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An exact sequence from the nth to the $(n-1)$-st fundamental group

by

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0. Introductory remark. For each pointed compactum (X, x) contained in the Hilbert cube we define, in each dimension $n > 0$, the approaching group $\underline{\pi}_n(X, x)$ and the inward group $I_n(X, x)$. Using category theory we show that $\underline{\pi}_n(X, x)$ and $I_n(X, x)$ depend only on the homotopy type of (X, x). We define an endomorphism from $I_n(X, x)$ to $I_n(X, x)$ whose kernel is the nth fundamental group, $\underline{\pi}_n(X, x)$ of Karol Borsuk. There is an epimorphism from $\underline{\underline{\pi}}_{n-1}(X, x)$ to $\underline{\pi}_{n-1}(X, x)$ whose kernel equals the coimage of the above endomorphism. Thus there is an exact sequence of groups and homomorphisms

$$0 \to \underline{\pi}_n(X, x) \to I_n(X, x) \to I_n(X, x) \to \underline{\underline{\pi}}_{n-1}(X, x) \to \underline{\pi}_{n-1}(X, x) \to 0 .$$

We work out the above sequence in full when $n = 1$ and X is the 3-adic solenoid Σ_3.

Consider the system of neighbourhoods U of X in the Hilbert Cube with inclusion mappings. It is known that $\underline{\pi}_n(X, x)$ can also be obtained by applying the nth homotopy functor π_n to this system and passing to the inverse limit, i.e.

$$\underline{\pi}_n(X, x) = \varprojlim \pi_n(U, x) .$$

1. Notation. $R, R^n, R^+, J, J^+, I, I^\omega, S^n, E^n, p_0$ denote respectively, the real numbers, Euclidean n-space, the non negative real numbers, the integers, the non negative integers, the closed interval $[0, 1]$, the Hilbert cube, the n-sphere, the n-ball and the point $(1, 0, 0, \ldots, 0) \in S^{n-1} \subset E^n$.

For each $n \geqslant 0$ let q_n be the identification mapping from $I \times S^n$ to $I \times S^n/(I \times \{p_0\}) \cup (\{0, 1\} \times S^n)$ to S^{n+1} and let h_n be a homeomorphism from $I \times S^n/(I \times \{p_0\}) \cup (\{0, 1\} \times S^n)$ to S^{n+1} such that

$$h_n \circ q_n\big((I \times \{p_0\}) \cup (\{0, 1\} \times S^n)\big) = \{p_0\} \subset S^{n+1}.$$

For each $n \geqslant 0$, let r_n be the identification mapping from $S^n \times I$ to $S^n \times I/S^n \times \{1\}$ and let k_n be a homeomorphism from $S^n \times I/S^n \times \{1\}$ to E^{n+1} such that $k_n \circ r_n(S^n \times \{1\}) = \{a\}$, where $a = (0, 0, \ldots, 0, 0)$ the centre of E^{n+1}. For each $n \geqslant 1$, let s_n be a continuous mapping from E^n to S^n such that $s_n(S^{n-1}) = \{p_0\}$ and s_n maps $E^n - S^{n-1}$ homeomorphicaly onto $S^n - \{p_0\}$.

We remark that, for each $n \geqslant 1$, (S^n, p_0) is a homotopy cogroup with a continuous comultiplication mapping ν^n from (S^n, p_0) to $(S^n, p_0) \vee \vee (S^n, p_0)$ and a continuous homotopy inverse mapping ψ^n from (S^n, p_0) to (S^n, p_0). We assume that for all $n \geqslant 1$, ν^{n+1} and ψ^{n+1} are derived by suspending ν^n and ψ^n. In other words, for all $n \geqslant 1$, $(t, e) \in I \times S^n$.

$$\nu^{n+1} \circ h_n \circ q_n(t, e) = \big((h_n \circ q_n), (h_n \circ q_n)\big)\big(t, \nu^n(e)\big), \tag{1.1}$$

$$\psi^{n+1} \circ h_n \circ q_n(t, e) = h_n \circ q_n\big(t, \psi^n(e)\big). \tag{1.2}$$

If f, g are continuous mappings from S^n to a topological space Y, then $f \times g$ denotes the continuous mapping $(f, g) \circ \nu^n$ from S^n to Y and f^{-1} the continuous mapping $f \circ \psi^n$, also from S^n to Y.

If Y and Z are sets, θ a function from $Y \times Z$ to I^ω then for each $y \in Y$ we denote by θ_y that function from Z to I^ω which carries each $z \in Z$ to $\theta(y, z) \in I^\omega$ and for each $z \in Z$ we denote by ${}_z\theta$ that function from Y to I^ω which carries each $y \in Y$ to $\theta(y, z) \in I^\omega$. As an example of this notation, if φ is a continuous mapping from $R^+ \times S^n \times I$ to I^ω then for $s \in R^+$, $t \in I$, ${}_t\varphi_s$ is that continuous mapping from S^n to I^ω which carries $e \in S^n$ to ${}_t\varphi_s(e) = \varphi(s, e, t) \in I^\omega$.

Let P be a topological space. If $\varkappa$ and λ are continuous mappings from $P \times S^n$ to I^ω then we denote by $\varkappa * \lambda$ and λ^{-1} respectively the continuous mappings from $P \times S^n$ to I^ω such that $(\varkappa * \lambda)_p = \varkappa_p * \lambda_p$ and $(\lambda^{-1})_p = (\lambda_p)^{-1}$, for all $p \in P$.

If K is a category, the objects of K are denoted by $\operatorname{Ob} K$, the morphisms by $\operatorname{Mor} K$ and for X, $Y \in \operatorname{Ob} K$ the morphisms of K with X as domain and Y as codomain are denoted by $\operatorname{Mor}_K(X, Y)$.

The set of compact neighbourhoods of the compactum $X \subset I^\omega$ are denoted by $\operatorname{Nhd}(X)$.

2. Definition. *Three categories* C, $\underline{\underline{C}}$, $\underline{C}$.

We will define 3 categories C, $\underline{\underline{C}}$ and $\underline{C}$ which have the same objects

$$\operatorname{Ob} C = \operatorname{Ob} \underline{\underline{C}} = \operatorname{Ob} \underline{C} = \{(X, x) |\ x \in X \subset I^\omega,\ X \text{ is compact}\}. \tag{2.1}$$

The *morphisms of* C are the set of base point preserving continuous mappings between pointed compacta with the usual composition. The identity continuous mapping at (X, x) is denoted $\operatorname{Id}_{(X,x)}$.

The *objects of* $\underline{\underline{C}}$ are defined above. We define the morphisms and composition in $\underline{\underline{C}}$ as follows. A continuous mapping f from $R^+ \times I^\omega$ to I^ω is a member of $\operatorname{Mor}_{\underline{\underline{C}}}\big((X, x), (Y, y)\big)$ and is referred to as an approaching mapping from (X, x) to (Y, y) if

$$f(R^+ \times \{x\}) = \{y\}, \tag{2.2}$$

(2.3) given any $V \in \operatorname{Nhd}(Y)$ there is a $U \in \operatorname{Nhd}(X)$ and an $r \in R^+$ such that $f([r, \infty) \times U) \subset V$.

The composition gf in $\underline{\underline{C}}$ of

$$f \in \operatorname{Mor}_{\underline{\underline{C}}}\big((X, x), (Y, y)\big) \text{ and } g \in \operatorname{Mor}_{\underline{\underline{C}}}\big((Y, y), (Z, z)\big)$$

is defined by $gf(r, i) = g\big(r, f(r, i)\big)$ for each $(r, i) \in R^+ \times I^\omega$. As gf is easily seen to satisfy (2.2) and (2.3) above $gf \in \operatorname{Mor}_{\underline{\underline{C}}}\big((X, x), (Z, z)\big)$. No confusion will result if we denote by $\operatorname{Id}_{(X,x)}$ the identity element of $\operatorname{Mor}_{\underline{\underline{C}}}\big((X, x), (X, x)\big)$. $\operatorname{Id}_{(X,x)}$ $(r, i) = i$, for each $(r, i) \in R^+ \times I^\omega$.

The *category* $\underline{C}$ was first defined in [1] by K. Borsuk. We now describe $\underline{C}$ in a manner suited to our purposes. $\operatorname{Ob} \underline{C}$ has been defined above. Denoting by $f|\ ; J^+ \times I^\omega \to I^\omega$ the restriction of the continuous mapping $f; R^+ \times I^\omega \to I^\omega$,

$$\operatorname{Mor}_{\underline{C}}\big((X, x), (Y, y)\big) = \big\{f|\ ; f \in \operatorname{Mor}_{\underline{\underline{C}}}\big((X, x), (Y, y)\big)\big\}. \tag{2.4}$$

Composition in $\underline{C}$ is $(g|)(f|) = (gf)|$. Clearly $\operatorname{Id}_{(X,x)}|$ is the identity element of $\operatorname{Mor}_{\underline{C}}\big((X, x), (X, x)\big)$ and no confusion will be caused if we simply denote this morphism by $\operatorname{Id}_{(X,x)}$.

We will also find useful the original definition of $\operatorname{Mor}_{\underline{C}}\big((X, x), (Y, y)\big)$ given by K. Borsuk in [1]. A fundamental sequence f from (X, x) to (Y, y) is a sequence $f = \{f_n\}_{n \geqslant 0}$ of continuous mappings f_n from I^ω to I^ω such that $f_n(x) = y$ for all $n \geqslant 0$ and such that given any $V \in \operatorname{Nhd}(Y)$ there is a $U \in \operatorname{Nhd}(X)$ and a $j \in J^+$ such that $f_n(U) \subset V$ for all $n \geqslant j$. From this point of view composition in $\underline{C}$ is defined

$$(gf)_n = g_n \circ f_n. \tag{2.5}$$

3. Definition. *Homotopy in* C, $\underline{\underline{C}}$ *and* $\underline{C}$.

Two morphisms $f, g \in \operatorname{Mor}_C\big((X, x), (Y, y)\big)$ are said to be *pointed homotopic* if there is a continuous mapping H from $X \times I$ to Y such that ${}_0H = f$, ${}_1H = g$ and $H(\{x\} \times I) = \{y\}$. In this case we write $H; f \simeq g$. Pointed homotopy is well known to be an equivalence relation on the morphisms of C compatible with composition in C. The class of $f \in \operatorname{Mor} C$ is denoted by $[f]$. There is therefore a category $\mathcal{K}C$ with $\operatorname{Ob} \mathcal{K}C = \operatorname{Ob} C$ and morphisms classes of morphisms of C. Composition in $\mathcal{K}C$ is $[g] \circ [f] = [g \circ f]$ and $[\operatorname{Id}_{X,x}] \in \operatorname{Mor}_{\mathcal{K}C}\big((X, x), (X, x)\big)$ is the identity element.

Two morphisms $f, g \in \mathrm{Mor}_{\underline{\underline{C}}}\big((X, x)(Y, y)\big)$ are said to be *pointed approaching homotopic* if there is a continuous mapping H from $R^+ \times I^\omega \times I$ to I^ω such that

(3.1) $$_0H = f\,, \quad {}_1H = g\,,$$

(3.2) $$H(R^+ \times \{x\} \times I) = \{y\}\,,$$

(3.3) given any $V \in \mathrm{Nhd}(Y)$ there is a $U \in \mathrm{Nhd}(X)$ and an $r \in R^+$ such that $H\big([r, \infty) \times U \times I\big) \subset V$.

In this case we write $H; f \simeq g$ (approaching). Pointed approaching homotopy is an equivalence relation on the morphisms of C. If

$$f \in \mathrm{Mor}_{\underline{\underline{C}}}\big((W, w), (X, x)\big), \quad g, h \in \mathrm{Mor}_{\underline{\underline{C}}}\big((X, x), (Y, y)\big),$$
$$k \in \mathrm{Mor}_{\underline{\underline{C}}}\big((Y, y), (Z, z)\big)$$

and $H; g \simeq h$ (approaching), then $Hf; gf \simeq hf$ (approaching), and $kH; kg \simeq kh$ (approaching) where Hf and kH are continuous mappings from $R^+ \times I^\omega \times I$ to I^ω defined by $_t(Hf) = ({}_tH)f$ and $_t(kH) = k({}_tH)$, for each $t \in I$. Thus the equivalence relation of pointed approaching homotopy is compatible with composition in $\underline{\underline{C}}$. Denoting the pointed approaching homotopy class of f by $[f]$ we may form a new category $\mathcal{H}\underline{\underline{C}}$ whose objects are the same as those of $\underline{\underline{C}}$ and whose morphisms are classes of morphisms of $\underline{\underline{C}}$. Composition in $\mathcal{H}\underline{\underline{C}}$ is $[g] \circ [f] = [gf]$. The identity element of $\mathrm{Mor}_{\mathcal{H}\underline{\underline{C}}}\big((X, x), (X, x)\big)$ is $[\mathrm{Id}_{(X,x)}]$. $\mathcal{H}\underline{\underline{C}}$ is called the *pointed approaching homotopy category*.

We next define homotopy on $\underline{C}$ and use this concept of homotopy to describe the (pointed) fundamental category. These ideas were first defined in [1] by K. Borsuk.

Two morphisms $f, g \in \mathrm{Mor}_{\underline{C}}\big((X, x)(Y, y)\big)$ are said to be *poir fundamentaly homotopic* if there is a continuous mapping H fr $J^+ \times I^\omega \times I$ to I^ω such that

(3.4) $$_0H = f\,, \quad {}_1H = g\,,$$

(3.5) $$H(J^+ \times \{x\} \times I) = \{y\}\,,$$

(3.6) given any $V \in \mathrm{Nhd}(Y)$ there is a $U \in \mathrm{Nhd}(X)$ and $j \in J^+$ such that $H_n(U \times I) \subset V$, for each $n \geqslant j$.

In this case we write $H; f \simeq g$ (fundamental). Fundamental homotopy is an equivalence relation on the morphisms of $\underline{C}$ compatible with the composition of $\underline{C}$ and the class of f is denoted $[f]$. As above we get a new category $\mathcal{H}\underline{C}$ called the (*pointed*) *fundamental category*, $\mathrm{Ob}\,\mathcal{H}\underline{C} = \mathrm{Ob}\,\underline{C}$, morphisms in $\mathcal{H}\underline{C}$ are classes of morphisms in $\underline{C}$ and composition in $\mathcal{H}\underline{C}$ is $[g] \circ [f] = [gf]$.

Two equivalent objects of $\mathcal{H}\underline{\underline{C}}$ are said to have the *same shape*.

4. Remark. *Comparison of* $\mathcal{H}C$, $\mathcal{H}\underline{\underline{C}}$ *and* $\mathcal{H}\underline{C}$.

If $(X, x), (Y, y) \in \mathrm{Ob}\,C$ and if f is a continuous mapping from $R^+ \times I^\omega$ to I^ω such that $f_r = \varphi;\, I^\omega \to I^\omega$, for all $r \in R^+$ and $\varphi|_X = \theta$ a continuous mapping from (X, x) to (Y, y) then we say that f is *generated by* θ. Such a mapping f is an approaching mapping from (X, x) to (Y, y) since $f(R^+ \times \{x\}) = \{\varphi(x)\} = \{y\}$ and if $V \in \mathrm{Nhd}(Y)$ there is, by the continuity of φ, a $U \in \mathrm{Nhd}(X)$ such that $\varphi(U) = f(R^+ \times U) \subset V$. Since any continuous mapping θ from (X, x) to (Y, y) can be extended to a continuous mapping φ from I^ω to I^ω it follows that each $\theta \in \mathrm{Mor}_C\big((X, x), (Y, y)\big)$ generates at least one $f \in \mathrm{Mor}_{\underline{\underline{C}}}\big((X, x)(Y, y)\big)$.

It is clear that $\mathrm{Id}_{(X,x)} \in \mathrm{Mor}\,C$ generates $\mathrm{Id}_{(X,x)} \in \mathrm{Mor}\,\underline{\underline{C}}$ and that if $\theta \in \mathrm{Mor}_C\big((X, x)(Y, y)\big)$, $\theta' \in \mathrm{Mor}_C\big((Y, y), (Z, z)\big)$ generate $f \in \mathrm{Mor}_{\underline{\underline{C}}}\big((X, x), (Y, y)\big)$ and $g \in \mathrm{Mor}_{\underline{\underline{C}}}\big((X, x), (Y, y)\big)$ respectively then $\theta' \circ \theta \in \mathrm{Mor}\,C$ generates $gf \in \mathrm{Mor}\,\underline{\underline{C}}$. Moreover if $\alpha, \beta \in \mathrm{Mor}_C\big((X, x), (Y, y)\big)$ generate $a, b \in \mathrm{Mor}_{\underline{\underline{C}}}\big((X, x), (Y, y)\big)$ and if $H; \alpha \simeq \beta$ then there is a continuous mapping K from $I^\omega \times I$ to I^ω such that $_0K = a_0$, $_1K = b_0$ and $K|_{X \times I} = H$. Defining the mapping L from $R^+ \times I^\omega \times I$ to I^ω by $L_r = K$, for all $r \in R^+$, then using the compactness of X it is easy to see that $L; a \simeq b$ (approaching).

From the above observations it follows that there is a functor $\mathcal{H}E$ from $\mathcal{H}C$ to $\mathcal{H}\underline{\underline{C}}$ taking $(X, x) \in \mathrm{Ob}\,\mathcal{H}C$ to $\mathcal{H}E(X, x) = (X, x) \in \mathrm{Ob}(\mathcal{H}\underline{\underline{C}})$ and $[\theta] \in \mathrm{Mor}\,\mathcal{H}C$ to $\mathcal{H}E([\theta]) = [f] \in \mathrm{Mor}\,\mathcal{H}\underline{\underline{C}}$, where $\mathcal{H}E([\theta]) = [f]$ is well defined to be the approaching homotopy class of any mapping f generated by θ.

It is immediate from definition 2 above that there is a functor R from $\underline{\underline{C}}$ to $\underline{C}$ carrying $(X, x) \in \mathrm{Ob}\,\underline{\underline{C}}$ to $R(X, x) = (X, x) \in \mathrm{Ob}\,\underline{C}$ and $f \in \mathrm{Mor}\,\underline{\underline{C}}$ to $R(f) = f| \in \mathrm{Mor}\,\underline{C}$. By the definition of $\mathrm{Mor}\,\underline{C}$ in (2.4) above it is clear that R considered as a function from the set $\mathrm{Mor}\,\underline{\underline{C}}$ to the set $\mathrm{Mor}\,\underline{C}$ is surjective.

If $H; f \simeq g$ (approaching), then denoting by $H|$ the restriction of H to $J^+ \times I^\omega \times I$ it is clear that $H|; f| \simeq g|$ (fundamental). Thus there is a functor $\mathcal{H}R$ from $\mathcal{H}\underline{\underline{C}}$ to $\mathcal{H}\underline{C}$ taking $(X, x) \in \mathrm{Ob}\,\mathcal{H}\underline{\underline{C}}$ to $\mathcal{H}R(X, x) = (X, x) \in \mathrm{Ob}\,\mathcal{H}\underline{C}$ and taking $[f] \in \mathrm{Mor}\,\mathcal{H}\underline{\underline{C}}$ to $\mathcal{H}R([f]) = [R(f)] = [f|] \in \mathrm{Mor}\,\mathcal{H}\underline{C}$. The surjectiveness of R above implies that $\mathcal{H}R$ considered as function from the set $\mathrm{Mor}\,\mathcal{H}\underline{\underline{C}}$ to the set $\mathrm{Mor}\,\mathcal{H}\underline{C}$ is surjective.

5. Definition. *The approaching functor.*

A continuous mapping ξ from $R^+ \times S^n$ to I^ω is said to be an *approaching n-mapping of* $(X, x) \in \mathrm{Ob}\,\mathcal{H}\underline{\underline{C}}$ iff

(5.1) $$\xi(R^+ \times \{p_0\}) = \{x\}\,.$$

(5.2) given $V \in \mathrm{Nhd}(X)$ there is an $r \in R^+$ such that $\xi([r, \infty) \times S^n) \subset V$.

If ξ and ξ' are approaching n-mappings of (X, x) then we say that ξ is *approaching homotopic to* ξ' iff there is a continuous mapping Φ from $R^+ \times S^n \times I$ to I^ω such that

(5.3) $$_0\Phi = \xi , \quad _1\Phi = \xi' ,$$

(5.4) $$\Phi(R^+ \times \{p_0\} \times I) = \{x\} ,$$

(5.5) given $V \in \mathrm{Nhd}(X)$ there is an $r \in R^+$ such that $\Phi([r, \infty) \times S^n \times \times I) \subset V$.

In this case we write $\Phi; \xi \simeq \xi'$ (approaching). Approaching homotopy is an equivalance relation on the set of all approaching n-mappings of (X, x) and the class of ξ will be written $\langle\xi\rangle$. The set of classes of approaching n-mappings of (X, x) will be denoted $\underline{\underline{\pi}}_n(X, x), n \geqslant 0$.

We denote by c the approaching n-mapping of (X, x) such that $c(R^+ \times S^n) = \{x\}$. If ξ and η are approaching n-mappings of (X, x) then ξ^{-1} and $\xi * \eta$ are also approaching n-mappings of (X, x). If $\xi \simeq \xi'$ (approaching) and $\eta \simeq \eta'$ (approaching) then $\xi * \eta \simeq \xi' * \eta'$ (approaching). Thus, when $n \geqslant 1$, we can compose classes of approaching mappings as follows, $\langle\xi\rangle * \langle\eta\rangle = \langle\xi * \eta\rangle$. If ξ, ξ' and ξ'' are approaching n-mappings of (X, x) the following remarks are easily proven.

(5.6) $$\langle\xi\rangle * \langle c\rangle = \langle\xi\rangle , \qquad n \geqslant 1 ,$$

(5.7) $$\langle\xi\rangle * \langle\xi^{-1}\rangle = \langle c\rangle = \langle\xi^{-1}\rangle * \langle\xi\rangle , \qquad n \geqslant 1 ,$$

(5.8) $$(\langle\xi\rangle * \langle\xi'\rangle) * \langle\xi''\rangle = \langle\xi\rangle * (\langle\xi'\rangle * \langle\xi''\rangle) , \qquad n \geqslant 1 ,$$

(5.9) $$\langle\xi\rangle * \langle\xi'\rangle = \langle\xi'\rangle * \langle\xi\rangle , \qquad n \geqslant 2 .$$

Thus $\underline{\underline{\pi}}_n(X, x)$ is a set group or abelian group according as $n = 0, 1$ or $n \geqslant 2$.

If ξ is an approaching n-mapping of (X, x) and $f \in \mathrm{Mor}_{\underline{C}}((X, x), (Y, y))$ there is a continuous mapping $f\xi$ from $R^+ \times S^n$ to I^ω defined by $f\xi(r, a) = f(r, \xi(r, a))$, for each $(r, a) \in R^+ \times S^n$. We observe that if $V \in \mathrm{Nhd}(Y)$ there is a $U \in \mathrm{Nhd}(X)$ and $r_1 \in R^+$ such that $f([r_1, \infty) \times U) \subset V$ and there is an $r_2 \in R^+$ such that $\xi([r_2, \infty) \times S^n) \subset U$. Thus

$$f\xi([r_1 + r_2, \infty) \times S^n) \subset f([r_1, \infty) \times \xi([r_2, \infty) \times S^n)) \subset f([r_1, \infty) \times U) \subset V .$$

From these 2 observations it follows that $f\xi$ is an approaching n-mapping of (Y, y).

It is easy to see that if ξ and η are approaching n-mappings of (X, x), $f, g \in \mathrm{Mor}_{\underline{C}}((X, x)(Y, y))$, $h \in \mathrm{Mor}_{\underline{C}}((Y, y), (Z, z))$, then

(5.10) $$f(\xi * \eta) = (f\xi) * (f\eta) .$$

(5.11) $\Phi; \xi \simeq \eta$ (approaching) implies $\Psi; f\xi \simeq f\eta$, where Ψ is defined by $_t\Psi = f(_t\Phi)$, for each $t \in I$.

(5.12) $H; f \simeq g$ (approaching) implies that $\chi; f\xi \simeq g\xi$ (approaching) where χ is defined by $_t\chi = f(_tH)$, for each $t \in I$.

(5.13) $$(hf)\xi = h(f\xi) .$$

From the above it follows that $\underline{\underline{\pi}}_n$ is a functor from $\mathcal{HC}$ to the category of sets, groups or abelian groups according as $n = 0, 1$ or $n \geqslant 2$. For $[f] \in \mathrm{Mor}_{\mathcal{HC}}((X, x), (Y, y))$, $\langle\xi\rangle \in \underline{\underline{\pi}}_n(X, x)$,

$$\underline{\underline{\pi}}_n([f])(\langle\xi\rangle) = \langle f\xi\rangle \in \underline{\underline{\pi}}_n(Y, y) .$$

Since $\underline{\underline{\pi}}_n$ is a functor from $\mathcal{HC}$ it follows that $\underline{\underline{\pi}}_n(X, x)$ is invariant up to equivalence of objects in the approaching category. Composing the functor $\underline{\underline{\pi}}_n$ with the functor $\mathcal{HE}$ of 4 above to obtain the functor $\underline{\underline{\pi}}_n \circ \mathcal{HE}$ from $\mathcal{HC}$ we see that (a fortiori) $\underline{\underline{\pi}}_n(X, x) = \underline{\underline{\pi}}_n \circ \mathcal{HE}(X, x)$ is invariant up to homotopy type of pointed compacta.

6. Definition. *The inward functor.*

A continuous mapping ξ from $J^+ \times S^n$ to I^ω is said to be an *inward n-mapping of* $(X, x) \in \mathrm{Ob}\,\mathcal{HC}$ iff

(6.1) $$\xi(J^+ \times \{p_0\}) = \{x\} ,$$

(6.2) given $V \in \mathrm{Nhd}(X)$ there is a $j_0 \in J^+$ such that $\xi_j(S^n) \subset V$, for all $j \geqslant j_0$.

If ξ and ξ' are inward n-mappings of (X, x) then we say that ξ is *inward homotopic to* ξ' iff there is a continuous Φ from $J^+ \times S^n \times I$ to I^ω such that

(6.3) $$_0\Phi = \xi , \quad _1\Phi = \xi' ,$$

(6.4) $$\Phi(J^+ \times \{p_0\} \times I) = \{x\} ,$$

(6.5) given $V \in \mathrm{Nhd}(X)$ there is a $j_0 \in J^+$ such that $\Phi_j(S^n \times I) \subset V$, for all $j \geqslant j_0$.

In this case we write $\Phi; \xi \simeq \xi'$ (inwardly).

The set of classes of inward n-mappings will be denoted by $I_n(X, x)$, for each $n \geqslant 0$. As in 5 above $I_n(X, x)$ is a set group or abelian group according as $n = 0, 1$ or $n \geqslant 2$. The identity element of $I_n(X, x)$ is denoted by $\langle c\rangle$ where $c(j, e) = x$, for all $(j, e) \in J^+ \times S^n$. Multiplication in $I_n(X, x)$ is $\langle\xi\rangle * \langle\eta\rangle = \langle\xi * \eta\rangle$.

If ξ is an inward n-mapping of (X, x) and $f \in \mathrm{Mor}_{\underline{C}}((X, x), (Y, y))$ there is an inward n-mapping $f\xi$ of (Y, y) defined by $(f\xi)_n = f_n \circ \xi_n$, for each $n \geqslant 0$. Thus proceeding as in 5 above we see that for each $n \geqslant 0$ there is a functor I_n from the category $\mathcal{HC}$ to the category of sets, groups or abelian groups according as $n = 0, 1$ or $n \geqslant 2$, where $I_n([f])(\langle\xi\rangle)$

$= \langle f\xi \rangle$ for each $[f] \in \mathrm{Mor}_{\mathcal{HC}}((X, x), (Y, y))$, $\langle \xi \rangle \in I_n(X, x)$. Since I_n is a functor from $\mathcal{HC}$ we see that $I_n(X, x)$ is *shape invariant*.

7. Definition. *The fundamental functor.*

The concept of approximative n-mapping of (X, x) or approximative sequence of (S^n, p_0) towards (X, x) was defined as follows by K. Borsuk "A sequence of maps ξ_k; $(S^n, p_0) \to (I^\omega, x)$ will said to be an approximative sequence of (S^n, p_0) towards (X, x) iff, for each neighbourhood V of X the homotopy $\xi_k \simeq \xi_{k+1}$ in (V, x) holds for almost all k" (see [1], (13.1)). Clearly each approximative sequence of (S^n, p_0) towards (X, x), (or approximative n-mapping) is an inward n-mapping and in [1] it is shown that

(7.1) ξ is an approximative n-mapping of (X, x) and $\xi \simeq \xi'$ (inwardly) implies that ξ' is an approximative n-mapping,

(7.2) ξ is an approximative n-mapping of (X, x) and $f \in \mathrm{Mor}_{\mathcal{C}}((X, x), (Y, y))$ implies $f\xi$ is an approximative n-mapping of (Y, y),

(7.3) ξ, η are approximative n-mappings of (X, x), $n \geqslant 1$ implies that $\xi * \eta$ is an approximative n-mapping of (X, x).

Thus there is a functor $\underline{\pi}_n$ from $\mathcal{C}$ to the category of sets, groups or abelian groups according as $n = 0, 1$ or $n \geqslant 2$ where for each $n \geqslant 0$ and each $(X, x) \in \mathrm{Ob}\,\mathcal{C}$, $\underline{\pi}_n(X, x)$ is that subset of $I_n(X, x)$ such that $\langle \xi \rangle \in \underline{\pi}_n(X, x)$ iff ξ is an approximative n-mapping of (X, x), and for each $f \in \mathrm{Mor}_{\mathcal{C}}((X, x), (Y, y))$, $\underline{\pi}_n([f])$ is $I_n([f])$ restricted to $\underline{\pi}_n(X, x)$. $\underline{\pi}_n$ is called the *n-th fundamental functor* (see [1]).

In [1] it is remarked that $\underline{\pi}_n$ being a functor from $\mathcal{HC}$ $\underline{\pi}_n(X, x)$ is shape invariant.

An equivalent useful method of defining the concept of approximative n-mapping is as follows.

(7.4) An inward n-mapping ξ of (X, x) is said to be an *approximative n-mapping of* (X, x) iff there is an approaching n-mapping η of (X, x) such that $\eta|_{J^+ \times S^n} = \xi$.

In [3] it is shown that, for all $n \geqslant 0$, $\underline{\pi}_n(X, x)$ is the inverse limit of the system with objects $\{\pi_n(U, x)\}_{U \in \mathrm{Nhd}(X)}$ and morphisms induced by inclusion between neighbourhoods. An indication of this proof is given in Appendix 21.

8. Remark. From now on (X, x) denotes a fixed compactum with base point, contained in I^ω.

9. Definition. *The homomorphism i_n; $\underline{\pi}_n(X, x) \to I_n(X, x)$.*

For all $n \geqslant 0$ we denote by i_n the inclusion mapping i_n; $\underline{\pi}_n(X, x) \subset I_n(X, x)$. When $n \geqslant 1$ this is a homomorphism between groups.

10. Definition. *Advancing a function.*

Let X and Y be sets and f a function from $J^+ \times X$ to Y. Then there is a function $A(f)$ also from $J^+ \times X$ to Y which takes $(n, t) \in J^+ \times X$ to $A(f)(n, t) = f(n+1, t) \in Y$. $A(f)$ is called the *advancement of f*.

11. Definition. *The advancing endomorphism, A_n; $I_n(X, x) \to I_n(X, x)$.*

Let ξ and η be inward n-mappings of (X, x). We observe that, when $n \geqslant 1$, $A(\xi * \eta) = A(\xi) * A(\eta)$, and, for all $n \geqslant 0$, if Φ; $\xi \simeq \eta$ (inwardly) then $A(\Phi)$; $A(\xi) \simeq A(\eta)$ (inwardly). Thus, for each $n \geqslant 0$, there is a function A_n, from $I_n(X, x)$ to $I_n(X, x)$, which takes each $\langle \xi \rangle \in I_n(X, x)$ to $A_n(\langle \xi \rangle) = \langle A(\xi) \rangle \in I_n(X, x)$ and which when $n \geqslant 1$ is an endomorphism called the *n-th advancing endomorphism of* (X, x).

12. Remark. For each $n \geqslant 0$ we denote by Id_n the identity function from $I_n(X, x)$ to $I_n(X, x)$, $\mathrm{Id}_n(\langle \xi \rangle) = \langle \xi \rangle$, for $\langle \xi \rangle \in I_n(X, x)$. In the case $n \geqslant 2$, $I_n(X, x)$ is abelian and so $\mathrm{Id}_n - A_n$ is a homomorphism from $I_n(X, x)$ to $I_n(X, x)$.

In the case $n = 1$ there is a function which we denote by $\mathrm{Id}_1 * A_1^{-1}$ from $I_1(X, x)$ to $I_1(X, x)$ which takes $\langle \xi \rangle \in I_1(X, x)$ to $\mathrm{Id}_1 * A_1^{-1}(\langle \xi \rangle) = \langle \xi \rangle * (A_1\langle \xi \rangle)^{-1} \in I_1(X, x)$. Since $I_1(X, x)$ may not be an abelian group $\mathrm{Id}_1 * A_1^{-1}$ is not in general a homomorphism.

13. Definition. *The homomorphism δ_n; $I_n(X, x) \to \underline{\pi}_{n-1}(X, x)$.*

Let ξ be an inward n-mapping where $n \geqslant 1$. Denote by $B(\xi)$ the continuous mapping from $R^+ \times S^{n-1}$ to I^ω which, for all $j \in J^+$ and $(r, e) \in [j, j+1] \times S^{n-1}$ takes (r, e) to $B(\xi)(r, e) = \xi_j \circ h_{n-1} \circ q_{n-1}(r-j, e) \in I^\omega$. By the definition of h_{n-1} and q_{n-1}, in notational remark 1, there is no ambiguity in this definition of $B(\xi)$. By (6.5), $B(\xi)$ is an approaching $(n-1)$-mapping of (X, x).

Let ξ and η be inward n-mappings for $n \geqslant 2$, then, by 1.1, $B(\xi * \eta) = B(\xi) * B(\eta)$. Again if ξ and η are inward n-mappings for $n \geqslant 1$ and Φ; $\xi \simeq \eta$ (inwardly) then defining the continuous mapping Ψ from $R^+ \times S^{n-1} \times I$ by ${}_t\Psi = B({}_t\Phi)$, $0 \leqslant t \leqslant 1$, we see that Ψ, $B(\xi) \simeq B(\eta)$ (approaching). Thus there is, for each $n \geqslant 1$, a function δ_n from $I_n(X, x)$ to $\underline{\pi}_{n-1}(X, x)$ which takes each $\langle \xi \rangle \in I_n(X, x)$ to $\delta_n(\langle \xi \rangle) = \langle B(\xi) \rangle \in \underline{\pi}_{n-1}(X, x)$. Moreover, when $n \geqslant 2$, δ_n is a homomorphism between groups.

14. Definition. *The homomorphism γ_n from $\underline{\pi}_n(X, x)$ to $\underline{\pi}_n(X, x)$.*

For each $n \geqslant 0$, there is a function γ_n from $\underline{\pi}_n(X, x)$ to $\underline{\pi}_n(X, x)$ which assigns to each $\langle\xi\rangle \in \underline{\pi}_n(X, x)$, $\langle\xi|_{J^+\times S^n}\rangle \in \underline{\pi}_n(X, x)$. By (7.4) γ_n is surjective, $n \geqslant 0$. For $n \geqslant 1$, γ_n is a homomorphism between groups.

15. Lemma. *When* $n \geqslant 2$, *Kernel* $(\mathrm{Id}_n - A_n) = \underline{\pi}_n(X, x)$. *Also, Kernel* $(\mathrm{Id}_1 * A_1^{-1}) = \underline{\pi}_1(X, x)$.

Proof. When $n \geqslant 2$,

$\langle\xi\rangle \in \mathrm{Kernel}(\mathrm{Id}_n - A_n)$

iff $\langle\xi\rangle = A_n(\langle\xi\rangle)$.

iff $\xi \simeq A(\xi)$ (inwardly).

iff There is a continuous mapping Φ from $J^+\times S^n \times I$ to I^ω s.t. ${}_0\Phi = \xi$, ${}_1\Phi = A(\xi)$ and given $V \in \mathrm{Nhd}(X)$ there is an $N \in J^+$ s.t. $\Phi_j(S^n \times I) \subset V$, for all $j \geqslant N$.

iff Given $V \in \mathrm{Nhd}(X)$, there is an $N \in J^+$ s.t. ξ_j is homotopic to $(A(\xi))_j = \xi_{j+1}$ in V, for all $j \geqslant N$.

iff ξ is an approximative n-mapping (see 8 above).

iff $\langle\xi\rangle \in \underline{\pi}_n(X, x)$.

The case $n = 1$ is dealt with in a similar fashion, Q.E.D.

16. Lemma. Kernel$(\gamma_{n-1}) = $ Image(δ_n), *for all* $n \geqslant 1$.

Proof. Let $\langle\xi\rangle \in I_n(X, x)$. Then

$$\gamma_{n-1}\circ\delta_n(\langle\xi\rangle) = \gamma_{n-1}(\langle B(\xi)\rangle) = \langle B(\xi)|_{J^+\times S^{n-1}}\rangle.$$

But

$$B(\xi)(\{j\}\times S^{n-1}) = \xi_j \circ h_{n-1}\circ q_{n-1}(\{j-j\}\times S^{n-1})$$
$$= \xi_j \circ h_{n-1}\circ q_{n-1}(\{0\}\times S^{n-1}) = \xi_j(\{p_0\}) = \{x\}.$$

Thus $B(\xi)|_{J^+\times S^{n-1}} = c$ and $\gamma_{n-1}\circ\delta_n(\langle\xi\rangle) = \langle c\rangle = 0 \in \underline{\pi}_{n-1}(X, x)$. This shows that Image$(\delta_n) \subset$ Kernel(γ_{n-1}).

On the other hand, let $\langle\xi\rangle \in$ Kernel(γ_{n-1}). Then $\gamma_{n-1}(\langle\xi\rangle) = \langle\xi|_{J^+\times S^{n-1}}\rangle = \langle c\rangle \in \underline{\pi}_{n-1}(X, x)$. Thus there is a continuous mapping Φ from $J^+\times S^{n-1}\times I$ to I^ω s.t. $\Phi;\ \xi|_{J^+\times S^{n-1}} \simeq c$ (inwardly). For each $j \geqslant 0$ let ϱ_j be a retraction from $[j, j+1]\times I$ to $([j, j+1]\times\{0\})\cup(\{j, j+1\}\times I)$. Let $\tilde\varrho$ be that retraction from $R^+\times I$ to $(R^+\times\{0\})\cup(J^+\times I)$ s.t. $\varrho|_{[j,j+1]\times I} = \varrho_j$, for all $j \geqslant 0$. Let $\tilde{\varrho}$ be that retraction from $R^+\times S^{n-1}\times I$ to $(R^+\times S^{n-1}\times\{0\})\cup(J^+\times S^{n-1}\times I)$ which takes $(s, e, t) \in R^+\times S^{n-1}\times I$ to $\tilde\varrho(s, e, t) = (s', e, t') \in (R^+\times S^n\times\{0\})\cup(J^+\times S^n\times I)$ where $(s', t') = \varrho(s, t)$. Now let Ψ be that continuous mapping from $(R^+\times S^{n-1}\times\{0\})\cup(J^+\times S^{n-1}\times I)$ to I^ω s.t. $\Psi|_{R^+\times S^{n-1}\times\{0\}} = \xi$ and $\Psi|_{J^+\times S^{n-1}\times I} = \Phi$. Let $\Gamma = \Psi\circ\tilde\varrho$. Then, denoting ${}_1\Gamma$ by η, $\Gamma;\ \xi \simeq \eta$ (approaching). Thus $\langle\xi\rangle = \langle\eta\rangle \in \underline{\pi}_{n-1}(X, x)$.

Next we will define an inward n-mapping η' of (X, x) s.t. $B(\eta') = \eta$. Since $\eta|_{J^+\times S^{n-1}} = {}_1\Gamma|_{J^+\times S^{n-1}} = \Psi\circ\tilde\varrho|_{J^+\times S^{n-1}\times\{1\}} = \Psi|_{J^+\times S^{n-1}\times\{1\}} = {}_1\Phi = c$, there is, for each $j \geqslant 0$, an unique continuous mapping θ_j from $I\times S^{n-1}/I\times\{p_0\}\cup\{0, 1\}\times S^{n-1}$ to I^ω s.t. for each $(t, e) \in I\times S_{n-1}$, $\theta_j\circ q_{n-1}(t, e) = \eta(t+j, e)$. Define the continuous mapping η' from $J^+\times S^n$ to I^ω by $\eta'_j = \theta_j\circ h_{n-1}^{-1}$, for all $j \in J^+$. Since η is an approaching $(n-1)$-mapping of (X, x) it follows from (5.2) that η' is an inward n-mapping of (X, x). Moreover, for each $j \in J^+$, given $(r, e) \in [j, j+1]\times S^{n-1}$,

$$B(\eta')(r, e)$$
$$= \eta'_j\circ h_{n-1}\circ q_{n-1}(r-j, e), \qquad \text{by definition of } B(\eta') \text{ (see 13)},$$
$$= \theta_j\circ h_{n-1}^{-1}\circ h_{n-1}\circ q_{n-1}(r-j, e), \qquad \text{by definition of } \eta', \text{ above}$$
$$= \theta_j\circ q_{n-1}(r-j, e)$$
$$= \eta(r-j+j, e), \qquad \text{by definition of } \theta_j, \text{ above}$$
$$= \eta(r, e).$$

Thus $B(\eta') = \eta$.

Therefore $\delta_n(\langle\eta'\rangle) = \langle B(\eta')\rangle = \langle\eta\rangle = \langle\xi\rangle$. Therefore Kernel$(\gamma_{n-1}) \subset$ Image(δ_n). Combining this with it's converse above, we have our result. Q.E.D.

17. Lemma. *For all* $n \geqslant 2$, Kernel$(\delta_n) = $ Image$(\mathrm{Id}_n - A_n)$. *Also*, Kernel$(\delta_1) = $ Image$(\mathrm{Id}_1 * A_1^{-1})$.

Proof. Let $\langle\xi\rangle \in I_n(X, x)$, $n \geqslant 1$. In all cases we will show that, if $\eta = \xi*(A(\xi))^{-1}$, then $\delta_n(\langle\eta\rangle) = \langle B(\eta)\rangle = 0 \in \underline{\pi}_{n-1}(X, x)$. Now $\eta_j = \xi_j * \xi_{j+1}^{-1}$, for all $j \geqslant 0$. Thus $B(\eta)$ is that continuous mapping from $R^+\times S^{n-1}$ such that, for all $j \in J^+$,

(17.1) $B(\eta)(r, e) = \xi_j\circ h_{n-1}\circ q_{n-1}(2(r-j), e)$, for $(r, e) \in [j, j+\frac{1}{2}]\times S_{n-1}$

and

(17.2) $B(\eta)(r, e) = \xi_{j+1}\circ h_{n-1}\circ q_{n-1}(2(j-r+1), e)$, for $(r, e) \in [j+\frac{1}{2}, j+1]\times S^{n-1}$.

Let $\{V_k\}_{k\in J^+}$ be a sequence of neighbourhoods of X in I^ω such that $V_{k+1} \subset V_k$, for all $k \in J^+$ and such that $\bigcap_{k\in J^+} V_k = X$. Since ξ is an inward n-mapping of (X, x) we can find a sequence of integers $\{j_k\}_{k\in J^+}$ tending to infinity such that, for each $k \in J^+$ and all $j \geqslant j_k$, $\xi(\{j\}\times S^n) \subset V_k$. By 17.1 and 17.2,

(17.3) $B(\eta)([i-\frac{1}{2}, i+\frac{1}{2}]\times S^{n-1}) \subset V_k$, for all $k \in J^+$, $i \geqslant j_k$.

Denoting the set $\{j+\frac{1}{2}|\ j \in J^+\}$ by $J^+ + \frac{1}{2}$, we define a continuous mapping Φ from $(R^+ \times S^{n-1}) \cup ((J^+ + \frac{1}{2}) \times E^n)$ to I^ω by $\Phi|_{R^+\times S^{n-1}} = B(\eta)$ and $\Phi((J^+ + \frac{1}{2}) \times E^n) = \{x\} \subset I^\omega$. By 17.3

$$\Phi\big(([i-\tfrac{1}{2}, i+\tfrac{1}{2}] \times S^{n-1}) \cup (\{i-\tfrac{1}{2}, i+\tfrac{1}{2}\} \times E^n)\big) \subset V_k, \quad \text{for all } i \geqslant j_k. \tag{17.4}$$

Now the pair $([i-\frac{1}{2}, i+\frac{1}{2}] \times E^n, [i-\frac{1}{2}, i+\frac{1}{2}] \times S^{n-1} \cup \{i-\frac{1}{2}, i+\frac{1}{2}\} \times E^n)$ is homotopically equivalent to the pair (E^{n+1}, S^n). By 17.1 and 17.2 and the definition of Φ, $\Phi|_{[i-\frac{1}{2},i+\frac{1}{2}]\times S^{n-1} \cup \{i-\frac{1}{2},i+\frac{1}{2}\}\times E^n}$ corresponds to the continuous mapping $\xi_i^{-1} * \xi_i$ from S^n to V_k, for all $i \geqslant j_k$. But $\xi_i^{-1} * \xi_i$, being homotopic to the constant mapping from S^n to V_k for all $i \geqslant j_k$ can be extended to a mapping from E^{n+1} to V_k, for all $i \geqslant j_k$. Thus for each $i \geqslant 1$ $\Phi|_{[i-\frac{1}{2},i+\frac{1}{2}]\times S^{n-1} \cup \{i-\frac{1}{2},i+\frac{1}{2}\}\times E^n}$ can be extended to a continuous mapping Ψ^i from $[i-\frac{1}{2}, i+\frac{1}{2}] \times E^n$ to I^ω, such that

$$\Psi^i([i-\tfrac{1}{2}, i+\tfrac{1}{2}] \times E^n) \subset V_k, \quad \text{for all } i \geqslant j_k. \tag{17.5}$$

Since $[0, \frac{1}{2}] \times S^{n-1} \cup \{\frac{1}{2}\} \times E^n$ is a retract of $[0, \frac{1}{2}] \times E^n$, $\Phi|_{[0,\frac{1}{2}]\times S^{n-1} \cup \{\frac{1}{2}\}\times E^n}$ can be extended to a mapping Ψ^0 from $[0, \frac{1}{2}] \times E^n$ to I^ω. Define the continuous mapping Ψ from $R^+ \times E^n$ to I^ω by $\Psi|_{[0,\frac{1}{2}]\times E^n} = \Psi^0$ and $\Psi|_{[i-\frac{1}{2},i+\frac{1}{2}]\times E^n} = \Psi^i$, for all $i \geqslant 1$. Note that

$$\Psi|_{(R^+\times S^{n-1}) \cup ((J^+ + \frac{1}{2})\times E^n)} = \Phi. \tag{17.6}$$

Let $\overline{a, p_0}$ denote the set $\{(1-t)a + tp_0|\ 0 \leqslant t \leqslant 1\}$ i.e. the line segment from the center $a = (0, 0, \ldots, 0)$ of E^n to $p_0 \in S^{n-1}$. Let T be any continuous mapping from $R^+ \times E^n$ to $R^+ \times E^n$ such that

(17.7) $T|_{R^+\times S^{n-1} \cup (J^+ + \frac{1}{2})\times E^n}$ is the identity continuous mapping.

(17.8) $T([0, \frac{1}{2}] \times E^n) = [0, \frac{1}{2}] \times E^n$,
$T([i-\frac{1}{2}, i+\frac{1}{2}] \times E^n) = [i-\frac{1}{2}, i+\frac{1}{2}] \times E^n$, for all $i \geqslant 1$,

(17.9) $T|_{[0,\frac{1}{2}]\times \overline{a,p_0}}$ is a retraction from $[0, \frac{1}{2}] \times \overline{a, p_0}$ to $[0, \frac{1}{2}] \times \{p_0\} \cup \{\frac{1}{2}\} \times \overline{a, p_0}$ and $T|_{[i-\frac{1}{2},i+\frac{1}{2}]\times\overline{a,p_0}}$ is a retraction from $[i-\frac{1}{2}, i+\frac{1}{2}] \times \overline{a, p_0}$ to $[i-\frac{1}{2}, i+\frac{1}{2}] \times \{p_0\} \cup \{i-\frac{1}{2}, i+\frac{1}{2}\} \times \overline{a, p_0}$, for all $i \geqslant 1$.

Now

(17.10)
$$\begin{aligned}
&\Psi \circ T(R^+ \times \{a\}) \\
&\subset \Psi\big((R^+ \times \{p_0\}) \cup ((J + \tfrac{1}{2}) \times \overline{a, p_0})\big), && \text{by (17.9)} \\
&\subset \Phi\big((R^+ \times \{p_0\}) \cup (J^+ + \tfrac{1}{2}) \times E^n\big), && \text{by (17.6), (17.7)} \\
&= B(\eta)(R^+ \times \{p_0\}) \cup \Phi\big((J^+ + \tfrac{1}{2}) \times E^n\big), && \text{by definition of } \Phi \\
&= \{x\}, && \text{by definition of } B(\eta) \text{ and } \Phi.
\end{aligned}$$

Thus we may, without ambiguity, define a continuous mapping Γ from $R^+ \times S^{n-1} \times I$ to I^ω, by $\Gamma(s, e, t) = \Psi \circ T(s, k_{n-1} \circ r_{n-1}(e, t))$, for each $(s, e, t) \in R^+ \times S^{n-1} \times I$ (r_{n-1}, k_{n-1} defined in notational remark 1). By (17.6) and (17.7) and the definition of Φ, ${}_0\Gamma = \Psi|_{R^+\times S^{n-1}} = \Phi|_{R^+\times S^{n-1}} = B(\eta)$. By (17.10) ${}_1\Gamma = c$. By (17.4), (17.8), $\Gamma([j_k, \infty) \times S^{n-1} \times I) \subset V_k$. Thus $\Gamma; B(\eta) \simeq c$ (approaching). Therefore $\delta_n(\langle\eta\rangle) = B(\eta) = \langle c\rangle = 0$ $\in \underline{\pi}_{n-1}(X, x)$. Thus, Image$(\mathrm{Id}_n - A_n)$, in the case $n \geqslant 2$, and Image$(\mathrm{Id}_n * A_1^{-1})$, in the case $n = 1$, are both contained in Kernel(δ_n).

On the other hand, let $\langle\xi\rangle \in$ Kernel(δ_n), then there is a continuous mapping Φ from $R^+ \times S^n \times I$ to I^ω s.t. $\Phi; B(\xi) \simeq c$ (approaching). Since $\Phi(R^+ \times S^{n-1} \times \{1\}) = \{x\} \subset I^\omega$ we may define a continuous mapping Ψ from $R^+ \times E^n$ to I^ω as follows. For each $(p, e, t) \in R^+ \times S^{n-1} \times I$, $\Psi(p, k_{n-1} \circ r_{n-1}(e, t)) = \Phi(p, e, t)$. By (5.5), Ψ has the following property,

(17.11) given $V \in \mathrm{Nhd}(X)$, there is an $N \in J^+$ s.t. $\Psi([j, j+1] \times E^n) \subset V$, for all $j \geqslant N$.

Since

$$\Psi(J^+ \times S^{n-1}) = \Phi(J^+ \times S^{n-1} \times \{0\}) = B(\xi)(J^+ \times S^{n-1}) = \xi(J^+ \times \{p_0\}) = \{x\},$$

there is for each $j \in J^+$ a continuous mapping τ_j from S^n to I^ω s.t. $\tau_j \circ s_n = \Psi_j$ (see notational remark 1, for definition of s_n). The pair $([j, j+1] \times E^n, [j, j+1] \times S^{n-1} \cup \{j, j+1\} \times E^n)$ is homotopicaly equivalent to the pair (E^{n+1}, S^n) and $\Psi|_{[j,j+1]\times S^{n-1} \cup \{j,j+1\}\times E^n}$ can be considered to be the mapping $\tau_j^{-1} * \xi_j * \tau_{j+1}$ from S^n to I, for all $j \in J^+$. Then, if V and N are as in (17.11), by (17.11) $\tau_j^{-1} * \xi_j * \tau_{j+1}$ can be extended to a mapping of E^{n+1} to V, for all $j \geqslant N$, i.e. $\tau_j^{-1} * \xi_j * \tau_{j+1}$ is homotopic to the constant mapping of S^n, to V, in V, for all $j \geqslant N$. Compounding the τ_j, $j \geqslant 0$, we get an inward n-mapping τ of (X, x) s.t. $\tau^{-1} * \xi * A(\tau) \simeq c$ (inwardly). Therefore $\langle\tau^{-1} * \xi * A(\tau)\rangle = \langle c\rangle$, $\langle\xi\rangle = \langle\tau\rangle * (A(\tau))^{-1} = (\mathrm{Id}_n - A_n)(\langle\xi\rangle)$, in the case $n \geqslant 2$, or $(\mathrm{Id}_1 * A_1^{-1})(\langle\tau\rangle)$, in the case $n = 1$.

Therefore Kernel$(\delta_n) \subset$ Image$(\mathrm{Id}_n - A_n)$ when $n \geqslant 2$ and is contained in Image$(\mathrm{Id}_1 * A_1^{-1})$ when $n = 1$. This remark with its converse above proves the lemma. Q.E.D.

18. THEOREM. *Let (X, x) be a pointed compactum contained in the Hilbert cube. Then, for all $n \geqslant 2$,*

$$0 \to \underline{\pi}_n(X, x) \xrightarrow{i_n} I_n(X, x) \xrightarrow{\mathrm{Id}_n - A_n} I_n(X, x) \xrightarrow{\partial_n} \underline{\pi}_{n-1}(X, x) \xrightarrow{\gamma_{n-1}} \underline{\pi}_{n-1}(X, x) \to 0$$

is an exact sequence of groups and homomorphisms, and also

$$0 \to \underline{\pi}_1(X, x) \xrightarrow{i_1} I_1(X, x) \xrightarrow{\mathrm{Id}_1 * A_1^{-1}} I_1(X, x) \xrightarrow{\partial_1} \underline{\pi}_0(X, x) \xrightarrow{\gamma_0} \underline{\pi}_0(X, x) \to 0$$

is an exact sequence, where $\underline{\pi}_1(X, x)$ and $I_1(X, x)$ are groups, and i_1 is a homomorphism.

Proof. From it's definition, i_n is a monomorphism, for all $n \geqslant 1$ and by 14 above γ_n is surjective for all $n \geqslant 0$. The theorem now follows directly from these remarks and lemmas 15, 16 and 17. Q.E.D.

19. Example. If Σ_3 is the 3-adic solenoid of van-Dantzig and $\sigma \in \Sigma_3$ we will show that in the exact sequence

$$0 \to \underline{\pi}_1(\Sigma_3, \sigma) \to I_1(\Sigma_3, \sigma) \to I_1(\Sigma_3, \sigma) \to \underline{\underline{\pi}}_0(\Sigma_3, \sigma) \to \underline{\pi}_0(\Sigma_3, \sigma) \to 0$$

$\underline{\pi}_0(\Sigma_3, \sigma)$ and $\underline{\pi}_1(\Sigma_3, \sigma)$ are both trivial but that the other 3 objects in the sequence are non trivial.

It is convenient and there is no essential difference so we work this example in R^3 instead of I^ω. We start by giving a description of an embedding of Σ_3 in R^3 and of a sequence $\{U_n\}_{n\geqslant 0}$ of neighbourhoods of Σ_3 s.t. $U_{n+1} \subset U_n$, for all $n \geqslant 0$, and such that $\bigcap_0^\infty U_n = \Sigma_3$.

In R^3 consider the disc $D = \{(x_1+2)^2 + x_2^2 \geqslant 1,\, x_3 = 0\}$ and the solid torus U_1 obtained by revolving D around the x_1-axis. In D consider the disc $D_0 = \{(x_1+\frac{5}{2})^2 + x_2^2 \leqslant .01,\, x_3 = 0\}$ and the discs D_1 and D_2 obtained from D_0 by revolving D around its center by the angles $2\pi/3$ and $4\pi/3$ respectively. D_0, D_1 and D_2 are disjoint since .01 is small. Now assume as D_1 revolves around the x_1-axis it also revolves around its own cer in such a way that as one revolution around the x_1-axis is complete becomes D_1, D_1 becomes D_2 and D_2 becomes D_0. Then the discs D_0, D_1, D_2 sweep out a solid torus U_2 which runs 3 times around the inside of the solid torus U_1. Let θ be any continuous mapping from U_1 to U_1 which takes U_1 homeomorphically onto $U_2 \subset U_1$. Then $U_2 = \theta(U_1)$. Define $U_3 = \theta \circ \theta(U_1)$ and in general $U_n = \theta^{n-1}(U_1)$. Define $U_0 = R^3$. Then $U_{n+1} \subset U_n$ for all $n \geqslant 0$ and $\bigcap_0^\infty U_n = \Sigma_3$.

Let $\sigma \in \Sigma_3$. Denote by inc_n the inclusion mapping $U_{n+1} \subset U_n$, for all $n \geqslant 0$, $j \geqslant 0$, denote by $\pi_j(\mathrm{inc}_n)$ the function induced by inc_n from $\pi_j(U_{n+1}, \sigma)$ to $\pi_j(U_n, \sigma)$.

Now U_1 is a homotopy 1-sphere and since θ is a homeomorphism from U_1 onto $U_2 = \theta(U_1)$ it follows that U_2 and by induction each U_n, $n \geqslant 1$, is a homotopy 1-sphere. Theorefore each object of the system

$$\{\pi_0(\mathrm{inc}_n);\ \pi_0(U_{n+1}, \sigma) \to \pi_0(U_n, \sigma)\}_{n\geqslant 1}$$

is trivial and therefore the inverse limit of this system, which by appendix 21 is $\underline{\pi}_0(\Sigma_3, \sigma)$ is trivial. Again each object of the system

$$\{\pi_1(\mathrm{inc}_n);\ \pi_1(U_{n+1}, \sigma) \to \pi_1(U_n, \sigma)\}_{n\geqslant 1}$$

equals $\pi_1(S^1, p_0) = Z$, the group of integers under addition, and for each $n \geqslant 1$, $\pi_1(\mathrm{inc}_n)$ is the homeomorphism from $\pi_1(U_{n+1}, \sigma) = Z$ to $\pi_1(U_n, \sigma)$

$= Z$, which takes $j \in Z$ to $3j \in Z$. Therefore the inverse limit of the latter system, which by appendix 21 is $\underline{\pi}_1(\Sigma_3, \sigma)$ is $\bigcap_{n\geqslant 1} 3^n Z$ which is trivial.

We take the point of view that $\pi_1(U_0, \sigma) = 0$ and $\pi_1(U_n, \sigma) = 3^{n-1}Z \subset \pi_1(U_1, \sigma) = Z$, for all $n \geqslant 1$. An inward 1-mapping ξ of (Σ_3, σ) is a sequence $\{\xi_j\}_{j\geqslant 0}$ of continuous mappings ξ_j from S^1 to R^3 such that given any $N \in J^+$, $\xi_j(S^1) \subset U_N$ for almost all j and thus the homotopy class, $\langle \xi_j \rangle$ of ξ_j in U_1 is an integer a_j divisible by 3^N for almost all j. Consider the set of sequences $\{a_j\}_{j\geqslant 0}$ of integers a_j which for each $N \in J^+$, are divisible by 3^N, for almost all j. There is an equivalence relation on this set, $\{a_j\}_{j\geqslant 0} \simeq \{b_j\}_{j\geqslant 0}$ iff there is an $M \in J^+$ s.t. $a_j = b_j$, for all $j \geqslant M$. Denote the class of $\{a_j\}_{j\geqslant 0}$ by $\langle\{a_j\}\rangle$. After partitioning inward 1-mappings by the inward homotopy relation we see that $I_1(\Sigma_3, \sigma)$ is the set of classes of such sequences of integers with addition $\langle\{a_j\}\rangle + \langle\{b_j\}\rangle = \langle\{a_j + b_j\}\rangle$.

Since $I_1(\Sigma_3, \sigma)$ is abelian $\mathrm{Id}_1 * A_1^{-1}$ can be written $\mathrm{Id}_1 - A_1$ and $\mathrm{Im}(\mathrm{Id}_1 - A_1)$ is a subgroup of $I_1(\Sigma_3, \sigma)$ and so in this particular case $\underline{\underline{\pi}}_0(\Sigma_3, \sigma) = I_1(\Sigma_3, \sigma)/\mathrm{Im}(\mathrm{Id}_1 - A_1)$ is also a group. To show that $I_1(\Sigma_3, \sigma)$ and $\underline{\underline{\pi}}_0(\Sigma_3, \sigma)$ are both non trivial it is necessary only to show that $\underline{\underline{\pi}}_0(\Sigma_3, \sigma)$ is non trivial.

We will show that there does not exist $\langle\{a_j\}\rangle \in I_1(\Sigma_3, \sigma)$ such that $(\mathrm{Id} - A_1)(\langle\{a_j\}\rangle) = \langle\{a_j - a_{j+1}\}\rangle = \langle\{3^j\}\rangle \in I_1(\Sigma_3, \sigma)$. Suppose such an $\langle\{a_j\}\rangle$ does exist then we can find $M \in J^+$ s.t. $a_j - a_{j+1} = 3^j$, for all $j \geqslant M$. Then for all $p-1 > M$ we get,

$$a_M - a_p = \sum_{j=M}^{p-1} (a_j - a_{j+1}) = \sum_{j=M}^{p-1} 3^j = \tfrac{1}{2}(3^p - 3^M).$$

Thus $3^p - 2a_p = 2a_M + 3^M$, for all $p > M+1$, and $2a_M + 3^M \neq 0$ since 3^M is not divisible by 2. Chose $N \in J^+$ s.t. $2a_m + 3^M$ is not divisible by 3^N. Let p be so large that $p > N$ and a_p is divisible by 3^N. Then 3^N divides $3^p - 2a_p = 2a_M + 3^M$, which is a contradiction.

To sum up we have shown that, $\underline{\pi}_0(\Sigma_3, \sigma) = 0$ and $\underline{\pi}_1(\Sigma_3, \sigma) = 0$ but none of the other three terms in the low dimensional sequence of theorem 30 is trivial. We remark that if $a \in S^n\Sigma_3$, the n'th suspension of Σ_3, then the exact sequence of theorem 30 beginning with $\underline{\pi}_{n+1}(S^n\Sigma_3, a)$ is the sequence we have just described.

20. Remark. If (X, A, x) is a pointed pair of compacta contained in I^ω, then we can develop 3 long sequences $\underline{\pi}(X, A, x)$, $I(X, A, x)$ and $\underline{\underline{\pi}}(X, A, x)$.

$$\to \underline{\pi}_{n+1}(X, A, x) \to \underline{\pi}_n(A, x) \to \underline{\pi}_n(X, x) \to \underline{\pi}_n(X, A, x) \to$$

$$\to I_{n+1}(X, A, x) \to I_n(A, x) \to I_n(X, x) \to I_n(X, A, x) \to$$

$$\to \underline{\underline{\pi}}_{n+1}(X, A, x) \to \underline{\underline{\pi}}_n(A, x) \to \underline{\underline{\pi}}_n(X, x) \to \underline{\underline{\pi}}_n(X, A, x) \to$$

and then, as in theorem 18, we can develop a 5 term exact sequence of long sequences and commutative ladders.

$$0 \to \underline{\pi}(X, A, x) \to I(X, A, x) \to I(X, A, x) \to S^3\underline{\underline{\pi}}(X, A, x) \to S^3\underline{\pi}(X, A, x) \to 0$$

where if C is a graded module then S^3C is that graded module with $(S^3C)_n = C_{n-3}$. $\underline{\underline{\pi}}(X, A, x)$ is exact (see [3]) and it is easy to show that $I(X, A, x)$ is exact. Using this set up it is possible to prove that if (X, A, x) is a movable pointed pair of compacta then $\underline{\pi}(X, A, x)$ is exact. The concept of movable compactum was defined by K. Borsuk in [2].

21. Appendix. *For each* $n \geq 0$, $\underline{\pi}_n(X, x)$ *is the inverse limit* L *of the system* $\{\pi_n(\mathrm{inc}(U, U')); \pi_n(U, x) \to \pi_n(U', x)\}_{U \subset U', U, U' \in \mathrm{Nhd}(X)}$ *where for* $U \subset U'$ *both neighbourhoods of* X $\mathrm{inc}(U, U')$ *is the inclusion mapping* $U \subset U'$.

Proof. If f is a continuous mapping from (S^n, p_0) to (U, x) denote its homotopy class by $[f] \in \pi_n(U, x)$, then L is the set of lists $\{[a_U]\}_{U \in \mathrm{Nhd}(X)}$ where for each $U \in \mathrm{Nhd}(X)$, $[a_U] \in \pi_n(U, x)$ and if $U \subset U'$, $U, U' \in \mathrm{Nhd}(X)$, $\pi_n(\mathrm{inc}(U, U'))([a_U]) = [a_{U'}]$.

If $\{U_n\}_{n \geq 0}$ is a nested sequence of neighbourhoods of X such that $\bigcap_{n \geq 0} U_n = X$ there is a morphism

$$\Psi;\ L \to \underline{\pi}_n(X, x), \qquad \{[a_U]\} \to \langle\{a_{U_n}\}\rangle$$

which has as 2 sided inverse the morphism

$$\Phi;\ \underline{\pi}_n(X, x) \to L, \qquad \langle\{a_n\}\rangle \to \{[b_U]\}$$

where b_U is defined as follows. Given $U \in \mathrm{Nhd}(X)$ there is an $N(U) \in J^+$ such that a_n is homotopic to a_{n+1} in U, for all $n \geq N(U)$, define $b_U = a_{N(U)}$. Q.E.D.

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The realization of dimension function d_2 (*)

by

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K. Nagami and J. H. Roberts [6] introduced the metric-dependent dimension function d_2 and posed the following question, which we will call the Realization Question. Let (X, ϱ) be a metric space with $d_2(X, \varrho) < \dim X$ and let k be an integer with $d_2(X, \varrho) \leq k \leq \dim X$. Does there exist a topologically equivalent metric σ for X with $d_2(X, \sigma) = k$? For each Cantor n-manifold (K_n, ϱ) with $n \geq 3$, Nagami and Roberts described a subset (X_n, ϱ) with the property that $d_2(X_n, \varrho) = [n/2]$ and $\dim X_n \geq n-1$. This paper answers the above question in the affirmative for these spaces (X_n, ϱ) where $K_n = I^n$ (n-cube). The question remains unanswered for arbitrary metric spaces.

Definition. Let (X, ϱ) be a non-empty metric space and let n be a non-negative integer. $d_2(X, \varrho) \leq n$ if (X, ϱ) satisfies the condition: For any collection $\mathcal{C} = \{(C_i, C_i'): i = 1, \ldots, n+1\}$ of $n+1$ pairs of closed sets with $\varrho(C_i, C_i') > 0$ for each $i = 1, \ldots, n+1$, there exist closed sets B_i, $i = 1, \ldots, n+1$, such that (i) B_i separates X between C_i and C_i' for each $i = 1, \ldots, n+1$ and (ii) $\bigcap_{i=1}^{n+1} B_i = \emptyset$.

If $d_2(X, \varrho) \leq n$ and the statement $d_2(X, \varrho) \leq n-1$ is false, we set $d_2(X, \varrho) = n$. The empty set $\emptyset$ has $d_2(\emptyset) = -1$.

Definition. Let X be a topological space, $g\colon X \times X \to R$ a real valued function, and let A and B be two subsets of X. Let

$$g(A, B) = \inf\{|g(x, y)|:\ x \in A,\ y \in B\}.$$

This real number $g(A, B)$ will be called the *g-distance between A and B*.

Definition. Let I^n denote the Euclidean n-cube, let $p, q \in I^n$ and let $A \subseteq I^n$. We define $\mathrm{Join}(p, q)$ to be the collection of all the points

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