References

[1] P. S. Alexandroff et P. S. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. K. Akademie Amsterdam 14 (1929), pp. 1–96.
[2] R. H. McDowell, *Extension of functions from dense subspaces*, Duke Math. J. 25 (1958), pp. 297–304.
[3] R. Engelking, *Outline of General Topology*, Amsterdam 1968.
[4] С. В. Фомин, *К теории расширений топологических пространств*, Матем. Сборник 50 (1940), pp. 285–294.
[5] H. Herrlich, *Topologische Reflexionen und Coreflexionen*, Berlin 1968.
[6] M. Katětov, *Über H-abgeschlossene und bikompakte Räume*, Časopis Matem. Fys. 69 (1940), pp. 36–49.
[7] — *On H-closed extensions of topological spaces*, Časopis Matem. Fys. 72 (1947), pp. 17–32.
[8] A. Błaszczyk and J. Mioduszewski, *On factorization of maps through τX*, Coll. Math. 23 (1971), pp. 45–52.
[9] J. Mioduszewski and L. Rudolf, *H-closed and extremally disconnected Hausdorff spaces*, Rozprawy Mat. 66 (1969), pp. 1–52.
[10] L. Rudolf, *θ-continuous extensions of maps on τX*, Fund. Math. 74 (1972), pp. 111–131.
[11] Н. Величко, *О продолжении отображений топологических пространств*, Сибирск. Мат. Ж. 6 (1) (1965), pp. 64–69.

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A short proof of Hausdorff's theorem on extending metrics

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The aim of this paper is to give a short proof of the following theorem:

THEOREM. *Let X be a metrizable topological space and let $A\subset X$ be a closed set. Then, for every metric ϱ on A which induces the relative topology on A, there is a metric $\bar{\varrho}$ on X which is an extension of ϱ and is compatible with the topology of X. If moreover X is complete-metrizable and ϱ is a complete metric on A, then the extension $\bar{\varrho}$ above can be obtained to be a complete metric on X.*

The first part of the theorem was proved by F. Hausdorff [5] in 1930 (cf. also [6]) and independently by R. H. Bing ([3], Theorem 5) in 1947, whereas the remark concerning the "complete" case was made by P. Bacon [2] in 1968. Let us note that R. Arens ([1], Theorem 3.3) gave in 1952 a relatively simple proof of Hausdorff's theorem; his arguments were based on a close examination of "Dugundji's retraction".

The proof we are going to present involves (besides other well-known facts) the use of the following lemma of V. L. Klee ([7], pp. 36).

LEMMA. *Let E and F be normed linear spaces and let $K\subset E\times\{0\}$ and $L\subset\{0\}\times F$ be closed subsets of $E\times F$. Then, for every homeomorphism $f\colon K\xrightarrow{\text{onto}}L$, there is an extension of $\bar{f}$ to a homeomorphism $\bar{f}\colon E\times F\xrightarrow{\text{onto}}E\times F$.*

Proof. Denote by p_E and p_F the natural projections of $E\times F$ onto E and F, respectively. Since F is an ANR($\mathfrak{M}$) ([4], Theorem 4.1), the function $p_F\circ f\colon K\to F$ can be extended to $\lambda\colon E\times\{0\}\to F$. We put $f_1(\alpha,\beta)=\big(\alpha,\beta+\lambda(\alpha,0)\big)$, $(\alpha,\beta)\in E\times F$; f_1 is then a homeomorphism of $E\times F$ onto itself satisfying $f_1(\alpha,\beta)=\big(\alpha,p_F\circ f(\alpha,\beta)\big)$ for $(\alpha,\beta)\in K$. Similarly, there is a homeomorphism $f_2\colon E\times F\xrightarrow{\text{onto}}E\times F$ such that $f_2(\alpha,\beta)=\big(p_E\circ f^{-1}(\alpha,\beta),\beta\big)$ for $(\alpha,\beta)\in L$. We then have $f_2\circ f(\alpha,\beta)=f_1(\alpha,\beta)$ for $(\alpha,\beta)\in K$, whence $\bar{f}=f_2^{-1}\circ f_1$ is the desired extension of f.

Now we pass to the proof of the theorem; Bacon's remark will be considered in brackets.

Proof of the theorem. By [8], (A, ϱ) can be isometrically imbedded as a closed subset of a [complete] normed linear space $(F, \|\ \|_F)$; let $h_0\colon A \to F$ be the embedding (1). Similarly, there is a topological embedding $g_0\colon X \to E$ of X onto a closed subset of a [complete] normed linear space $(E, \|\ \|_E)$. We put $\|(\alpha, \beta)\| = \|\alpha\|_E + \|\beta\|_F$ for $(\alpha, \beta) \in E \times F$ and we set $g = g_0 \times 0\colon X \to E \times \{0\}$, $h = 0 \times h_0\colon A \to \{0\} \times F$, $K = g(A)$ and $L = h(A)$. By the Lemma, the homeomorphism $f = h \circ g^{-1}\colon K \xrightarrow{\text{onto}} L$ can be extended to a homeomorphism $\bar{f}\colon E \times F \xrightarrow{\text{onto}} E \times F$. Obviously, $u = \bar{f} \circ g$ is a topological embedding of X onto a closed subset of $E \times F$ and, since $h(A) \subset \{0\} \times F$, the restriction $u_{|A} = h$ is an isometric embedding of (A, ϱ) into the [complete] normed linear space $(E \times F, \|\ \|)$. The metric

$$\bar{\varrho}(x_1, x_2) = \|u(x_1) - u(x_2)\|, \qquad x_1, x_2 \in X,$$

is the extension of ϱ we have been looking for.

References

[1] R. Arens, *Extension of functions on fully normal spaces*, Pacific Journ. of Math. 2 (1952), pp. 11–22.

[2] P. Bacon, *Extending a complete metric*, Amer. Math. Monthly 75 (1968), pp. 642–643.

[3] R. H. Bing, *Extending a metric*, Duke Math. J. 14 (1947), pp. 511–519.

[4] J. Dugundji, *An extension of Tietze's theorem*, Pacific Journ. of Math. 1 (1951), pp. 353–367.

(1) Added in proof. For the sake of completeness we insert here a construction of h_0 (it seems to be somewhat simplier than that given by E. Michael [8]). Fix $p \in A$, denote by Z the set of all finite subsets of A, and let E be the Banach space of bounded real functions on Z (with the supremum norm). Define $h_0\colon A \to E$ by

$$h_0(x) = f_x \in E, \quad \text{where} \quad f_x(S) = \operatorname{dist}_\varrho(x, S) - \operatorname{dist}_\varrho(p, S) \text{ for } S \in Z.$$

Given $a, b \in A$ and $S \in Z$ we have $|(f_a - f_b)(S)| \leqslant \varrho(a, b) = |(f_a - f_b)(\{b\})|$, what implies $\|f_a\| \leqslant \varrho(p, a)$ and $\|f_a - f_b\| = \varrho(a, b)$. Therefore it remains to show that h_0 is closed when considered as a map into $F =$ the linear span of $h_0(A)$. This is because if $(x_n) \in A^\infty$ is a sequence such that $\lim\limits_n h_0(x_n) = \sum\limits_{i=1}^m \lambda_i f_{a_i}$ for some $\lambda_1, \ldots, \lambda_m \in R$ and $a_1, \ldots, a_m \in A$, then setting $S = \{a_1, \ldots, a_m, p\}$ we get $\lim\limits_n \operatorname{dist}_\varrho(x_n, S) = \lim\limits_n f_{x_n}(S) = (\sum\limits_{i=1}^m \lambda_i f_{a_i})(S) = 0$; thus (x_n) must contain a subsequence converging to a point of $\{a_1, \ldots, a_m, p\}$. Our argument shows also that $h_0(A \setminus \{p\})$ is a linearly independent set (observe that $h_0(x) \in \operatorname{span}\{h_0(a_i)\colon i \leqslant m\}$ implies $x \in \{a_1, \ldots, a_m, p\}$). The embedding constructed is very similar to the classical embedding of Kuratowski-Kunugui, see Fund. Math. 25 (1935), p. 543, and Proc. Imp. Acad. Japan 11 (1935), p. 351.

[5] F. Hausdorff, *Erweiterung einer Homöomorphie*, Fund. Math. 16 (1930), pp. 353–360.

[6] — *Erweiterung einer stetigen Abbildung*, Fund. Math. 30 (1938), pp. 40–47.

[7] V. L. Klee, *Some topological properties of convex sets*, Trans. Amer. Math. Soc. 78 (1955), pp. 30–45.

[8] E. Michael, *A short proof of the Arens-Eells embedding theorem*, Proc. Amer. Math. Soc. 15 (1964), pp. 415–416.

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