An exact sequence from the $n$th to the $(n-1)$-st fundamental group

by

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0. Introductory remark. For each pointed compactum $(X, x)$ contained in the Hilbert cube we define, in each dimension $n > 0$, the approaching group $\pi_n(X, x)$ and the inward group $I_n(X, x)$. Using category theory we show that $\pi_n(X, x)$ and $I_n(X, x)$ depend only on the homotopy type of $(X, x)$. We define an endomorphism from $I_n(X, x)$ to $I_{n-1}(X, x)$ whose kernel is the $n$th fundamental group, $\pi_n(X, x)$ of Karol Borsuk. There is an epimorphism from $\pi_n(X, x)$ to $\pi_{n-1}(X, x)$ whose kernel equals the colimit of the above endomorphism. Thus there is an exact sequence of groups and homomorphisms

$$0 \to \pi_n(X, x) \to I_n(X, x) \to I_{n-1}(X, x) \to \pi_{n-1}(X, x) \to 0.$$ 

We work out the above sequence in full when $n = 1$ and $X$ is the 3-adic solenoid $\Sigma_3$.

Consider the system of neighbourhoods $U$ of $X$ in the Hilbert Cube with inclusion mappings. It is known that $\pi_n(X, x)$ can also be obtained by applying the $n$th homotopy functor $\pi_n$ to this system and passing to the inverse limit, i.e.

$$\pi_n(X, x) = \lim \pi_n(U, x).$$

1. Notation. $R, R^+, R^*, J, J^*, I, I^*, S^n, B^n, p_n$ denote respectively, the real numbers, Euclidean $n$-space, the non negative real numbers, the integers, the non negative integers, the closed interval $[0, 1]$, the Hilbert cube, the $n$-sphere, the $n$-ball and the point $(1, 0, 0, \ldots, 0) \in S^{n-1} \subset \mathbb{R}^n$.

For each $n \geq 0$ let $g_n$ be the identification mapping from $I \times S^n$ to $I \times S^n/(I \times \{p_n\}) \cup ([0, 1] \times S^n)$ to $S^{n+1}$ and let $h_n$ be a homeomorphism from $I \times S^n/(I \times \{p_n\}) \cup ([0, 1] \times S^n)$ to $S^{n+1}$ such that

$$h_n = g_n([(I \times \{p_n\}) \cup ([0, 1] \times S^n)) = \{p_n\} \subset S^{n+1}.$$
For each $n \geq 0$, let $r_n$ be the identification mapping from $S^n \times I$ to $S^n \times I / S^n \times \{1\}$ and let $\kappa_n$ be a homeomorphism from $S^n \times I / S^n \times \{1\}$ to $E^{n+1}$ such that $\kappa_n = r_n(S^n \times \{1\}) = \{a\}$, where $a = (0, 0, ..., 0, 0)$ the centre of $E^{n+1}$. For each $n \geq 1$, let $s_n$ be a continuous mapping from $E^n$ to $S^n$ such that $s_n(S^n \times \{1\}) = \{p_0\}$ and $s_n$ maps $S^n - S^n \cdot r_{n-1}$ homeomorphically onto $S^n - \{p_0\}$.

We remark that, for each $n \geq 1$, $(S^n, p_0)$ is a homotopy cogroup with a continuous multiplication mapping $\nu^n$ from $(S^n, p_0)$ to $(S^n, p_0) \vee (S^n, p_0)$ and a continuous homotopy inverse mapping $\eta^n$ from $(S^n, p_0)$ to $(S^n, p_0)$. We assume that for all $n \geq 1$, $\nu^{n+1}$ and $\eta^{n+1}$ are derived by suspending $\nu^n$ and $\eta^n$. In other words, for all $n \geq 1$, $(t, \epsilon) \mapsto \nu^{n+1}(t, \epsilon) = (\nu^n(t), \eta^n(t, \epsilon)),$

\begin{equation}
\nu^{n+1} \cdot \kappa_n \cdot \mu_n(t, \epsilon) = (\kappa_n \cdot \mu_n(t), \kappa_n \cdot \mu_n(\epsilon)),
\end{equation}

\begin{equation}
\eta^{n+1} \cdot \kappa_n \cdot \mu_n(t, \epsilon) = \kappa_n \cdot \mu_n(t, \eta^n(\epsilon)).
\end{equation}

If $f, g$ are continuous mappings from $S^n$ to a topological space $X$, then $f \times g$ denotes the continuous mapping $(f, g) : S^n \to X$ and $f^{-1}$ the continuous mapping $f^{-1} : X \to S^n$.

If $Y$ and $Z$ are sets, $\theta$ a function from $Y \times Z$ to $I^*$. Then for each $y \times z$, we denote by $\theta_{y, z}$ that function from $Z$ to $I^*$ that carries each $z \in \theta_{y, z} \in I^*$ and for each $z \in Z$ we denote by $\theta_z$ that function from $Z$ to $I^*$ that carries each $y \in \theta_z \in I^*$. As an example of this notation, if $\varphi$ is a continuous mapping from $E^n \times S^n \times I$ to $I'$ then for $x \in E^n, t \in I$, $\varphi_{x, t} = \varphi$ is that continuous mapping from $S^n \times I$ which carries $s \in S^n$ to $\varphi_{x, t}(s) = \varphi(s, t) \in I'$.

Let $P$ be a topological space. If $x$ and $I$ are continuous mappings from $P \times S^n \to I^*$ then we denote by $x \times I$ and $I \times I$ respectively the continuous mappings from $P \times S^n \times I^*$ to $I^*$ such that $(x \times I)_p = x_p \times I_p$ and $(I \times I)_p = (I_p \times I_p)$, for all $p \in P$.

If $K$ is a category, the objects of $K$ are denoted by Ob $K$, the morphisms by Mor $K$ and for $X, Y \in$ Ob $K$ the morphisms of $K$ with $X$ as domain and $Y$ as codomain are denoted by Mor $K(X, Y$).

The set of compact neighbourhoods of the compactum $X \subset I^*$ are denoted by Nhd $X$.

2. DEFINITION. Three categories $C, \overline{C}, \mathcal{C}$.

We will define 3 categories $C, \overline{C}$, and $\mathcal{C}$ which have the same objects.

\begin{equation}
\text{Ob } C = \text{Ob } \overline{C} = \text{Ob } \mathcal{C} = \{ (X, y) | x \in X \subset I^*, X \text{ is compact} \}.
\end{equation}

The morphisms of $C$ are the set of base point preserving continuous mappings between pointed compacta with the usual composition. The identity continuous mapping at $(x, y)$ is denoted $\text{Id}_{x,y}$.

The objects of $C$ are defined above. We define the morphisms and composition in $C$ as follows. A continuous mapping $f : R^+ \times I^*$ to $I^*$ is a member of Mor $C((X, x), (Y, y))$ and is referred to as an approaching mapping from $(X, x)$ to $(Y, y)$ if

\begin{equation}
f(R^+ \times \{x\}) = \{y\},
\end{equation}

\begin{equation}
given \text{ any } V \in \text{Nhd}(X) \text{ there is a } U \in \text{Nhd}(X) \text{ and an } r \in R^+ \text{ such that } f([r, \infty) \times U) \subset V.
\end{equation}

The composition $gf$ in $C$ of

\begin{equation}
f \in \text{Mor}_C((X, x), (Y, y)) \text{ and } g \in \text{Mor}_C((Y, y), (Z, z))
\end{equation}

is defined by $(gf)(r, s) = g(f(r), s)$ for each $(r, s) \in R^+ \times I^*$. As $g$ is easily seen to satisfy (2.2) and (2.3) above $g \in \text{Mor}_C((X, x), (Z, z))$. No confusion will result if we denote by $\text{Id}_{x,y}$ the identity element of $\text{Mor}_C((X, x), (X, x))$. $\text{Id}_{x,y}(r, s) = i$, for each $(r, s) \in R^+ \times I^*$.

The category $\mathcal{C}$ was first defined in [1] by K. Borsuk. We now describe $\mathcal{C}$ in a manner suited to our purposes. Ob $\mathcal{C}$ has been defined above. Denoting by $f^* : J^+ \times I^* \to I^*$ the restriction of the continuous mapping $f : R^+ \times I^* \to I^*$,

\begin{equation}
\text{Mor}_{\mathcal{C}}((X, x), (Y, y)) = \{ f | f \in \text{Mor}_C((X, x), (Y, y)) \}.
\end{equation}

Composition in $\mathcal{C}$ is $(gf)(r) = (gf)(r)$. Clearly $\text{Id}_{x,y}$ is the identity element of $\text{Mor}_{\mathcal{C}}((X, x), (Y, y))$ and no confusion will be caused if we simply denote this morphism by $\text{Id}_{x,y}$.

We will also find useful the original definition of $\text{Mor}_C((X, x), (Y, y))$ given by K. Borsuk in [1]. A fundamental sequence $f$ from $(X, x)$ to $(Y, y)$ is a sequence $f = (f_n)_{n \in \mathbb{N}}$ of continuous mappings $f_n$ from $I^*$ to $I^*$ such that $f_n(x) = y$ for all $n \geq 0$ and such that given any $V \in \text{Nhd}(X)$ there is a $U \in \text{Nhd}(X)$ and a $j \in J^*$ such that $f_n(U) \subset V$ for all $n \geq 0$. From this point of view composition in $\mathcal{C}$ is defined

\begin{equation}
(gf)(n) = g_n \circ f_n.
\end{equation}

3. DEFINITION. Homotopy in $C, \overline{C}$ and $\mathcal{C}$.

Two morphisms $f, g \in \text{Mor}_C((X, x), (Y, y))$ are said to be pointed homotopic if there is a continuous mapping $H$ from $X \times I$ to $Y$ such that $H = f$, $H = g$ and $H([x] \times I) = \{y\}$. In this case we write $H; f = g$. Pointed homotopy is well known to be an equivalence relation on the morphisms of $C$ compatible with composition in $C$. The class of $f \in \text{Mor}_C$ is denoted by $[f]$. There is therefore a category $\mathcal{C}$ with Ob $\mathcal{C} =$ Ob $C$ and morphisms $\mathcal{C}$ classes of morphisms of $C$. Composition in $\mathcal{C}$ is $[g] \cdot [f] = [g \circ f]$ and $\text{Id}_{x,y} \in \text{Mor}_{\mathcal{C}}((X, x), (X, x))$ is the identity element.
Two morphisms \( f, g \in \text{Mor}_C((X, x), (Y, y)) \) are said to be pointed approaching homotopic if there is a continuous mapping \( H \) from \( R^+ \times I \) to \( I^\ast \) such that

\[
\begin{align*}
\nu_H = f, & \quad \alpha_H = g, \\
H(R^+ \times \{x\} \times I) = \{y\}, & \quad \text{given any } V \in \text{Nhd}(Y) \text{ there is a } U \in \text{Nhd}(X) \text{ and an } r \in R^+ \text{ such that } H([r, \infty) \times U \times I) \subset V.
\end{align*}
\]

In this case we write \( H; f \simeq g \) (approaching). Pointed approaching homotopy is an equivalence relation on the morphisms of \( C \). If

\[
f \in \text{Mor}_C([W, w], (X, x)), \quad g, h \in \text{Mor}_C((X, x), (Y, y)), \quad k \in \text{Mor}_C((Y, y), (Z, z))
\]

and \( H; f \simeq h \) (approaching), then \( H; gf \simeq kg \) (approaching), and \( H; kH \) are continuous mappings from \( R^+ \times I \) to \( I^\ast \) defined by \( \langle Hf \rangle = (\alpha_H)f \) and \( \langle kH \rangle = k(\alpha_H) \), for each \( \epsilon I \). Thus the equivalence relation of pointed approaching homotopy is compatible with composition in \( C \). Denoting the pointed approaching homotopy class of \( f \) by \( \{f\} \) we may form a new category \( \mathcal{C}_q \) whose objects are the same as those of \( C \) and whose morphisms are classes of morphisms of \( C \). Composition in \( \mathcal{C}_q \) is \( \{fg\} \). The identity element of \( \text{Mor}_{\mathcal{C}_q}((X, x), (X, x)) \) is \( [id_{(X, x)}] \). \( \mathcal{C}_q \) is called the pointed approaching homotopy category.

We next define homotopy on \( C \) and use this concept of homotopy to describe the (pointed) fundamental category. These ideas were first defined in [1] by K. Borsuk.

Two morphisms \( f, g \in \text{Mor}_C((X, x), (Y, y)) \) are said to be pointed fundamentally homotopic if there is a continuous mapping \( H \) from \( R^+ \times I \) to \( I^\ast \) such that

\[
\begin{align*}
\nu_H = f, & \quad \alpha_H = g, \\
H(J^+ \times \{x\} \times I) = \{y\}, & \quad \text{given any } V \in \text{Nhd}(Y) \text{ there is a } U \in \text{Nhd}(X) \text{ and an } j \in J^+ \text{ such that } H([a, b] \times U \times I) \subset V, \text{ for each } a < b.
\end{align*}
\]

In this case we write \( H; f \simeq g \) (fundamental). Fundamental homotopy is an equivalence relation on the morphisms of \( C \) compatible with the composition of \( C \) and the class of \( f \) is denoted \( \{f\} \). As above we get a new category \( \mathcal{C}_Q \) called the (pointed) fundamental category, \( \text{Ob}\mathcal{C}_Q = \text{Ob}\mathcal{C} \), morphisms in \( \mathcal{C}_Q \) are classes of morphisms in \( C \) and composition in \( \mathcal{C}_Q \) is \( \{gf\} \).

4. Remark. Comparison of \( \mathcal{C}_Q, \mathcal{C}_q \) and \( \mathcal{C} \).

If \( (X, x), (Y, y) \in \text{Ob}\mathcal{C} \) and if \( f \) is a continuous mapping from \( R^+ \times I^\ast \) to \( I^\ast \) such that \( f_r = \varphi; I^\ast \rightarrow I^\ast \), for all \( r \in R^+ \) and \( \varphi(x) = (Y, y) \) a continuous mapping from \( (X, x) \) to \( (Y, y) \), then we say that \( f \) is generated by \( \varphi \). Such a mapping \( f \) is an approaching mapping from \( (X, x) \) to \( (Y, y) \), since \( f(R^+ \times \{x\}) = \{y\} \) if \( \varphi(x) = (Y, y) \) and if \( V \in \text{Nhd}(Y) \), there is, by the continuity of \( \varphi \), a \( U \in \text{Nhd}(X) \) such that \( f(U) = f(R^+ \times U) \subset V \). Since any continuous mapping \( \theta \) from \( (X, x) \) to \( (Y, y) \) can be extended to a continuous mapping \( \varphi \) from \( I^\ast \) to \( I^\ast \) it follows that \( \theta \in \text{Mor}_C((X, x), (Y, y)) \) generates at least one \( f \in \text{Mor}_C((X, x), (Y, y)) \).

It is clear that \( \text{Id}_{(X, x)} \in \text{Mor}\mathcal{C} \) generates \( \text{Id}_{(X, x)} \in \text{Mor}_C \) and that if \( \theta \in \text{Mor}_C((X, x), (Y, y)), \theta' \in \text{Mor}_C((Y, y), (Z, z)) \), generate \( f \in \text{Mor}_C((X, x), (Z, z)) \), \( g \in \text{Mor}_C((Z, z), (Y, y)) \), respectively then \( \theta \circ \theta' \in \text{Mor}_C \) generates \( f \circ g \in \text{Mor}_C \). Moreover if \( a, \beta \in \text{Mor}_C((X, x), (Y, y)) \) generate \( a, \beta \in \text{Mor}_C((X, x), (Y, y)) \) and if \( H; a \simeq \beta \) then there is a continuous mapping \( \tilde{H} \) from \( R^+ \times I \) to \( I^\ast \) such that \( \tilde{H} = a = a, \tilde{K} = k = k, \text{ and } K_{[x, t]} = K \). Defining the mapping \( L \) from \( R^+ \times I \) to \( I^\ast \) by \( L_r = L \), for all \( r \in R^+ \), then using the compactness of \( X \), it is easy to see that \( I \alpha = a \) (approaching).

From the above observations it follows that there is a functor \( \mathcal{C}_E \) from \( \mathcal{C}_Q \) to \( \mathcal{C}_q \) taking \( (X, x) \in \text{Ob}\mathcal{C}_Q \) to \( (X, x) \in \text{Ob}\mathcal{C}_q \) and \( [f] \in \text{Mor}\mathcal{C}_q \) to \( [f] \in \text{Mor}\mathcal{C}_q \), where \( \mathcal{C}_E([f]) = [f] \) is well defined to be the approaching homotopy class of any mapping \( f \) generated by \( \theta \).

It is immediate from definition 2 above that there is a functor \( R \) from \( \mathcal{C}_Q \) to \( \mathcal{C}_q \) carrying \( (X, x) \in \text{Ob}\mathcal{C}_Q \) to \( (X, x) \in \text{Ob}\mathcal{C}_q \) and \( f \in \text{Mor}_C \) to \( f \in \text{Mor}_C \). By the definition of \( \text{Mor}\mathcal{C}_q \) in (2.4) above it is clear that \( R \) considered as a function from the set \( \text{Mor}\mathcal{C}_Q \) to the set \( \text{Mor}\mathcal{C}_q \) is surjective.

If \( H; f \simeq g \) (approaching), then denoting by \( H \) the restriction of \( H \) to \( J^+ \times I \) it is clear that \( H; f \simeq g \) (fundamental). Thus there is a functor \( \mathcal{C}_R \) from \( \mathcal{C}_Q \) to \( \mathcal{C}_Q \) taking \( (X, x) \in \text{Ob}\mathcal{C}_Q \) to \( (X, x) \in \text{Ob}\mathcal{C}_Q \) and \( [f] \in \text{Mor}\mathcal{C}_Q \) to \( [f] \in \text{Mor}\mathcal{C}_Q \). The surjectivity of \( R \) above implies that \( \mathcal{C}_R \) considered as function from the set \( \text{Mor}\mathcal{C}_Q \) to the set \( \text{Mor}\mathcal{C}_Q \) is surjective.

5. Definition. The approaching functor.

A continuous mapping \( \xi \) from \( R^+ \times S^m \) to \( I^\ast \) is said to be an approaching \( n \)-mapping of \( (X, x) \in \text{Ob}\mathcal{C}_Q \) if

\[
\xi((R^+ \times \{p_0\}) \times \{y\}) = \{y\}, \quad \text{given } V \in \text{Nhd}(X) \text{ there is an } r \in R^+ \text{ such that } \xi([r, \infty) \times S^m) \subset V,
\]
If \( \xi \) and \( \xi' \) are approaching \( n \)-mappings of \((X, a)\) then we say that \( \xi \) is approaching homotopic to \( \xi' \) iff there is a continuous mapping \( \Phi \) from \( R^+ \times S^n \times I \) to \( I^n \) such that

\[
\phi = \xi, \quad \phi' = \xi',
\]

\[
\Phi(R^+ \times \{p^n\} \times I) = \{a\},
\]

given \( V \in \text{Nhd}(X) \) there is an \( r \in R^+ \) such that \( \Phi([r, \infty) \times S^n \times I) \subset V \).

In this case we write \( \Phi; \xi \simeq \xi' \) (approaching). Approaching homotopy is an equivalence relation on the set of all approaching \( n \)-mappings of \((X, a)\) and the class of \( \xi \) will be written \( \langle \xi \rangle \). The set of classes of approaching \( n \)-mappings of \((X, a)\) will be denoted \( \mathcal{P}(X, a) \), \( n \geq 0 \).

We denote by \( \ast \) the approaching \( n \)-mapping of \((X, a)\) such that \( c(R^+ \times S^n) = \{a\} \). If \( \xi \) and \( \eta \) are approaching \( n \)-mappings of \((X, a)\) then \( \xi^{-1} \) and \( \xi \ast \eta \) are also approaching \( n \)-mappings of \((X, a)\). If \( \xi \simeq \xi' \) (approaching) and \( \eta \simeq \eta' \) (approaching) then \( \xi \ast \eta \simeq \xi' \ast \eta' \) (approaching). Thus, when \( n \geq 1 \), we can compose classes of approaching mappings as follows, \( \langle \xi \rangle \ast \langle \eta \rangle = \langle \xi \ast \eta \rangle \). If \( \xi, \xi' \) and \( \xi'' \) are approaching \( n \)-mappings of \((X, a)\) the following remarks are easily proven.

\[
\langle \xi \rangle \ast \langle \xi' \rangle = \langle \xi \ast \xi' \rangle,
\]

\[
\langle \xi \rangle \ast \langle \xi'^{-1} \rangle = \langle \xi \ast \xi'^{-1} \rangle = \langle \xi'^{-1} \rangle \ast \langle \xi \rangle,
\]

\[
\langle \xi \rangle \ast \langle \xi' \rangle \ast \langle \xi'' \rangle = \langle \xi \ast \xi' \rangle \ast \langle \xi'' \rangle = \langle \xi' \rangle \ast \langle \xi \ast \xi'' \rangle,
\]

\[
\langle \xi \rangle \ast \langle \eta \rangle = \langle \xi \ast \eta \rangle = \langle \eta \ast \xi \rangle,
\]

\[
\langle \xi \rangle \ast \langle \eta \rangle = \langle \xi \ast \eta \rangle \simeq \langle \eta \ast \xi \rangle \simeq \langle \xi \ast \eta \rangle.
\]

Thus \( \mathcal{P}(X, a) \) is a set of group or abelian group according as \( n = 0, 1 \) or \( n \geq 2 \).

If \( \xi \) is an approaching \( n \)-mapping of \((X, a)\) and \( f \in \text{Mor}_C((X, a), (Y, y)) \) there is a continuous mapping \( f \xi \) from \( R^+ \times S^n \) to \( I^n \) defined by \( f \xi(r, a) = f(r, \xi(r, a)) \), for each \( (r, a) \in R^+ \times S^n \). We observe that if \( V \in \text{Nhd}(Y) \) there is a \( U \in \text{Nhd}(X) \) and \( r \in R^+ \) such that \( f([r, \infty) \times U) \subset V \) and there is an \( r \in R^+ \) such that \( f([r, \infty) \times U) \subset V \). Thus

\[
f \xi([r, \infty) \times S^n) \subset C(f([r, \infty) \times S^n)) \subset f([r, \infty) \times U) \subset V.
\]

From these 2 observations it follows that \( f \xi \) is an approaching \( n \)-mapping of \((Y, y)\).

It is easy to see that if \( \xi \) and \( \eta \) are approaching \( n \)-mappings of \((X, a)\), \( f, g \in \text{Mor}_C((X, a), (Y, y)) \), then \( h \in \text{Mor}_C((Y, y), (Z, z)) \), then

\[
f \xi \ast \eta = (f \xi) \ast (f \eta),
\]

where \( \Psi \) is defined by \( \Psi = f(\phi) \), for each \( t \in I \).

\[
(5.12) \quad f \phi = f \xi \ast g \phi \quad \text{for each} \quad t \in I.
\]

\[
(5.13) \quad (f \xi) \ast h = h (f \xi).
\]

From the above it follows that \( \mathcal{P}_n \) is a functor from \( \mathcal{C}_C \) to the category of sets, groups or abelian groups according as \( n = 0, 1 \) or \( n \geq 2 \). For \( [f] \in \text{Mor}_C((X, a), (Y, y)) \), \( \langle \xi \rangle \in \mathcal{P}_n(X, a) \),

\[
\mathcal{P}_n(f) = \langle f \xi \rangle \in \mathcal{P}_n(Y, y).
\]

Since \( \mathcal{P}_n \) is a functor from \( \mathcal{C}_C \) it follows that \( \mathcal{P}_n(X, a) \) is invariant up to equivalence of objects in the approaching category. Composing the functor \( \mathcal{P}_n \) with the functor \( \mathcal{E}_n \) above to obtain the functor \( \mathcal{P}_n \ast \mathcal{E}_n \) from \( \mathcal{C}_C \) we see that (a fortiori) \( \mathcal{P}_n(X, a) = \mathcal{P}_n \ast \mathcal{E}_n(X, a) \) is invariant up to homotopy type of pointed compacta.

6. Definition. The inward functor.

A continuous mapping \( \xi \) from \( J^+ \times S^n \) to \( I^n \) is said to be an inward \( n \)-mapping of \((X, a) \in \text{Ob}_\mathcal{C}_C \) iff

\[
(6.1) \quad \xi([J^+ \times \{p^n\}] = \{a\} \),
\]

\[
(6.2) \quad \text{given} \ V \in \text{Nhd}(X) \text{ there is a} \ j \in J^+ \text{ such that} \ \xi([S^n]) \subset V, \text{ for all} \ j \geq j_0.
\]

If \( \xi \) and \( \xi' \) are inward \( n \)-mappings of \((X, a)\) then we say that \( \xi \) is inward homotopic to \( \xi' \) iff there is a continuous \( \Phi \) from \( J^+ \times S^n \times I \) to \( I^n \) such that

\[
(6.3) \quad c = 0, \ 0 = \xi', \quad\Phi(J^+ \times \{p^n\} \times I) = \{a\},
\]

\[
(6.4) \quad \Phi(J^+ \times \{p^n\} \times I) = \{a\},
\]

\[
(6.5) \quad \text{given} \ V \in \text{Nhd}(X) \text{ there is a} \ j \in J^+ \text{ such that} \ \Phi([S^n] \times I) \subset V, \text{ for all} \ j \geq j_0.
\]

In this case we write \( \Phi; \xi \simeq \xi' \) (inwardly).

The set of classes of inward \( n \)-mappings will be denoted by \( \mathcal{I}_n(X, a) \), for each \( n \geq 0 \). As in 5 above \( \mathcal{I}_n(X, a) \) is a set group or abelian group according as \( n = 0, 1 \) or \( n \geq 2 \). The identity element of \( \mathcal{I}_n(X, a) \) is denoted by \( \langle 0 \rangle \) where \( o(j, e) = a \), for all \( (j, e) \in J^+ \times S^n \). Multiplication in \( \mathcal{I}_n(X, a) \) is \( \langle \xi \rangle \ast \langle \eta \rangle = \langle \xi \ast \eta \rangle \).
9. Definition. The homomorphism \( i_n : \tau_n(X, x) \to I_n(X, x) \).

For all \( n \geq 0 \) we denote by \( i_n \) the inclusion mapping \( i_n : \tau_n(X, x) \subset I_n(X, x) \). When \( n > 1 \) this is a homomorphism between groups.

10. Definition. Advancing a function.

Let \( X \) and \( Y \) be sets and \( f \) a function from \( J^+ \times X \) to \( Y \). Then there is a function \( A(f) \) also from \( J^+ \times X \) to \( Y \) which takes \( (n, t) \in J^+ \times X \) to \( A(f)(n, t) = f(n + 1, t) \in Y \). \( A(f) \) is called the advancement of \( f \).

11. Definition. The advancing endomorphism, \( A_\delta : I_n(X, x) \to I_n(X, x) \).

Let \( \delta \) and \( \eta \) be inward \( n \)-mappings of \((X, x)\). We observe that, when \( n \geq 1 \), \( A(\delta \circ \eta) = A(\delta) \circ A(\eta) \), and, for all \( n \geq 0 \), if \( \delta \equiv \eta \) (inwardly) then \( A(\delta) \equiv A(\eta) \) (inwardly). Thus, for each \( n \geq 0 \), there is a function \( A_n \) from \( I_n(X, x) \) to \( I_n(X, x) \), which takes each \( \langle \delta \rangle \in I_n(X, x) \) to \( A_n(\langle \delta \rangle) = \langle A(\delta) \rangle \in I_n(X, x) \) and which when \( n \geq 1 \) is an endomorphism called the \( n \)-th advancing endomorphism of \((X, x)\).

12. Remark. For each \( n \geq 0 \) we denote by \( I_0 \) the identity function from \( I_n(X, x) \) to \( I_n(X, x) \), \( I_0(\langle \delta \rangle) = \langle \delta \rangle \), for \( \langle \delta \rangle \in I_n(X, x) \). In the case \( n = 2 \), \( I(X, x) \) is abelian and so \( I_0 \) is an endomorphism from \( I_n(X, x) \) to \( I_n(X, x) \).

In the case \( n = 1 \) there is a function which we denote by \( I_1 \) from \( I_1(X, x) \) to \( I_1(X, x) \) which takes \( \langle \delta \rangle \in I_1(X, x) \) to \( I_1(\langle \delta \rangle) = \langle A(\delta) \rangle \in I_1(X, x) \). Since \( I_1(X, x) \) may not be an abelian group \( I_1 \) is not a general endomorphism.

13. Definition. The homomorphism \( \delta_n : I_n(X, x) \to \tau_{n-1}(X, x) \).

Let \( \delta \) be an inward \( n \)-mapping where \( n \geq 1 \). Denote by \( B(\delta) \) the continuous mapping from \( R^+ \times S^{n-1} \to I^* \) such that, for all \( j \in J^+ \) and \( (r, e) \in \langle j \rangle \times \langle j + 1 \rangle \times S^{n-1} \) takes \( (r, e) \) to \( B(\delta)(r, e) = \delta_j \circ \eta_{j-1} \circ \varphi_{j-1}(r-j, e) \in I^* \).

By definition of \( \varphi_{j-1} \), \( j \), and \( \eta_{j-1} \), in notational remark 1, there is no ambiguity in this definition of \( B(\delta) \). By (6.5), \( B(\delta) \) is an approximating \((n-1)\)-mapping of \((X, x)\).

Let \( \delta \) and \( \eta \) be inward \( n \)-mappings for \( n \geq 2 \), then by 1.1, \( B(\delta \circ \eta) = B(\delta) \circ B(\eta) \). Again if \( \delta \) and \( \eta \) are inward \( n \)-mappings for \( n \geq 1 \) and \( \delta \equiv \eta \) (inwardly) then defining the continuous mapping \( W \) from \( R^+ \times S^{n-1} \times \langle J^+ \rangle_1 \times \langle J^+ \rangle_2 \) by \( W(\varphi, \theta) = (\theta \circ \varphi, \varphi, \theta) \in \langle J^+ \rangle_1 \times \langle J^+ \rangle_2 \) we see that \( W(\varphi, \theta) \equiv \varphi \equiv \eta \) (approximating).

Thus there is, for each \( n \geq 1 \), a function \( \delta_n \) from \( I_n(X, x) \) to \( \tau_{n-1}(X, x) \) which takes each \( \langle \delta \rangle \in I_n(X, x) \) to \( \delta_n(\langle \delta \rangle) = \langle \delta \rangle \in \tau_{n-1}(X, x) \). Moreover, when \( n \geq 2 \), \( \delta_n \) is a homomorphism between groups.

14. Definition. The homomorphism \( \gamma_n \) from \( \tau_n(X, x) \) to \( \tau_{n-1}(X, x) \).
Next we will define an inward $n$-mapping $\eta'$ of $(X, x)$ s.t. $B(\eta') = \eta$. Since $\eta|_{S^m-1} = \eta_i|_{S^m-1} = \eta_i|_{S^m-1+1} = \eta_i|_{S^m-1+0} = \eta|_{S^m-1+0} = \theta$, there is for each $j \geq 0$, an unique continuous mapping $\theta_j$ from $I^* \times S^m-1 \times \{y_j\} \cup \{0, 1\} \times S^m-1$ to $I^*$ s.t. for each $(t, e) \in I \times S^m-1$, $\theta_j \circ g_{n-1}(t, e) = \eta_i(t, e)$. Define the continuous mapping $\eta'$ from $I^* \times S^m-1$ to $I^*$ by $\eta'_i = \theta_i \circ h_{n-1}^{-1}$, for all $j \in J^*$. Since $\eta$ is an approaching $(n-1)$-mapping of $(X, x)$ it follows from (5.2) that $\eta'$ is an inward $n$-mapping of $(X, x)$. Moreover, for each $j \in J^*$, given $(r, e) \in [j, j+1] \times S^m-1$,

$$B(\eta')(r, e) = \eta'_i \circ h_{n-1} \circ g_{n-1}(r, e),$$

by definition of $B(\eta')$ (see (13)),

$$\theta_i \circ h_{n-1}^{-1} \circ h_{n-1} \circ g_{n-1}(r, e),$$

by definition of $\theta_i$, above

$$\theta_i \circ g_{n-1}(r, e),$$

by definition of $\theta_i$ above

$$\eta(r, e).$$

Thus $B(\eta') = \eta$.

Therefore $d_n(\eta') = B(\eta'') = \eta$. Therefore $\text{Kernel}(\gamma_{n-1}) \subset \text{Image}(\delta_n)$. Combining this with it's converse above, we have our result.

17. Lemma. For all $n \geq 2$, $\text{Kernel}(\delta_n) = \text{Image}(\delta_n - A_n)$. Also, $\text{Kernel}(\delta_n) = \text{Image}(\delta_n - A_n)$.

Proof. Let $\langle \xi \rangle \in \text{Image}(\delta_n - A_n)$. Then $\xi \in \text{Kernel}(\gamma_{n-1})$. Thus $\text{Kernel}(\gamma_{n-1}) = \langle \xi \rangle$. Therefore $\text{Kernel}(\gamma_{n-1}) = \text{Image}(\delta_n - A_n)$.
Denoting the set \((j + \frac{1}{2}) \in J^+\) by \(J^+ + \frac{1}{2}\), we define a continuous mapping \(\Phi\) from \((E^+ \times S^m) - \{(j + \frac{1}{2}) \times E^p\}\) to \(I^\ast\) by \(\Phi(p, \phi, \eta) = (\eta, p)\), and \(\Phi(J^+ + \frac{1}{2}) \times E^p = \{\eta\} \times E^p\). By (17.3)

\[(17.4) \Phi[(i - \frac{1}{2}, i + \frac{1}{2}) \times S^m - \{(i - \frac{1}{2}, i + \frac{1}{2}) \times E^p\}] \subset V_2, \text{ for all } i \geq j_2.\]

Now the pair \((i - \frac{1}{2}, i + \frac{1}{2}) \times E^p\) is homotopically equivalent to the pair \((E^+ + \frac{1}{2}, S^m)\). By (17.1) and (17.3) and the definition of \(\Phi\), \(\Phi[(i - \frac{1}{2}, i + \frac{1}{2}) \times S^m, i \times E^p\) corresponds to the continuous mapping \(\xi^i \circ \xi^i\) from \(S^m\) to \(V_2\), for all \(i \geq j_2\). But \(\xi^i \circ \xi^i\), being homotopic to the constant mapping from \(S^m\) to \(V_2\) for all \(i \geq j_2\), can be extended to a mapping from \(E^+ + \frac{1}{2}\) to \(V_2\), for all \(i \geq j_2\). Thus for each \(i \geq 1\)

\[(17.5) \Phi[(i - \frac{1}{2}, i + \frac{1}{2}) \times E^p] \subset V_2, \text{ for all } i \geq j_2.\]

Since \([0, \frac{1}{2}) \times S^m - \{(i) \times E^p\}\) is a retract of \([0, \frac{1}{2}) \times E^p, \Phi[(i - \frac{1}{2}, i + \frac{1}{2}) \times E^p] \subset V_2\), for all \(i \geq j_2\). Let \(T\) be any continuous mapping from \(E^+ \times E^p\) to \(E^+ \times E^p\) such that

\[(17.7) T[p, \phi, \eta] = \Phi[p, \phi, \eta]\] is the identity continuous mapping.

\[(17.8) T[(0, \frac{1}{2}) \times E^p] = (0, \frac{1}{2}) \times E^p,\]

\[(17.9) T[(i - \frac{1}{2}, i + \frac{1}{2}) \times E^p] = (i - \frac{1}{2}, i + \frac{1}{2}) \times E^p, \text{ for all } i \geq 1,\]

\[(17.10) \Phi[(i - \frac{1}{2}, i + \frac{1}{2}) \times E^p] \subset V_2, \text{ for all } i \geq j_2.\]

Thus we may, without ambiguity, define a continuous mapping \(\Gamma\) from \(E^+ \times S^m - \{(i) \times E^p\}\) to \(I^\ast\) by \(\Gamma(s, e, \eta) = (s, e, \phi, \eta)\), for each \((s, e, \phi, \eta) \in E^+ \times S^m - \{(i) \times E^p\}\). By (17.6) and (17.7) and the definition of \(\Phi\), \(\Gamma[p, \phi, \eta] = \Phi[p, \phi, \eta] \circ \phi\), and \(\Gamma[(i - \frac{1}{2}, i + \frac{1}{2}) \times E^p] \subset V_2\), for all \(i \geq j_2\). Thus \(\Gamma[p, \phi, \eta](\eta) = (\eta, p)\), and \(\Gamma[(i - \frac{1}{2}, i + \frac{1}{2}) \times E^p] \subset V_2\), for all \(i \geq j_2\). Thus for each \(i \geq 1\)

\[(17.11) \Phi[(i - \frac{1}{2}, i + \frac{1}{2}) \times E^p] \subset V_2, \text{ for all } i \geq j_2.\]

Since \(\Phi[(i - \frac{1}{2}, i + \frac{1}{2}) \times E^p] = \Phi[(i - \frac{1}{2}, i + \frac{1}{2}) \times E^p] - \{(i) \times E^p\}\), there is a continuous mapping \(\tau_i\) from \(S^m\) to \(I^\ast\) s.t. \(\tau_i = \phi\), for each \(i \in J^+\) a continuous mapping \(\tau_i\) from \(S^m\) to \(I^\ast\) s.t. \(\tau_i = \phi\), for each \(i \in J^+\). Thus, if \(V\) and \(N\) are as in (17.11), by (17.11) \(\tau_i \circ \xi_i \circ \tau_i\) can be extended to a mapping of \(E^+ \times V\) to \(V\), for all \(i \geq j_2\), i.e. \(\tau_i \circ \xi_i \circ \tau_i\) is homotopic to the constant mapping of \(S^m\) to \(V\), for all \(i \geq j_2\). Compounding the \(\tau_i\), \(i \geq 0\), we get an inward n-mapping \(\tau\) of \((X, x)\) s.t. \(\tau_i \circ \xi_i \circ \tau_i\) is homotopic to \(\tau\) (inwardly). Therefore \(\tau_i \circ \xi_i \circ \tau_i\) is homotopic to \(\tau\) (inwardly). Therefore the homotopy \((\tau_i \circ \xi_i \circ \tau_i) an \in I^\ast\), in the case \(n \geq 2\), or \((\tau_i \circ \xi_i \circ \tau_i) an \in I^\ast\), in the case \(n = 1\), is contained in Image(\(\tau_i\)).

Therefore Kernel(\(\phi\)) \(\subset \) Image(\(\tau_i\)). This remark with its converse above proves the lemma. Q.E.D.

18. THEOREM. Let \((X, x)\) be a pointed compact space contained in the \(H^\ast\)ibert cube. Then, for all \(n \geq 2\).

\[0 \rightarrow \pi_n(X, x) \rightarrow I_n(X, x) \rightarrow I_{n-1}(X, x) \rightarrow \cdots \rightarrow \pi_1(X, x) \rightarrow \pi_0(X, x) \rightarrow 0\]

is an exact sequence of groups and homomorphisms, and also

\[0 \rightarrow \pi_n(X, x) \rightarrow I_n(X, x) \rightarrow I_{n-1}(X, x) \rightarrow \cdots \rightarrow \pi_1(X, x) \rightarrow \pi_0(X, x) \rightarrow 0\]

is an exact sequence, where \(\pi_1(X, x)\) and \(I_1(X, x)\) are groups, and \(\iota_i\) is a homomorphism.
Proof. From its definition, $i_\alpha$ is a monomorphism, for all $\alpha \geq 1$ and by 14 above $\gamma_\alpha$ is surjective for all $\alpha \geq 0$. The theorem now follows directly from these remarks and lemmas 15, 16 and 17. Q.E.D.

19. Example. If $\Sigma$ is the 3-adic solenoid of van-Dantzig and $\sigma \in \Sigma_3$, we will show that in the exact sequence

$$0 \rightarrow \pi_2(\Sigma_3, \sigma) \rightarrow \pi_1(\Sigma_3, \sigma) \rightarrow \pi_1(\Sigma_3, \sigma) \rightarrow \pi_0(\Sigma_2, \sigma) \rightarrow 0$$

$\pi_2(\Sigma_3, \sigma)$ and $\pi_0(\Sigma_3, \sigma)$ are both trivial but that the other 3 objects in the sequence are non-trivial.

It is convenient and there is no essential difference so we work this example in $\mathbb{R}^2$ instead of $\mathbb{R}^n$. We start by giving a description of an embedding of $\Sigma_3$ in $\mathbb{R}^2$ and of a sequence $(U_n)_{n \geq 0}$ of neighborhoods of $\Sigma_3$ s.t. $U_{n+1} \subseteq U_n$, for all $n \geq 0$, and such that $\bigcap_{n=0}^{\infty} U_n = \Sigma_3$.

In $\mathbb{R}^2$ consider the disc $D = \{(x + 2)^2 + x^2 \geq 1, x_a = 0\}$ and the solid torus $U_1$ obtained by revolving $D$ around the $x_1$-axis. In $D$ consider the disc $D_1 = \{(x + 2)^2 + x^2 \leq 0.1, x_2 = 0\}$ and the discs $D_2$ and $D_3$ obtained from $D_1$ by revolving $D$ around its center by the angles $2\pi/3$ and $4\pi/3$ respectively. $D_2$ and $D_3$ and $D_4$ are disjoint since $0.1$ is small. Now assume as $D_1$ revolves around the $x_2$-axis it also revolves around its own $x_1$ in such a way that as one revolution around the $x_2$-axis is complete becomes $D_2$, $D_2$ becomes $D_3$ and $D_3$ becomes $D_4$. Then the discs $D_2, D_3, D_4$ sweep out a solid torus $U_1$ which runs 3 times around the inside of the solid torus $U_1$. Let $\theta$ be any continuous mapping from $U_1$ to $U_1$ which takes $U_1$ homeomorphically onto $U_1 \cap U_1$, for all $n \geq 0$, and in general $U_n = \theta(U_n)$, define $U_n = \{ \theta(U_n) \}$ and in general $U_n = \theta^{-1}(U_n)$. Define $U_n = \mathbb{R}^2$. Then $U_{n+1} \subseteq U_n$ for all $n \geq 0$ and $\bigcap_{n=0}^{\infty} U_n = \Sigma_3$.

Let $\sigma \in \Sigma_3$. Denote by $\text{inci}_{\sigma}$ the inclusion mapping $U_{n+1} \subseteq U_n$, for all $n \geq 0$, $\sigma \geq 0$, denote by $\pi_i(\text{inci}_{\sigma})$ the function induced by $\text{inci}_{\sigma}$ from $\pi_i(U_{n+1}, \sigma)$ to $\pi_i(U_n, \sigma)$.

Now $U_n$ is a homogeneous 1-sphere and since $\theta$ is a homeomorphism from $U_1$ onto $U_\infty \cap U_\infty$ it follows that $U_n$ by induction each $U_n$, $n \geq 1$, is a homogeneous 1-sphere. Therefore each object of the system

$$(\pi_i(\text{inci}_{\sigma}); \pi_i(U_{n+1}, \sigma) \rightarrow \pi_i(U_n, \sigma))_{n \geq 0}$$

is trivial and therefore the inverse limit of this system, which by appendix 21 is $\pi_0(\Sigma_3, \sigma)$ is trivial. Again each object of the system

$$(\pi_i(\text{inci}_{\sigma}); \pi_i(U_{n+1}, \sigma) \rightarrow \pi_i(U_n, \sigma))_{n \geq 0}$$

equals $\pi_i(\mathbb{R}, \mathbb{R}) = \mathbb{Z}$, the group of integers under addition, and for each $n \geq 1, \pi_i(\text{inci}_{\sigma})$ is the homomorphism from $\pi_i(U_{n+1}, \sigma) \rightarrow \pi_i(U_n, \sigma)$

$= \mathbb{Z}$, which takes $j \in \mathbb{Z}$ to $3j \in \mathbb{Z}$. Therefore the inverse limit of the latter system, which by appendix 21 is $\pi_0(\Sigma_3, \sigma)$ is $\mathbb{Z}$ which is trivial.

We take the point of view that $\pi_1(U_0, \sigma) = 0$ and $\pi_2(U_1, \sigma) = \mathbb{Z}$ and $\pi_3(U_2, \sigma) = Z$, for all $n \geq 1$. An inward 1-mapping $\xi$ of $(\Sigma_3, \sigma)$ is a sequence $\{\xi_j\}_{j \geq 0}$ of continuous mappings $\xi_j$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that given any $N \leq J^+$, $\xi_j(\mathbb{R}^2) \subseteq U_N$ for almost all $j$ and thus the homotopy class, $\langle \xi_j \rangle_{j \geq 0}$ of $\xi_j$ in $U_1$ is an integer $\alpha_j$ divisible by $3^j$ for almost all $j$. Consider the set of sequences $(\alpha_j)_{j \geq 0}$ of integers $\alpha_j$ which for all $N \leq J^+$, are divisible by $3^j$, for almost all $j$. There is an equivalence relation on this set, $(\alpha_j)_{j \geq 0} \sim (\beta_j)_{j \geq 0}$ iff there is an $M \leq J^+$ s.t. $\alpha_j = \beta_j$ for all $j \geq M$. Denote the class of $(\alpha_j)_{j \geq 0}$ by $\langle \alpha_j \rangle$. After partitioning inward 1-mappings by the inward homotopy relation we see that $\langle \pi_3(U_1, \sigma) \rangle$ is the set of classes of such sequences of integers with addition $\langle \langle a \rangle \rangle + \langle \langle b \rangle \rangle = \langle \langle a + b \rangle \rangle$.

Since $\pi_3(U_1, \sigma)$ is abelian $\pi_1(\mathbb{R}^2, \mathbb{R}^2)$ can be written $\pi_1(\mathbb{R}^2, \mathbb{R}^2)$ and $\pi_3(\mathbb{R}^2, \mathbb{R}^2)$ is a subgroup of $\pi_3(U_1, \sigma)$ and so in this particular case $\pi_3(U_1, \sigma) = \pi_3(U_1, \sigma)/\pi_3(\mathbb{R}^2, \mathbb{R}^2)$ is also a group. To show that $\langle \pi_3(U_1, \sigma) \rangle$ and $\pi_3(\mathbb{R}^2, \mathbb{R}^2)$ are both non trivial it is necessary only to show that $\pi_3(U_1, \sigma)$ is non trivial.

We will show that there does not exist $\langle \langle a \rangle \rangle \in \pi_3(U_1, \sigma)$ such that $\langle \langle a_j \rangle \rangle \in \langle \langle a \rangle \rangle$ and $\langle \langle a_j \rangle \rangle \in \langle \langle a_j \rangle \rangle$. Suppose such an $\langle \langle a \rangle \rangle$ does exist then we can find $M \leq J^+$ s.t. $a_j - a_{j+1} \in 3^M$ for all $j \geq M$. Then for all $p-1 \geq M$ we get,

$$a_{p-1} = \sum_{j=0}^{p-1} (a_j - a_{j+1}) = \sum_{j=0}^{p-1} 3^j = \frac{3^p - 3^0}{3}$$

Thus $3^p - 2a_p = 2a_p + 3^0$, for all $p \geq M+1$, and $2a_p + 3^0 = 0$ since $3^0$ is not divisible by 2. Choose $N \leq J^+$ s.t. $2a_p + 3^0$ is not divisible by $3^0$. Let $p$ be large so that $p > N$ and $a_p$ is divisible by $3^N$. Then $3^N$ divides $3^p - 2a_p = 2a_p + 3^0$, which is a contradiction.

To sum up we have shown that $\pi_3(U_1, \sigma) = 0$ and $\pi_3(U_1, \sigma) = 0$ but none of the other three terms in the low dimensional sequence of theorem 30 is trivial. We remark that if $a \in \mathbb{R}^2$, then the exact sequence of theorem 30 beginning with $\pi_{n+1}(\mathbb{R}^2, \mathbb{R}^2)$ is the sequence we have just described.

20. Remark. If $(X, A, \sigma)$ is a pointed pair of compacta contained in $\mathbb{R}^n$, then we can develop 3 long sequences $\pi(X, A, \sigma), \pi(X, A, \sigma)$ and $\pi(X, A, \sigma)$.

$$\rightarrow \pi_3(X, A, \sigma) \rightarrow \pi_3(A, \sigma) \rightarrow \pi_3(X, A) \rightarrow \pi_3(X, A, \sigma) \rightarrow \pi_3(X, A, \sigma) \rightarrow \pi_3(X, A, \sigma) \rightarrow \pi_3(X, A, \sigma) \rightarrow \pi_3(X, A, \sigma)$$
and then, as in theorem 18, we can develop a 5-term exact sequence of long sequences and commutative ladders.

0 → π_n(X, A, x) → I(X, A, x) → I(X, A, x) → S^2π_n(X, A, x) → S^2π(0, X, A, x) → 0

where if C is a graded module then S^2C is the graded module with (S^2C)_n = C_{n−2}. π(X, A, x) is exact (see [3]) and it is easy to show that I(X, A, x) is exact. Using this set up it is possible to prove that if (X, A, x) is a movable pointed pair of compacta then π(X, A, x) is exact. The concept of movable compacta was defined by K. Borsuk in [2].

21. Appendix. For each n ≥ 0, π_n(X, x) is the inverse limit L of the system {π_n(U, x)}; π_n(U, x) → π_n(U', x) is the inclusion mapping U ⊂ U'.

Proof. If f is a continuous mapping from (S^n, p, q) to (U, x) denote its homotopy class by [f] ∈ π_n(U, x), then L is the set of lists ([aU], [aU']) where for each U ∈ Nh(X), [aU] ∈ π_n(U, x) and if U ⊂ U', U' ∈ Nh(X), π_n(U, x) → π_n(U', x) is the inclusion mapping U → U'.

If (U_n)_{n∈N} is a nested sequence of neighborhoods of X such that \bigcap_{n∈N} U_n = X there is a morphism

Ψ: L → π_n(X, x), \langle [aU], [aU'] \rangle → \langle [aU] \rangle

which has as 2 sided inverse the morphism

Φ: π_n(X, x) → L, \langle [aU] \rangle → \langle [bU], [bU'] \rangle

where b_U is defined as follows. Given U ∈ Nh(X) there is an X' such that a_U is homotopic to a_{U+1} in U, for all n ∈ N(U), define

b_U = a_{N(U)}. Q.E.D.

References


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The realization of dimension function d_k(·)

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K. Nagami and J. H. Roberts [6] introduced the metric-dependent dimension function d_k(·) and posed the following question, which we call the Realization Question. Let (X, φ) be a metric space with d_0(X, φ) < dim X and let k be an integer with d_0(X, φ) ≤ k ≤ dim X. Does there exist a topologically equivalent metric φ for X with d_k(X, φ) = k? For each Cantor n-manifold (K_n, φ) with n ≥ 3, Nagami and Roberts described a subset (K_n, φ) with the property that d_k(K_n, φ) = n [2] and dim K_n = n - 1. This paper answers the above question in the affirmative for these spaces (K_n, φ) where K_n = I^n (n-cube). The question remains unanswered for arbitrary metric spaces.

Definition. Let (X, φ) be a non-empty metric space and let n be a non-negative integer. d_n(X, φ) ≤ n if (X, φ) satisfies the condition:

For any collection C = (Ω_i, C_i): i = 1, ..., n, of pairs of closed sets with φ(C_i, C_i') > 0 for each i = 1, ..., n + 1, there exist closed sets B_i, i = 1, ..., n + 1, such that (i) B_i separates X between Ω_i and C_i' for each i = 1, ..., n + 1 and (ii) \bigcap_{i=1}^{n+1} B_i = Θ.

If d_n(X, φ) ≤ n and the statement d_n(X, φ) ≤ n − 1 is false, we set d_n(X, φ) = n. The empty set Θ has d_n(Θ) = −1.

Definition. Let X be a topological space, g: X × X → R a real valued function, and let A and B be two subsets of X. Let

g(A, B) = inf \{g(x, y): x ∈ A, y ∈ B\}.

This real number g(A, B) will be called the g-distance between A and B.

Definition. Let I^n denote the Euclidean n-cube, let p, q ∈ I^n and let A ⊂ I^n. We define Join(p, q) to be the collection of all the points

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