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Partial order and collapsibility of 2-complexes [1]

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A partial order $\leqslant$ on a topological space X is said to be *closed* if $\leqslant$ is a closed subset of $X\times X$ (where $X\times X$ has the product topology). An element $\theta \in X$ is called the *zero* of X if and only if $\theta\leqslant x$ for each $x\in X$. For each $x\in X$ the *principal ideal* determined by x is denoted by $L(x)$ and

$$L(x) = \{y\in X|\ y\leqslant x\}.$$

If $A\subset X$ we let

$$L(A) = \bigcup\{L(y)|\ y\in A\}.$$

A compact, Hausdorff space is said to be *acyclic* if and only if it has the Spanier cohomology groups of a space with exactly one point. In [2] it was proved that the Spanier cohomology groups coincide with the Čech cohomology groups on compact Hausdorff spaces. We shall need the following theorem of A. D. Wallace [6]:

THEOREM (Wallace). *Let X be a compact space with a closed partial order. If X has a zero and if all of the principal ideals of X are acyclic then X is acyclic.*

A metric for a metric space X is said to be *strongly convex* if and only if for each pair of distinct points x and y of X there is a unique line segment in X with endpoints x and y.

Warren White proved in [7] that a 2-complex K is collapsible if and only if $|K|$ admits a strongly convex metric ϱ. The following theorem weakens rather dramatically the condition that $|K|$ admit a strongly convex metric.

THEOREM. *Let K be a finite 2-complex. Then K is collapsible if and only if $|K|$ admits a closed partial order with zero and with acyclic principal ideals.*

Proof. ($\Rightarrow$) If K is collapsible then by White's theorem [7] $|K|$ admits a strongly convex metric ϱ. Let $\theta\in|K|$. Define $x\leqslant y$ in $|K|$ if and only

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if either $x = y$ or x lies on the unique line segment with endpoints θ and y. Then $\leqslant$ is easily seen to be a closed partial order on $|K|$ with zero θ and every principal ideal is either an arc or a point.

($\Leftarrow$) Suppose that $|K|$ admits a closed partial order $\leqslant$ with zero θ and with acyclic principal ideals. By Wallace's theorem $|K|$ is acyclic.

Just suppose that K is not collapsible. Then K contains a minimal closed 2-complex P such that K collapses to P. Then $|P|$ is acyclic since it is a retract of $|K|$. A well-known theorem of Borsuk states that a Peano continuum that is not unicoherent can be retracted onto some simple closed curve and hence is not acyclic. It follows that $|P|$ is unicoherent. Let Q be a non-degenerate cyclic element of $|P|$. Then Q is unicoherent and Q has no cutpoints (or local cutpoints). Since P does not admit any further collapsing it follows that Q has no free edges.

Since Q is a compact subset of $|K|$ and $\leqslant$ is a closed partial order on $|K|$, Q has a maximal element x_0 (see [5]).

Let U be an open neighbourhood of x_0 in $|K|$ such that the boundary of U in $|K|$ is a graph, $\overline{U}$ is the cone with vertex x_0 over the boundary of U and $\overline{U}$ contains no vertices of K distinct from x_0. Then the boundary of $U \cap Q$ in Q contains a simple closed curve C since Q is a 2-complex with no free edges and no cutpoints. Furthermore, C bounds a 2-cell in $Q \cap \overline{U}$.

Consider

$$L(Q) = \big(L(Q) \cap \overline{U}\big) \cup L\big(L(Q) - U\big).$$

Since the partial order on $|K|$ is closed and Q is compact it follows that $L(Q)$ and $L\big(L(Q)-U\big)$ are compact sets. By Wallace's theorem $L(Q)$ and $L\big(L(Q)-U\big)$ are acyclic. Since x_0 is a maximal element of Q, $x_0 \notin L\big(L(Q)-U\big)$.

Let

$$i_2\colon L\big(L(Q)-U\big) \cap \overline{U} \to L(Q) \cap \overline{U},$$

$$i_1\colon L\big(L(Q)-U\big) \cap \overline{U} \to L\big(L(Q)-U\big)$$

and

$$i\colon C \to L\big(L(Q)-U\big) \cap \overline{U}$$

be inclusions.

The Mayer–Vietoris sequence for the compact Hausdorff triple $\big(L(Q);\ L\big(L(Q)-U\big),\ L(Q) \cap \overline{U}\big)$ is exact (see [1] and [2]). We have the exact sequence

$$\ldots \to H^1\big(L\big(L(Q)-U\big)\big) \oplus H^1\big(L(Q) \cap \overline{U}\big) \xrightarrow{I^*} H^1\big(L\big(L(Q)-U\big) \cap \overline{U}\big) \to H^2\big(L(Q)\big) \to \ldots$$

where $I^*(h_1, h_2) = i_1^*(h_1) - i_2^*(h_2)$. Since $H^1\big(L\big(L(Q)-U\big)\big) = 0$ and $H^2\big(L(Q)\big) = 0$ it follows that i_2^* is onto $H^1\big(L\big(L(Q)-U\big) \cap \overline{U}\big)$.

It is not difficult to see that the simple closed curve C is a retract of $L\big(L(Q)-U\big) \cap \overline{U}$ since $x_0 \notin L\big(L(Q)-U\big) \cap \overline{U}$ and the boundary of U in $|K|$ is a retract of $\overline{U} - \{x_0\}$. Hence,

$$i^*\colon H^1\big(L\big(L(Q)-U\big) \cap \overline{U}\big) \to H^1(C)$$

is onto. Thus,

$$i^* \circ i_2^*\colon H^1\big(L(Q) \cap \overline{U}\big) \to H^1(C)$$

is onto. However, the injection

$$i_2 \circ i\colon C \to L(Q) \cap \overline{U}$$

is homotopic to a constant map since C bounds a 2-cell in $L(Q) \cap \overline{U}$. Thus $0 = (i_2 \circ i)^* = i^* \circ i_2^*$. This is a contradiction since $H^1(C) \neq 0$ and $i^* \circ i_2^*$ is onto $H^1(C)$. This proves that K is collapsible.

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