big(f(x)+\varepsilon(x), f(x)+\alpha_{m+1}(x)\big),$$

$$\max\big(f(x)-\varepsilon(x), f(x)-\alpha_{m+1}(x)\big) < f(b_m) < \min\big(f(x)+\varepsilon(x), f(x)+\alpha_{m+1}(x)\big).$$

We put $\psi(x) = \big\{\{a_m\}_{m=1}^{\infty}, \{b_m\}_{m=1}^{\infty}\big\}$. Let $D_L(f, x) \cap D_R(f, x) = \{0\}$; let there exist no two sequences for which (i)–(iv) hold true. As there exist two sequences $\{a_m\}_{m=1}^{\infty}$, $\{b_m\}_{m=1}^{\infty}$, $a_m \uparrow x$, $b_m \downarrow x$ such that $(f/\{a_1, b_1, \ldots\})'(x) = 0$, there is x the point of $M_f \cup N_f$. This contradicts our propositions on the $D(f)$. Hence there exist sequences with conditions as in the case of $D_L(f, x) \cap D_R(f, x) \subsetneq \{0\}$. Therefore these sequences satisfy (i)–(iv). We denote the set of these two sequences by $\psi(x)$.

We put $A_{n+1} = \bigcup_{x \in H_n} \psi(x)$. It remains to construct the functions $\underline{\psi}(x)$, $\overline{\psi}(x)$ with (v) for every point of H_n. Let x be any point of H_n. For $n = 1$ the construction of $\underline{\psi}(x)$, $\overline{\psi}(x)$ is easy. Suppose that $n > 1$, $y \in H_{n+1}$ such that $y \in p(x)$. Therefore there exist a sequence $\{a_m\}_{m=1}^{\infty} \in A_{n+1}$ and a natural number n_0 such that $y = a_{n_0}$ and $a_m \to x$. There exist real numbers γ_n, δ_n such that

a) $f(x) - \varepsilon(x) < \delta_m < f(a_m) < \gamma_m < f(x) + \varepsilon(x)$,

b) $\left|\dfrac{f(a_m) - \gamma_m}{a_m - x}\right| < \dfrac{1}{m}$, $\left|\dfrac{f(a_m) - \delta_m}{a_m - x}\right| < \dfrac{1}{m}$.

We put $\overline{\psi}(x)(a_m) = \gamma_m$, $\underline{\psi}(x)(a_m) = \delta_m$. Between a_k and a_{k+1} we define $\underline{\psi}(x)$ and $\overline{\psi}(x)$ continuous and monotonic. Then $\big(\underline{\psi}(x)\big)'(x) = \big(\overline{\psi}(x)\big)'(x) = \big(f/p(x)\big)'(x)$.

We have chosen the set A_{n+1} and the functions φ_{n+1}, α_{n+1}, $\underline{\psi}_{n+1}$, $\overline{\psi}_{n+1}$ from the conditions (i)–(v) for $n+1$.

We put $H = \bigcup_{i=0}^{\infty} H_i$. H is obviously bilaterally dense-in-itself. $H = H' \cup H''$, where H' is the set of points x such that $f/p(x)$ is increasing and $H'' = H - H'$. It is easy to prove that either H' or H'' contains a non-empty bilaterally dense-in-itself subset B. Then f/B is monotonic and differentiable; this completes the proof of Theorem 1.

References

[1] J. G. Ceder, *Differentiable roads for real functions*, Fund. Math. 65 (1969), pp. 351–358.

Reçu par la Rédaction le 17. 8. 1971

On shape and fundamental deformation retracts II

by

M. Moszyńska (Warszawa)

According to Fox (see [2]), two spaces X, Y are of the same homotopy type iff they are both imbeddable in some space Z as its deformation retracts.

The main result of this note is the following: For two compact metric spaces X, Y to be of the same shape it is necessary and sufficient that both X and Y be imbeddable in some compactum Z as its fundamental deformation retracts (Theorem 3.3).

We shall refer to 3.3 as the Fox Theorem for shapes.

The proof is based on some statements concerning maps of ANR-sequences. For the particular case of usual maps, these statements have been proved by the author in [6].

For terminology and notation, see [6].

1. Mapping cylinder for an arbitrary map of inverse sequences. The notion of mapping cylinder introduced in [6] § 3 for usual maps of inverse systems can be extended — in the case of inverse sequences — to arbitrary maps.

Take two inverse sequences of topological spaces, $\boldsymbol{X} = (X_n, p_n^{n+1})$, $\boldsymbol{Y} = (Y_n, q_n^{n+1})$, and a map $\boldsymbol{f} = (\varphi, f_n)\colon \boldsymbol{X} \to \boldsymbol{Y}$. By definition (see [3] or [6]), all the diagrams

$$\begin{array}{ccc} X_{\varphi(n)} & \xleftarrow{\;p_{\varphi(n)}^{\varphi(n+1)}\;} & X_{\varphi(n+1)} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ Y_n & \xleftarrow[\;q_n^{n+1}\;]{} & Y_{n+1} \end{array} \qquad \text{commute up to homotopy.}$$

Thus, there exist homotopies $k_n^{n+1}\colon X_{\varphi(n+1)} \times I \to Y_n$ such that

$$k_n^{n+1}(x, 0) = f_n p_{\varphi(n)}^{\varphi(n+1)}(x), \qquad k_n^{n+1}(x, 1) = q_n^{n+1} f_{n+1}(x)$$
$$\text{for} \quad x \in X_{\varphi(n+1)},\ n = 1, 2, \ldots$$

Let C_{f_n} be the mapping cylinder of f_n. Define

$$r_n^{n+1}\colon \quad C_{f_{n+1}} \to C_{f_n}$$