Recursions with uniquely determined topologies

by

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Let (X, τ) be a topological space and $C(X_\tau)$ the set of all continuous functions from X into X which are continuous with respect to τ. In a recent paper [6] J. C. Warndof treated the question: if τ_1 and τ_2 are both topologies for the same set X, when does $C(X_{\tau_1}) = C(X_{\tau_2})$ imply that $\tau_1 = \tau_2$. We wish to consider two analogous problems. Following Wallace [5] we define a recursion to be a continuous function $\mu\colon S\times X\to X$ such that both S and X are nonvoid Hausdorff spaces. It is the purpose of this paper to find conditions on S, X, and μ such that the topology of S is uniquely determined or the topology of X is uniquely determined. Note that $\mu\colon S\times X\to X$ is a recursion whenever X is locally compact, S is a subspace of $C(X)$ with the compact-open topology, and μ is the evaluation map. This is a special case of an act. An act is a recursion $\mu\colon S\times X\to X$ where in addition S is a topological semigroup and $\mu(st, x) = \mu(s, \mu(t, x))$ for all $s, t \in S$ and $x \in X$. The second half of this paper is concerned with showing that under certain conditions if $\mu\colon S\times X\to X$ is an act then S and X can be embedded in a topological semigroup S' such that $[\mu(s, x)]' = s'\cdot x'$ where y' is the element of S' identified with $y \in S\cup X$ and $\cdot$ is the multiplication in S'.

Definitions and notation. Let $\mu\colon S\times X\to X$ be a recursion. We will generally write sx for $\mu(s, x)$ and call X the *state space*. μ is said to be *effective* if $sx = tx$ for all $x \in X$ implies $s = t$. Note that if μ is effective, there is a natural embedding of the set S into $C(X)$ by mapping s into φ_s where $\varphi_s\colon X\to X$ is defined by $\varphi_s(x) = sx$. If $\{y_n\}$ is a net in a topological space Y, we say $\lim y_n = \infty$ if $\{y_n\}$ has no convergent subnets. A function $\mu\colon S\times X\to X$, where S and X are topological spaces, is said to be IP (*infinity preserving*) on $x_0 \in X$ if whenever $\{s_a\}$ is a net in S such that $\lim s_a = \infty$ then $\lim s_a x_0 = \infty$. μ is said to be IP if it is IP on x for all $x \in X$. It is said to be *weakly* IP if whenever $\{s_a\}$ is a net in S with $\lim s_a = \infty$,

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there is some $x \in X$ such that $\lim s_\alpha x = \infty$. Our definition of IP varies slightly from that given for group actions in [3]. A function that satisfies the definition of an act except for the topological criteria will be called an *algebraic act*. If $\mu\colon S \times X \to X$ is an act, we often say S *acts on* X. If S and T are topological semigroups and $h\colon S \to T$ is a homeomorphism that is in addition an algebraic isomorphism, h is called an *iseomorphism*. All topological spaces in this paper are assumed to be Hausdorff.

1. Unique topologies.

1.1. Theorem. *Let $\mu\colon S \times X \to X$ be a weakly IP, effective recursion. Then S is homeomorphic to the subspace $\{\varphi_s\colon X \to X \mid s \in S\} \subset C(X)$ where $C(X)$ has the compact open topology and $\varphi_s\colon x \to \mu(s, x)$.*

Proof: Let $T = \{\varphi_s \mid s \in S\}$. Since $\varphi_s = \mu|_{\{s\}\times X}$, φ_s is continuous and thus $T \subset C(X)$. Let $h\colon S \to T$ be defined by $h(s) = \varphi_s$ for all $s \in S$. We will show that h is a homeomorphism. h is clearly onto T and if $h(s_1) = h(s_2)$, then $s_1 x = s_2 x$ for all $x \in X$ which implies that $s_1 = s_2$ since μ is effective. Thus h is a bijection.

The notation $(K, V) = \{f \in C(X) \mid f(K) \subset V\}$ where K is compact and V is open is used to denote a subbasic open set of the compact open topology. We next show that h is continuous. Let $h(s) \in (K, V) \cap T$ a subbasic open set in T. Then $sK \subset V$. Choose U_0 open in S and V_0 open in X such that $s \in U_0$, $K \subset V_0$ and $U_0 V_0 \subset V$. This can be done since K is compact and μ is continuous. Let $t \in U_0$, then $tK \subset U_0 V_0 \subset V$ implies $\varphi_t \in (K, V)$ which in turn implies $h(U_0) \subset (K, V) \cap T$ and since $s \in U_0$ this means h is continuous.

To complete the proof, we now show that h is open. Let $0 \subset S$ be open and let $\varphi_s \in h(0)$. If we can find (A) $K_1, K_2, \ldots, K_n$ compact in X and $U_1, \ldots, U_n$ open in X such that $\varphi_s \in \bigcap \{(K_i, U_i) \mid i = 1, \ldots, n\} \cap T \subset h(0)$ then we are finished. Suppose the desired sets don't exist. Let $\mathcal{F}$ be the set of all finite intersections of subbasic open sets of T that contain φ_s. Thus if $F \in \mathcal{F}$ then $\varphi_s \in F$ and $F = \bigcap \{(K_i, U_i) \mid i = 1, \ldots, n\}$ for some n where each K_i is compact and each U_i is open in X. Let D be an index set for $\mathcal{F}$ and if $\alpha, \beta \in D$, define $\alpha < \beta$ if $F_\beta \subset F_\alpha$. Since $F_\alpha, F_\beta \in \mathcal{F}$ implies that $F_\alpha \cap F_\beta \in \mathcal{F}$, it follows that $(D, <)$ is a directed set. For each $\alpha \in D$, choose s_α such that $\varphi_{s_\alpha} \in F_\alpha$ but $s_\alpha \notin 0$. Since (A) does not occur, we can always make this choice.

Now $\{s_\alpha\}$ is a net in S. We first show that $\lim s_\alpha = \infty$. Suppose $\{s_\beta\}$ were a subnet of $\{s_\alpha\}$ which converged to s_0. Then $s_0 \in S \setminus 0$ since $\{s_\alpha\} \in S \setminus 0$ which is closed. Thus $s_0 \neq s$ since $s \in 0$. By the effectiveness of μ, there exist $x \in X$ such that $s_0 x \neq sx$. Choose U, V open in X such that $s_0 x \in U$, $sx \in V$ and $U \cap V = \varphi$ and then select U' open in S such that $s_0 \in U'$ and $U'x \subset U$. Finally let $\delta \in D$ be such that $F_\delta = (x, V)$. Then $\alpha > \delta$ implies $\varphi_{s_\alpha} \in F_\delta$ which means $s_\alpha x \in V$ and thus $s_\alpha \notin U'$. But this contradicts the fact that a subnet of $\{s_\alpha\}$ converges to $s_0 \in U'$. Therefore $\{s_\alpha\}$ has no convergent subnets, i.e. $\lim s_\alpha = \infty$.

We now show that $\lim s_\alpha = \infty$ contradicts the fact that S is weakly IP on X. Let $x \in X$ and V any open set in X with $x \in V$. Choose δ so that $F_\delta = (x, V)$ then for $\alpha > \delta$, $\varphi_{s_\alpha} \in F_\delta$ which implies $s_\alpha x \in V$. Thus $\lim s_\alpha x = x$ for all $x \in X$ which contradicts the fact that S is weakly IP on X. Therefore (A) may not be denied which means that h is an open map and hence is a homeomorphism.

We can now solve the first problem posed in the introduction.

1.2. Corollary. *Let S be a set, X a topological space and $\mu\colon S \times X \to X$ a function such that $\mu(s, x) = \mu(t, x)$ for all $x \in X$ implies $s = t$. Suppose τ_1 and τ_2 are both topologies for S such that when S is endowed with either topology μ becomes a weakly IP recursion. Then $\tau_1 = \tau_2$.*

1.3. Corollary. *Let S, X and μ be as above and suppose τ_1 and τ_2 are both compact topologies for S such that when S is endowed with either topology μ is a recursion. Then $\tau_1 = \tau_2$.*

Proof. Since every net in a compact space has a convergent subnet, μ is IP when S is endowed with either topology.

The following example illustrates the fact that Theorem 1.1 is not in general true without the hypothesis that μ is weakly IP, even in the case of semigroup actions with strong restrictions on μ. Let $\mu\colon S \times X \to X$ be an act. μ is said to be *transitive* if $Sx = X$ for all $x \in X$. A semigroup is said to have *left zero multiplication* if $xy = x$ for all $x, y \in S$.

1.4. Example. Let I be the unit interval with left zero multiplication and define $\mu\colon I \times I \to I$ by $\mu(i, j) = i$. Then μ is a transitive and effective algebraic act. Let τ_1 be the usual topology for I and B a base for τ_1. Let τ_2 be the topology on I generated by $B \cup \{\{0\}, \{1\}\}$. Both τ_1 and τ_2 are locally compact topologies for I and both $\mu\colon (I, \tau_1) \times (I, \tau_1) \to (I, \tau_1)$ and $\mu\colon (I, \tau_2) \times (I, \tau_1) \to (I, \tau_1)$ are continuous. But $\tau_1 \neq \tau_2$.

We now turn to the similar question for the state space of a recursion. That is, if τ_1 and τ_2 are two topologies for X, what restrictions on S, X and μ imply $\tau_1 = \tau_2$? From the results above, one might guess that if S was a compact semigroup and τ_1, τ_2 were both compact topologies for X such that $\mu\colon S \times (X, \tau_1) \to (X, \tau_1)$ and $\mu\colon S \times (X, \tau_2) \to (X, \tau_2)$ were both continuous effective acts with $SX = X$, then $\tau_1 = \tau_2$. This is not true, as the following example demonstrates.

1.5. Example. Let $I = [0, 1]$ with the usual topology and multiplication. Let $A_1 = [-1, 0]$ with the usual topology and $A_2 = [-1, 0]$ with a different compact Hausdorff topology than A_1. Let $X = [-1, 1]$ and define $\tau_i = \{U \subset X\colon U \cap I$ is open in I and $U \cap A_i$ is open in $A_i\}$ for $i = 1, 2$. Then (X, τ_1) and (X, τ_2) are both compact Hausdorff spaces.

Define $\mu\colon I\times X\to X$ by: $\mu(i,x)=x$ if $x\in[-1,0]$ and $\mu(i,x)=$ the usual product in I of i and x if $x\in[0,1]$. $\mu|_{I\times[0,1]}$ is continuous because it is just multiplication in the semigroup I. $\mu|_{I\times[-1,0]}$ is clearly continuous from its definition no matter which topology is used on $[-1,0]$. μ is effective since $\mu(i,1)=i$. Thus, I is a compact connected abelian semigroup that acts effectively on both (X,τ_1) and (X,τ_2); yet $\tau_1\neq\tau_2$.

The preceding example indicates that for the state space some restriction in addition to IP and effective is needed on the recursion.

1.6. THEOREM. *Let S be a topological space, X a set, and $\mu\colon S\times X\to X$ a function such that $\mu(S\times\{x_0\})=X$ for some $x_0\in X$. Let τ_1 and τ_2 be two topologies for X such that when X is endowed with either topology, μ is* IP *on x_0 and $\mu|_{S\times\{x_0\}}$ is continuous. Then $\tau_1=\tau_2$.*

Proof. We use nets and the notation ${}_S\lim$, ${}_1\lim$, ${}_2\lim$ to indicate limits taken in S, (X,τ_1) and (X,τ_2) respectively. Suppose that the set $F\subset X$ is closed in τ_1 but not in τ_2. Then there is a net $\{x_a\}\subset F$ such that ${}_2\lim x_a=x_1\in X-F$. For each a, choose $s_a\in S$ such that $s_ax_0=x_a$ and note that ${}_2\lim(s_ax_0)=x_1$, which implies ${}_S\lim s_a\neq\infty$ because μ is IP on x_0. Thus, there is a convergent subnet $\{s_\beta\}$ of $\{s_a\}$. Let ${}_S\lim s_\beta=s_1$ and observe that $s_1x_0=({}_S\lim s_\beta)x_0={}_2\lim(s_\beta x_0)={}_2\lim(s_ax_0)={}_2\lim x_a=x_1$ since $\{s_\beta x_0\}$ is a subnet of $\{s_ax_0\}$ and $\mu|_{S\times\{x_0\}}\colon S\times\{x_0\}\to(X,\tau_2)$ is continuous. Let $x_\beta=s_\beta x_0$ for each β. Then the continuity of $\mu|_{S\times\{x_0\}}\colon S\times\{x_0\}\to(X,\tau_1)$ implies that $x_1=s_1x_0=({}_S\lim s_\beta)x_0={}_1\lim(s_\beta x_0)={}_1\lim x_\beta$ and $\{x_\beta\}\subset F$ converges to s_1 in τ_1. But F is closed in τ_1 and $x_1\in X-F$ is a contradiction. Thus, every set that is closed in τ_1 is closed in τ_2. Similarly, every set closed in τ_2 is closed in τ_1 so that $\tau_1=\tau_2$.

When restricted to an action by a compact semigroup, we obtain the following corollary:

1.7. COROLLARY. *Let S be a compact semigroup and $\mu\colon S\times X\to X$ be an algebraic act such that $Sx_0=X$ for some $x_0\in X$. Let τ_1 and τ_2 be two topologies for X such that when X is endowed with either topology μ is continuous. Then $\tau_1=\tau_2$.*

Proof. Since S is compact, there is no net in S such that $\lim s_a=\infty$. Therefore S is IP on X with any topology for X and the result follows from Theorem 1.6.

To see that some condition like IP is necessary in Theorem 1.6 even when the recursion is a transitive and effective act, consider the following example.

1.8. EXAMPLE. Let $S=[0,1]$ with the discrete topology and left zero multiplication. Let τ_1 and τ_2 be two different compact, Hausdorff topologies for $X=[0,1]$. Define $\mu\colon S\times(X,\tau_i)\to(X,\tau_i)$ by $\mu(s,i)=s$. Then, μ is clearly a continuous transitive effective act for $i=1,2$ and S is locally compact, but $\tau_1\neq\tau_2$.

2. Embedding actions in semigroups. Let $\mu\colon S\times X\to X$ and $\nu\colon T\times Y\to Y$ be acts. Let $\varphi\colon S\to T$ be an iseomorphism and $\psi\colon X\to Y$ a homeomorphism such that the following diagram commutes.

$$\begin{array}{ccc} S\times X & \xrightarrow{\ \mu\ } & X \\ {\scriptstyle\varphi\times\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ T\times Y & \xrightarrow[\ \nu\ ]{} & Y \end{array}$$

Then the triple (S,X,μ) is said to be *equivalent to* (T,Y,ν) and μ is said to be *equivalent to* ν. Let (S,X,μ) be an action triple as above and T a topological semigroup with multiplication $M\colon T\times T\to T$. Let φ be an iseomorphism from S into T and h a homeomorphism from X into T such that the diagram below commutes.

$$\begin{array}{ccc} S\times X & \xrightarrow{\ \mu\ } & X \\ {\scriptstyle\varphi\times h}\downarrow & & \downarrow{\scriptstyle h} \\ T\times T & \xrightarrow[\ M\ ]{} & T \end{array}$$

Then we say the triple (S,X,μ) is *embedded in the semigroup* T.

A well known embedding theorem due independently to Stadtlander [4] and Day and Wallace [1] is the following:

2.1. THEOREM. *Let S be a compact abelian semigroup acting effectively on a space X so that $Sx_0=X$ for some $x_0\in X$. Then X is homeomorphic to S and the action of S on X is equivalent to multiplication in S.*

In this section, we prove a generalization of Theorem 2.1 and then go on to show that if (S,X,μ) is an action triple with S and X both compact, we can construct a compact semigroup T such that X is homeomorphic to $K(T)$, the minimal ideal of T, and (S,X,μ) is embedded in T.

2.2. THEOREM. *Let S be an abelian topological semigroup that acts on the space X such that the action is* IP *on x_0 and effective and $Sx_0=X$ for some $x_0\in X$. Then X is homeomorphic to S and the action of S on X is equivalent to multiplication in S.*

Proof. Let m represent the multiplication in S and define $h\colon S\to X$ by $h(s)=sx_0$. h is clearly continuous and onto. Now $h(s_1)=h(s_2)$ means $s_1x_0=s_2x_0$ and since any $x\in X$ is of the form sx_0, we have

$$s_1x=s_1sx_0=s(s_1x_0)=s(s_2x_0)=s_2sx_0=s_2x$$

for all $x\in X$ which implies that $s_1=s_2$ since the action is effective. To see that h is a homeomorphism, we have only to show it is open. Let O be an arbitrary open set in S and sx_0 an arbitrary element of $h(O)$, then $s\in O$ since h is injective. We need (B) a U open in X such that $sx_0\in U\subset f(O)$.

Suppose no such U exists and let $\mathcal{F}$ be the collection of all open neighborhoods of sx_0 with D as an index set for $\mathcal{F}$. Thus $\mathcal{F} = \{F_\alpha | \ \alpha \in D\}$ where each F_α is an open neighborhood of sx_0. Define the relation $<$ on D by $\alpha < \beta$ if $F_\beta \subset F_\alpha$, then as in the proof of (1.1), $(D, <)$ is a directed set. For each $\alpha \in D$ choose $s_\alpha \in S$ such that $s_\alpha x_0 \in F_\alpha$ but $s_\alpha x_0 \notin f(O)$. This is possible since we are assuming (B) does not occur. Now $\{s_\alpha x_0\}$ is a net in X and $\lim (s_\alpha x_0) = sx_0$ hence the net $\{s_\alpha\}$ has a convergent subnet, $\{s_\beta\}$ in S because the action was IP. Let $s_1 = \lim s_\beta$, then since $\{s_\beta\} \subset S \backslash O$ a closed set, we have $s_1 \in S \backslash O$ which implies $s_1 \neq s$. But

$$s_1 x_0 = (\lim s_\beta) x_0 = \lim (s_\beta x_0) = \lim (s_\alpha x_0) = sx_0$$

which contradicts the fact that h is one to one. Therefore h is an open map and hence a homeomorphism. The commutativity of the following diagram shows that the action of S on X is equivalent to multiplication in S:

$$\begin{array}{ccc} S \times X & \xrightarrow{\mu} & X \\ {\scriptstyle i \times h^{-1}} \downarrow & & \downarrow {\scriptstyle h^{-1}} \\ S \times S & \xrightarrow[m]{} & S \end{array}$$

Let (S, X, μ) be an action triple such that X is locally compact and μ is IP and effective. It is well known that $C(X)$ with the compact-open topology and composition for multiplication is a topological semigroup. We will now exhibit an embedding of (S, X, μ) into a subsemigroup, T, of $C(X)$ such that if $\varphi\colon S \to T$ and $h\colon X \to T$ are the embedding functions then $T = \varphi(S) \cup h(X)$ and $K(T) = h(X)$.

Let $\varphi\colon S \to C(X)$ be defined by $\varphi(s) = \varphi_s$ where $\varphi_s(x) = sx$, then as in (1.1) φ is a homeomorphism into $C(X)$. It is also a homomorphism since

$$\varphi(s_1 s_2)(x) = \varphi_{s_1 s_2}(x) = (s_1 s_2)x = s_1(s_2 x) = [\varphi(s_1) \circ \varphi(s_2)]x$$

for all $x \in X$. Thus $\varphi(S)$ is a subsemigroup of $C(X)$ iseomorphic to S. For each $y \in X$, let $c_y\colon X \to X$ be defined by $c_y(x) = y$ for all $x \in X$ and define $h\colon X \to C(X)$ by $h(x) = c_x$ then h is a homeomorphism of X into $C(X)$ [2]. Let $T = \varphi(S) \cup h(X)$. We will show that T is a subsemigroup of $C(X)$. Let $t_1, t_2 \in T$. If $t_1, t_2 \in \varphi(S)$ then $t_1 \circ t_2 \in \varphi(S)$, if $t_1, t_2 \in h(X)$ then $t_1 = c_{y_1}$, $t_2 = c_{y_2}$ and $t_1 \circ t_2 = c_{y_1} \circ c_{y_2} = c_{y_1} \in T$ if $t_1 \in \varphi(S)$ and $t_2 \in h(X)$ then $t_1 = \varphi_{s_1}$, $t_2 = c_{y_2}$ and

$$t_1 \circ t_2(x) = \varphi_{s_1}\big(c_{y_2}(x)\big) = \varphi_{s_1}(y_2) = s_1 y_2 = c_{s_1 y_2}(x)$$

for all $x \in X$, thus $t_1 \circ t_2 = c_{s_1 y_2} \in T$. Finally if $t_1 \in h(X)$ and $t_2 \in \varphi(S)$ then $t_1 = c_{y_1}$, $t_2 = \varphi_{s_2}$ and $t_1 \circ t_2 = c_{y_1} \circ \varphi_{s_2} = c_{y_1} \in T$. Therefore T is a subsemigroup of $C(X)$. We also see that if $k \in h(X)$ then $k \circ t \in h(X)$ and $t \circ k \in h(X)$ for any $t \in T$ making $h(X)$ an ideal of T. $h(X)$ is the minimal ideal of T because $k \circ t = k$ for all $k \in h(X)$ and all $t \in T$. To see that (S, X, μ) is embedded in T consider the following diagram:

$$\begin{array}{ccc} S \times X & \xrightarrow{\mu} & X \\ {\scriptstyle \varphi \times h} \downarrow & & \downarrow {\scriptstyle h} \\ T \times T & \xrightarrow[\circ]{} & T \end{array}$$

Since $h(sx) = c_{sx}$ and $\varphi_s \circ c_x = c_{sx}$, we see that the diagram commutes. Much more can be said about this construction in special cases. For example, if S and X are both compact then T is compact; if S has identity 1 and $SX = X$, then $\varphi(1)$ is an identity for T; and if S and X are both connected and some element of S acts as a constant (there exists $s_0 \in S$ such that $s_0 X$ is a point) then T is connected. We summarize a special case of the above in the following theorem.

2.3. Theorem. *Let (S, X, μ) be an action triple with S and X compact. Then there exists a compact semigroup T with (S, X, μ) embedded in T and such that if S' is the subsemigroup of T identified with S and X' the subspace identified with X, we have $S' \cup X' = T$ and $X' = K(T)$.*

References

[1] J. M. Day and A. D. Wallace, *Multiplication induced in the state space of an act*, Math. Systems Theory 1 (1968), pp. 305–314.

[2] J. Dugundji, *Topology*, 1966.

[3] T. H. McH. Hanson, *Actions that fiber and vector semigroups*, to appear.

[4] D. P. Stadtlander, *Semigroup Actions on Topological Spaces*, Dissertation, Pennsylvania State University 1966.

[5] A. D. Wallace, *Recursions with semigroup state spaces*, Revue Romaine de Math. Pure et App. 12 (1967), pp. 1411–1415.

[6] J. C. Warndof, *Topologies uniquely determined by their continuous self maps*, Fund. Math. 66 (1969), pp. 25–43.

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