We conclude the paper with some open problems. Is it true that

HB ⇒ BPI?, or

KM ⇒ BPI?, or

SKM ⟺ KM?

Postscript (January 12, 1972). After this paper was submitted, we received a preprint of a review of [1] by W. A. J. Luxemburg in which the results of the present paper are arrived at independently. Corollary 2.3 has also been proved independently by Peter Renz. We have also been informed by Professor Luxemburg that D. Pincus has recently answered the first two of our open problems in the negative.

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Extending maps from dense subspaces

by

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The main result of the present paper is a generalization of the Tajmanov theorem, which claims that a continuous map $f\colon X\to Y$ of a space X into a compact Hausdorff space Y can be extended by a continuous map $*f\colon *X\to Y$ onto an extension $*X$ of X iff for each pair of closed disjoint subsets A and A' of Y we have $\mathrm{Cl}_{*X}f^{-1}(A)\cap\mathrm{Cl}_{*X}f^{-1}(A')=\emptyset$ (see [3]). Certain results generalizing this theorem were obtained in [2] and [11]. The main theorem of [2] affords a description of the greatest subset X_f lying between X and $*X$, onto which a given continuous map $f\colon X\to Y$ can be continuously extended, which however is external and rather complicated and needs complete regularity of all spaces in question. Meanwhile, the generalization of the Tajmanov theorem given in [11], which depends on replacing the compact space Y in it by an H-closed Urysohn space, the closed sets A and A' by regularly closed ones and the continuity of $*f$ by θ-continuity, is an immediate consequence of the Tajmanov theorem since H-closed Urysohn spaces are known to be exactly those which have a compact minimalization [6] (the minimalization of a Hausdorff topology $\mathcal{T}$ on X is the Hausdorff topology $\mu\mathcal{T}$ on X generated by regularly open sets of $\mathcal{T}$; the identity $(X,\mu\mathcal{T})\to(X,\mathcal{T})$ is θ-continuous). Besides, the cardinal disadvantage of the quoted results (and so far as I know, these are the strongest ones towards generalization of the Tajmanov theorem) is that they are useless in the theory of H-closed spaces, since genuine difficulties appear in this theory when the spaces are not only non-regular (a regular H-closed space is compact [1]) but also non-Urysohn ones since just then they do not admit a contraction to a compact Hausdorff space [6].

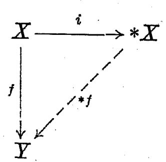
In looking for a generalization of the Tajmanov theorem, it seems simplest to give an answer to the following question: under which conditions has a map $f\colon X\to Y$ a continuous extension $*f\colon *X\to Y$ on a certain extension $*X$ of X? It is, however, hopeless to expect the existence of a continuous extension $*f$ in this general situation, particularly in the case where $*X$ is compact and $\mathrm{Cl}_Y f(X)$ is not compact, which may happen for non-compact Y. Besides, the clasical notion of continuity does not seem to be natural when non-regular spaces are in question, and it has often been replaced, either from necessity or for convenience, by the notion of θ-continuity, introduced in [4] and often used, in particular, in the theory of H-closed spaces (the definitions are given in the Preliminaries).

The main theorem yields a sufficient condition for a θ-continuous map $f\colon X\to Y$ to possess a θ-continuous extension $*f\colon *X\to Y$ on a certain extension $*X$ of X and contains the Tajmanov theorem and also some new extension theorems. Further, it is shown that θ-continuous maps can possess certain defects, which lead to a bad categorial behaviour of θ-continuous maps. A class of very θ-continuous maps (in the sense of weakly continuous maps, see III.2) is defined, and a Tajmanov-type theorem for such maps is given, i.e. a necessary and sufficient condition for a map $f\colon X\to Y$ to possess an extension $*f\colon *X\to Y$ on a given extension $*X$, both f and $*f$ being very θ-continuous maps in this sense. This theorem is applied to obtain a new result on extending maps onto the Katětov extension.

This paper is a dissertation presented at the Mathematical Institute of the Polish Academy of Sciences in 1971. I wish to express my cordial thanks to my teachers Professors Bronisław Knaster and Jerzy Mioduszewski, to whom I owe a debt of gratitude for their patient guidance and inspiring consultations, in which they have both gone out of their way to make this paper more valuable.

I. Preliminaries

The term $i\colon X\to *X$ *is an extension of* X *to* $*X$, shortly $*X$ *is an extension of* X is here used in the common sense, i.e. the map i is an embedding of the space X onto a dense subset of the space $*X$. In what follows, $i(X)$ is identified with X and denoted also by X. Extending a map $f\colon X\to Y$ onto an extension $*X$ of X is understood as usual, i.e. a map $*f\colon *X\to Y$ is a $*$-extension of f whenever $*f(x)=f(x)$ for each $x\in X$, in other words the diagram



is commutative.

The symbol Cl is used to denote the closure operator and Cl_* always stands for the closure operator in $*X$. The symbol U_x stands for an open neighbourhood of a point x of the space under consideration, while $\mathcal{U}_x$ denotes the family of all such neighbourhoods.

A Hausdorff space X is H-closed (Alexandroff and Urysohn [1]) whenever each centred family $\mathcal{U}$, consisting of open sets of X, has an adherence point, i.e. $\bigcap\{\mathrm{Cl}\,U\colon U\in\mathcal{U}\}\neq\emptyset$. Equivalently, X is H-closed whenever each covering $\mathcal{T}$ of X, consisting of open sets of X, contains a finite quasi-subcovering, i.e. $X=\mathrm{Cl}\,U_1\cup\ldots\cup\mathrm{Cl}\,U_n$ for some elements $U_1,\ldots,U_n\in\mathcal{T}$.

Each Hausdorff space X has a maximal H-closed extension (Katětov [6]), abbreviated to τX, which is constructed by adding to X all ultrafilters consisting of open sets of X without adherence points and the topology in τX is generated by all open sets of X and all sets of the form $\{x\}\cup U$ where x is an ultrafilter from $\tau X-X$ and U is an element of x.

A map $f\colon X\to Y$ is called θ-*continuous* (Fomin [4]) whenever for each $x\in X$ and each open neighbourhood U_y of $y=f(x)$ there exists an open neighbourhood U_x of x such that $f(\mathrm{Cl}\,U_x)\subset\mathrm{Cl}\,U_y$. Observe that each continuous map is θ-continuous and these notions coincide when Y is a regular space.

II. θ-continuous extensions of maps

1. $\theta*$-proper maps. Let $*X$ be an extension of X. Given a map $f\colon X\to Y$, denote by $\mathcal{U}(f,\mathcal{U}_x)$ the filter of open sets of Y, generated by the centred family of sets $f(U_x\cap X)$, U_x running over the family $\mathcal{U}_x$ of open (in $*X$) neighbourhoods of a point $x\in *X-X$, i.e. $\mathcal{U}(f,\mathcal{U}_x)=\{U\colon U$ is open in Y and $U\supset f(U_x\cap X)$ for some $U_x\in\mathcal{U}_x\}$.

A map $f\colon X\to Y$ is called $\theta*$-*proper* whenever for each $x\in *X-X$

$$\bigcap\{\mathrm{Cl}\,U\colon U\in\mathcal{U}(f,\mathcal{U}_x)\}\neq\emptyset,\tag{1}$$

that is, $\mathcal{U}(f,\mathcal{U}_x)$ has an adherence point.

Observe that in the case of Y being H-closed, each map $f\colon X\to Y$ is $\theta*$-proper with respect to each extension $*X$ of X. This is not true in general: the identity map $X\to X$ of a Hausdorff space X is not $\theta*$-proper with respect to any proper Hausdorff extension $*X$ of X and such an extension exists whenever X is not H-closed.

The $\theta*$-proper maps appear when looking for θ-continuous extensions of maps, more exactly:

THEOREM 1.1. *If* $*f\colon *X\to Y$ *is a* θ*-continuous extension of* $f\colon X\to Y$ *on* $*X$, *then*

$$*f(x)\in\bigcap\{\mathrm{Cl}\,U\colon\ U\in\mathfrak{U}(f,\mathfrak{U}_x)\}\quad \textit{for each } x\in *X-X\,. \tag{2}$$

Proof. Suppose, on the contrary, that $y=*f(x)\notin \mathrm{Cl}\,U$ for some $U\in\mathfrak{U}(f,\mathfrak{U}_x)$. Take $U_y=Y-\mathrm{Cl}\,U$, an open neighbourhood of y. Then there exists, by the θ-continuity of $*f$, an open neighbourhood U_x of x such that $*f(\mathrm{Cl}\,U_x)\subset\mathrm{Cl}\,U_y\subset Y-U$ and all the more $*f(U_x\cap X)=f(U_x\cap X)\subset Y-U$. But U, being an element of $\mathfrak{U}(f,\mathfrak{U}_x)$, contains a set of the form $f(U_{x'}\cap X)$ for some $U_{x'}\in\mathfrak{U}_x$. Now $f(U_x\cap X)\cap\cap f(U_{x'}\cap X)\subset(Y-U)\cap U=\emptyset$, which implies $f(U_x\cap U_{x'}\cap X)=\emptyset$, a contradiction.

The theorem proved above shows that in looking for a theorem on θ-continuous $*$-extensions similar to the Tajmanov theorem a restriction to $\theta*$-proper maps is necessary. But not every $\theta*$-proper map $f\colon X\to Y$ possesses a θ-continuous extension $*f\colon *X\to Y$ (even in the case of compact Y, each $f\colon X\to Y$ is $\theta*$-proper, however, a θ-continuous $*f\colon *X\to Y$ which becomes by the regularity of Y simply continuous, does not always exist — a consequence of the Tajmanov theorem). A $\theta*$-proper map, in order to possess a θ-continuous $*$-extension, must fulfil additional conditions, which imply the condition of the Tajmanov theorem in the case covered by it.

2. $*$-free maps. Let $*X$ be an extension of X. A map $f\colon X\to Y$ is called $*$*-free* whenever

(3) for each $x\in *X-X$ and each $y\in Y$ and each regularly closed set $A\subset Y$ such that $y\notin A$ there exists an open neighbourhood U_y of y such that

$$x\notin \mathrm{Cl}_*f^{-1}(\mathrm{Cl}\,U_y)\cap\mathrm{Cl}_*f^{-1}(A)\,.$$

The next theorem justifies the notion of $*$-freeness.

THEOREM 2.1. *A continuous map* $f\colon X\to Y$ *of a space* X *into a compact Hausdorff space* Y *is* $*$*-free iff for each pair of closed disjoint subsets* A *and* A' *of* Y *there is*

$$\mathrm{Cl}_*f^{-1}(A)\cap\mathrm{Cl}_*f^{-1}(A')=\emptyset\,. \tag{4}$$

Proof of the non-trivial implication (3)$\Rightarrow$(4). Observe that in view of the normality of Y it suffices to prove (4) for A and A' being disjoint regularly closed sets. Let x be an arbitrary point of $*X-X$. For each $a\in A$ there exists by (3) an open neighbourhood U_a such that

$$x\notin\mathrm{Cl}_*f^{-1}(\mathrm{Cl}\,U_a)\cap\mathrm{Cl}_*f^{-1}(A'). \tag{5}$$

Since $\{U_a\colon a\in A\}$ is an open covering of the compact set A a finite subcovering $\{U_{a_1},\ldots,U_{a_n}\}$ can be chosen, i.e.

$$A\subset U_{a_1}\cup\ldots\cup U_{a_n}\subset\mathrm{Cl}(U_{a_1}\cup\ldots\cup U_{a_n})\,. \tag{6}$$

From (5) it follows that

$$x\notin\mathrm{Cl}_*f^{-1}(\mathrm{Cl}(U_{a_1}\cup\ldots\cup U_{a_n}))\cap\mathrm{Cl}_*f^{-1}(A')\,. \tag{7}$$

Thus (6) together with (7) yields $x\notin\mathrm{Cl}_*f^{-1}(A)\cap\mathrm{Cl}_*f^{-1}(A')$, x being an arbitrary point of $*X-X$, and in consequence

$$(*X-X)\cap\mathrm{Cl}_*f^{-1}(A)\cap\mathrm{Cl}_*f^{-1}(A')=\emptyset\,. \tag{8}$$

But $X\cap\mathrm{Cl}_*f^{-1}(A)=\mathrm{Cl}_Xf^{-1}(A)=f^{-1}(A)$, the set A being closed and f being continuous, whence

$$X\cap\mathrm{Cl}_*f^{-1}(A)\cap\mathrm{Cl}_*f^{-1}(A')=\emptyset\,. \tag{9}$$

Now (8) and (9) imply (4) for A and A' being regularly closed sets and the theorem is proved.

Although the Tajmanov condition can be generalized in several ways, the $*$-freeness seems to be the most reasonable generalization.

Call a $\theta*$-proper and $*$-free map simply $*$*-proper*. Examine some properties of $*$-proper maps, needed in the proof of the main theorem. To do this, prove a rather technical but useful lemma.

LEMMA 2.2. *Let* $*X$ *be an arbitrary extension of* X *and let* $f\colon X\to Y$ *be an arbitrary set-theoretic map. Then for each* $x\in *X$ *and each* $y\in Y$

$$y\in\bigcap\{\mathrm{Cl}\,U\colon\ U\in\mathfrak{U}(f,\mathfrak{U}_x)\}\quad \textit{iff}\quad x\in\bigcap\{\mathrm{Cl}_*f^{-1}(\mathrm{Cl}\,U_y)\colon\ U_y\in\mathfrak{U}_y\}\,.$$

Proof. Let $y\in\bigcap\{\mathrm{Cl}\,U\colon\ U\in\mathfrak{U}(f,\mathfrak{U}_x)\}$. This is equivalent to

$$U_y\cap U\neq\emptyset\quad\text{for each } U_y\in\mathfrak{U}_y \text{ and each } U\in\mathfrak{U}(f,\mathfrak{U}_x)\,. \tag{10}$$

Observe that from (10) it follows that

$$\mathrm{Cl}\,U_y\cap f(U_x\cap X)\neq\emptyset\quad\text{for each } U_y\in\mathfrak{U}_y \text{ and each } U_x\in\mathfrak{U}_x \tag{11}$$

(in the opposite case $\mathrm{Cl}\,U_y\cap f(U_x\cap X)=\emptyset$ for some $U_y\in\mathfrak{U}_y$ and some $U_x\in\mathfrak{U}_x$; thus the open set $U=Y-\mathrm{Cl}\,U_y\supset f(U_x\cap X)$ is an element of $\mathfrak{U}(f,\mathfrak{U}_x)$ for which $U_y\cap U=\emptyset$, contrary to (10)).

Moreover, (11) implies that $\mathrm{Cl}\,U_y\cap U\neq\emptyset$ for each $U_y\in\mathfrak{U}_y$ and each $U\in\mathfrak{U}(f,\mathfrak{U}_x)$ and, this inequality being equivalent to $U_y\cap U\neq\emptyset$ since U is an open set, (10) follows. Thus (10) is equivalent to (11). But (11) is equivalent to

$$U_x\cap X\cap f^{-1}(\mathrm{Cl}\,U_y)\neq\emptyset\quad\text{for each } U_x\in\mathfrak{U}_x \text{ and each } U_y\in\mathfrak{U}_y \tag{12}$$

and this is equivalent, in view of $X \cap f^{-1}(\mathrm{Cl}\, U_y) = f^{-1}(\mathrm{Cl}\, U_y)$, to

$$(13) \qquad U_x \cap f^{-1}(\mathrm{Cl}\, U_y) \neq \emptyset \quad \text{for each } U_x \in \mathfrak{U}_x \text{ and each } U_y \in \mathfrak{U}_y .$$

Finally, (13) is equivalent to $x \in \bigcap \{\mathrm{Cl}_* f^{-1}(\mathrm{Cl}\, U_y)\colon U_y \in \mathfrak{U}_y\}$ and the lemma is proved.

Now a uniqueness property of $*$-proper maps is easy to verify.

THEOREM 2.3. *Let $*X$ be an arbitrary extension of X. Then for each $*$-proper map $f\colon X \to Y$ into a Hausdorff space Y the intersection $\bigcap \{\mathrm{Cl}\, U\colon U \in \mathfrak{U}(f, \mathfrak{U}_x)\}$ is a one-point set for each $x \in *X - X$.*

Proof. Let x be an arbitrary point of $*X - X$ and suppose that $y \in \bigcap \{\mathrm{Cl}\, U\colon U \in \mathfrak{U}(f, \mathfrak{U}_x)\}$ (such a point y exists since f is a $\theta*$-proper map). By 2.2,

$$(14) \qquad x \in \mathrm{Cl}_* f^{-1}(\mathrm{Cl}\, U_y) \quad \text{for each } U_y \in \mathfrak{U}_y .$$

Take an arbitrary point y' of Y, different from y. Since Y is a Hausdorff space, there exists an open neighbourhood $U_{y'}$ of y' such that $y \notin \mathrm{Cl}\, U_{y'}$. The map f being $*$-free, there exists an open neighbourhood U_y of y such that

$$(15) \qquad x \notin \mathrm{Cl}_* f^{-1}(\mathrm{Cl}\, U_y) \cap \mathrm{Cl}_* f^{-1}(\mathrm{Cl}\, U_{y'}) .$$

From (14) and (15) it follows that $x \notin \mathrm{Cl}_* f^{-1}(\mathrm{Cl}\, U_{y'})$ and this implies in view of 2.2 that $y' \notin \bigcap \{\mathrm{Cl}\, U\colon U \in \mathfrak{U}(f, \mathfrak{U}_x)\}$, y' being an arbitrary point of Y different from y, and this ends the proof.

The converse theorem is true for Y being H-closed. More precisely

THEOREM 2.4. *A map $f\colon X \to Y$ into an H-closed space Y is $*$-proper iff the intersection $\bigcap \{\mathrm{Cl}\, U\colon U \in \mathfrak{U}(f, \mathfrak{U}_x)\}$ is a one-point set for each $x \in *X - X$.*

Proof. The "only if" implication following from 2.3, prove the lacking "if", that is assume that $\bigcap \{\mathrm{Cl}\, U\colon U \in \mathfrak{U}(f, \mathfrak{U}_x)\}$ is a one-point set for each $x \in *X - X$. It suffices to prove that f is $*$-free. To do this, let x be an arbitrary point of $*X - X$ and let y be a point of Y such that

$$(16) \qquad x \in \bigcap \{\mathrm{Cl}_* f^{-1}(\mathrm{Cl}\, U_y)\colon U_y \in \mathfrak{U}_y\} .$$

It remains to show that for each regularly closed set $A \subset Y$, if $y \notin A$ then $x \notin \mathrm{Cl}_* f^{-1}(A)$. Given such a set A, observe that for each $a \in A$ there exists an open neighbourhood U_a such that

$$(17) \qquad x \notin \mathrm{Cl}_* f^{-1}(\mathrm{Cl}\, U_a)$$

since y is, by assumption, the only point satisfying (16). Regularly closed subsets of H-closed spaces are known to be H-closed also [6]; thus a finite quasi-subcovering of the covering $\{U_a\colon a \in A\}$ can be chosen, that is $A \subset \mathrm{Cl}\, U_{a_1} \cup \ldots \cup \mathrm{Cl}\, U_{a_n}$. Then (17) implies that $x \notin \mathrm{Cl}_* f^{-1}(A)$ and the theorem is proved.

3. Main theorem. In looking for θ-continuous extensions of maps there is no reason to consider extensions of continuous maps only. However, at most θ-continuous maps can possess θ-continuous extensions since the restriction of maps preserves θ-continuity.

THEOREM 3.1. *Let $*X$ be an arbitrary extension of X. Then each θ-continuous $*$-proper map $f\colon X \to Y$ possesses a θ-continuous extension $*f\colon *X \to Y$. The extension of f is unique when Y is a Hausdorff space.*

Proof. Define

$$(18) \qquad *f(x) \in \bigcap \{\mathrm{Cl}\, U\colon U \in \mathfrak{U}(f, \mathfrak{U}_x)\} \quad \text{for each } x \in *X - X;$$

This definition is correct since f is $\theta*$-proper.

To prove θ-continuity of $*f$, the following formula will be useful:

$$(19) \qquad *f\big((*X - X) \cap \mathrm{Cl}_* f^{-1}(\mathrm{Cl}\, U)\big) \subset \mathrm{Cl}\, U \quad \text{for each open set } U \subset Y .$$

To prove (19) suppose on the contrary that for some open $U \subset Y$ and some $x \in (*X - X) \cap \mathrm{Cl}_* f^{-1}(\mathrm{Cl}\, U)$ there is $*f(x) = y \notin \mathrm{Cl}\, U$. The map f being $*$-free, there exists an open neighbourhood U_y of y such that

$$(20) \qquad x \notin \mathrm{Cl}_* f^{-1}(\mathrm{Cl}\, U_y) \cap \mathrm{Cl}_* f^{-1}(\mathrm{Cl}\, U) .$$

From the definition (18) of $*f$ on points of the remainder it follows by 2.2 that

$$(21) \qquad x \in \mathrm{Cl}_* f^{-1}(\mathrm{Cl}\, U_y) .$$

Then (20) together with (21) implies $x \notin \mathrm{Cl}_* f^{-1}(\mathrm{Cl}\, U)$, contrary to the choice of x, and so (19) is proved.

Now pass to the proof of the θ-continuity of $*f$.

First consider an arbitrary point x of X. Take $y = *f(x) = f(x)$ and let U_y be an arbitrary open neighbourhood of y. The map f being θ-continuous, there exists an open neighbourhood U_x of x such that $f(\mathrm{Cl}\, U_x) \subset \mathrm{Cl}\, U_y$; in other words

$$(22) \qquad \mathrm{Cl}\, U_x \subset f^{-1}(\mathrm{Cl}\, U_y) .$$

Define $U_x^* = \bigcup \{U\colon U \text{ is open in } *X \text{ and } U \cap X = U_x\}$, an open neighbourhood of x in $*X$. Then $\mathrm{Cl}_*\, U_x^* = \mathrm{Cl}_*(U_x^* \cap X) = \mathrm{Cl}_*\, U_x$ and from (22) it follows that $\mathrm{Cl}_*\, U_x \subset \mathrm{Cl}_* f^{-1}(\mathrm{Cl}\, U_y)$; thus

$$(23) \qquad \mathrm{Cl}_*\, U_x^* \subset \mathrm{Cl}_* f^{-1}(\mathrm{Cl}\, U_y) .$$

Calculate

$$*f(\mathrm{Cl}_*\,U_x^*) = *f\big((*X-X)\cap \mathrm{Cl}_*\,U_x^*\big)\cup *f(X\cap \mathrm{Cl}_*\,U_x^*)$$
$$= *f\big((*X-X)\cap \mathrm{Cl}_*\,U_x^*\big)\cup f(\mathrm{Cl}\,U_x)\subset \mathrm{Cl}\,U_y\,,$$

the inclusion of the first summand being a consequence of (19) and (23) and of the second one it follows by (22). Thus $*f$ is θ-continuous at points of X.

Now prove the θ-continuity of $*f$ at points of the remainder. Consider an arbitrary point $x\in *X-X$ and let U_y be an arbitrary r.o. neighbourhood (r.o. stands for regularly open) of $y = *f(x)$. It suffices to prove θ-continuity by using r.o. neighbourhoods of y only, of course. Then $y\notin Y-U_y$ and from the formula (19), applied to the r.c. (regularly closed) set $Y-U_y$, it follows that

$$x\notin \mathrm{Cl}_* f^{-1}(Y-U_y)\,. \tag{24}$$

Thus $U_x = *X-\mathrm{Cl}_*f^{-1}(Y-U_y)$ is an open neighbourhood of x in $*X$. But $Y = \mathrm{Cl}\,U_y\cup(Y-U_y)$; thus $*X = \mathrm{Cl}_*f^{-1}(\mathrm{Cl}\,U_y)\cup \mathrm{Cl}_*f^{-1}(Y-U_y)$ and U_x is disjoint, by definition, with the second summand; thus $U_x\subset \mathrm{Cl}_*f^{-1}(\mathrm{Cl}\,U_y)$ and hence

$$\mathrm{Cl}_*\,U_x\subset \mathrm{Cl}_*f^{-1}(\mathrm{Cl}\,U_y)\,. \tag{25}$$

Moreover, $U_x\cap X = X-\mathrm{Cl}f^{-1}(Y-U_y)\subset X-f^{-1}(Y-U_y) = f^{-1}(U_y)$; thus

$$\mathrm{Cl}(U_x\cap X)\subset \mathrm{Cl}f^{-1}(U_y)\,. \tag{26}$$

But $f\big(\mathrm{Cl}f^{-1}(U)\big)\subset \mathrm{Cl}\,U$ holds for each open $U\subset Y$ since f is θ-continuous. To prove this, suppose on the contrary that $f(x)\notin \mathrm{Cl}\,U$ for some $x\in \mathrm{Cl}f^{-1}(U)$. For $U_{f(x)} = Y-\mathrm{Cl}\,U$ take U_x such that $f(\mathrm{Cl}\,U_x)\subset \mathrm{Cl}\,U_{f(x)}\subset Y-U$. It follows that $\mathrm{Cl}\,U_x\cap f^{-1}(U) = \emptyset$, and all the more $x\notin \mathrm{Cl}f^{-1}(U)$, contrary to the choice of x. The formula proved above, together with (26), yields

$$f\big(\mathrm{Cl}(U_x\cap X)\big)\subset \mathrm{Cl}\,U_y\,. \tag{27}$$

Calculate

$$*f(\mathrm{Cl}_*\,U_x) = *f\big((*X-X)\cap \mathrm{Cl}_*\,U_x\big)\cup *f(X\cap \mathrm{Cl}_*\,U_x)$$
$$= *f\big((*X-X)\cap \mathrm{Cl}_*\,U_x\big)\cup f\big(\mathrm{Cl}(U_x\cap X)\big)\subset \mathrm{Cl}\,U_y\,,$$

the inclusion of the first summand following from (19) and (25) and of the second one from (27), which proves the θ-continuity of $*f$ at points of $*X-X$; this ends the proof of the θ-continuity of the defined extension $*f$.

The uniqueness of the θ-continuous extension of a $*$-proper map into a Hausdorff space is a direct consequence of 1.1 and 2.3.

Remark. It has been proved, in fact, that

(*) for each $x\in *X-X$ and for each r.o. neighbourhood U_y of $y = *f(x)$ there exists an open neighbourhood U_x of x such that $*f(\mathrm{Cl}\,U_x)\subset \mathrm{Cl}\,U_y$ and $*f(U_x\cap X)\subset U_y$.

The latter inclusion follows from the definition $U_x = *X-\mathrm{Cl}_*f^{-1}(Y-U_y)$.

Suppose the map $f\colon X\to Y$ in Theorem 3.1 to be $*$-free with respect to all closed sets, which means that the set A in the definition of $*$-freeness is assumed to be an arbitrary closed, not only r.c., set. Then Theorem 3.1 remains valid and $*f$ has the property (*) with respect to each open, not necessarily r.o., neighbourhood U_y of $y = *f(x)$. To prove this assertion observe that on account of the assumption on f formula (19) is true for arbitrary closed sets. In consequence, given an arbitrary open neighbourhood U_y of y, formula (24) holds; thus $U_x = *X-\mathrm{Cl}_*f^{-1}(Y-U_y)$ is a neighbourhood of x with property (*). This property of $*f$ is very important (see III.2 and IV.1).

The assumption of f being $*$-proper, which is sufficient for the existence of a (unique) θ-continuous extension $*f$ of f, is not necessary, as shown in

Example 1. Let Y_0 be the plane subset consisting of the points $(-1,0)$, $(1,0)$ and points $(\pm 1/n,\,1/m)$ and $(0,1/m)$ for $n, m = 1, 2, \ldots$ with topology generated by all one-point sets $\{(\pm 1/n, 1/m)\}$ and sets

$$U^k_{(0,1/m)} = \{(0,1/m)\}\cup\{(\pm 1/n, 1/m)\colon\ n\geqslant k\}\quad \text{for}\quad m = 1, 2, \ldots,$$
$$U^k_{(-1,0)} = \{(-1,0)\}\cup\{(-1/n, 1/m)\colon\ m\geqslant k \text{ and } n = 1, 2, \ldots\},$$
$$U^k_{(1,0)} = \{(1,0)\}\cup\{(1/n, 1/m)\colon\ m\geqslant k \text{ and } n = 1, 2, \ldots\},$$

where $k = 1, 2, \ldots$

The space Y_0 was constructed by Urysohn and proved to be H-closed in [1]. The set $U = \{(1/n, 1/2m)\colon n, m = 1, 2, \ldots\}$ is regularly open; thus Y_0-U is H-closed as a regularly closed subset of the H-closed space Y_0.

Let N denote the set of natural numbers with discrete topology and $*N = N\cup\{*\}$ — its Alexandroff one-point compactification. Define $i\colon N\to Y_0-U$ by $i(n) = (0, 1/n)$. Then both $(-1,0)$ and $(1,0)$ are adherence points of $\mathfrak{U}(i, \mathfrak{U}_*)$, $\mathfrak{U}_*$ being the filter of co-finite sets of $*N$, containing the point $*$. Thus i is not $*$-proper in consequence of 2.4. But the extension $*i$ of i, given by the formula $*i(*) = (-1,0)$, is θ-continuous, and it is easy to verify that this is the only θ-continuous extension of i on $*N$, and this proves that not only $*$-proper maps possess unique θ-continuous extensions.

Theorem 3.1 yields an answer to the more difficult half of the problem of the characterization of extendability of maps under sufficiently large and reasonable assumptions. The properties of the map $*f$ depend simultaneously on those of the map f and of the spaces $*X$ and Y, and this connection is too deep to be expressed in this general situation by using natural and simple conditions.

However, under various restrictions either on the space Y or on the map f or on the structure of the extension $*X$ the converse theorem becomes true. Note two Tajmanov-type theorems following immediately from 3.1 (another one will be proved in IV).

COROLLARY 1. *A continuous map* $f\colon X\to Y$ *into a regular space* Y *possesses a continuous extension* $*f\colon *X\to Y$ *iff* f *is* $*$*-proper.*

Proof of $\Rightarrow$. Suppose $*f$ is a continuous $*$-extension of f. Then f is $\theta*$-proper by 1.1. To prove that f is $*$-free observe that the continuity of $*f$ implies $*f\big(\mathrm{Cl}_*f^{-1}(A)\big)\subset \mathrm{Cl}A$ for each $A\subset Y$. So, given $y\in Y$ and a r.c. set $A\subset Y$ such that $y\notin A$, it follows that

$$*f\big(\bigcap\{\mathrm{Cl}_*f^{-1}(\mathrm{Cl}\,U_y)\colon U_y\in\mathcal{U}_y\}\cap \mathrm{Cl}_*f^{-1}(A)\big)\subset\bigcap\{\mathrm{Cl}\,U_y\colon U_y\in\mathcal{U}_y\}\cap A = \{y\}\cap A=\emptyset;$$

thus no point of $*X-X$ belongs to $\bigcap\{\mathrm{Cl}_*f^{-1}(\mathrm{Cl}\,U_y)\colon U_y\in\mathcal{U}_y\}\cap\mathrm{Cl}_*f^{-1}(A)$ and this proves the $*$-freeness of f.

Proof of $\Leftarrow$. Suppose f is a continuous $*$-proper map. Then the θ-continuous $*$-extension of f, which exists by 3.1, is simply continuous, the space Y being regular.

Remark. For Y being a compact space, Corollary 1 together with 2.1 implies the Tajmanov theorem.

COROLLARY 2. *An open* θ*-continuous map* $f\colon X\to Y$ *possesses a* θ*-continuous extension* $*f\colon *X\to Y$ *iff* f *is* $*$*-proper.*

Proof of the non-trivial implication $\Rightarrow$. It suffices to prove that f is $*$-free. Suppose $x\in\bigcap\{\mathrm{Cl}_*f^{-1}(\mathrm{Cl}\,U_y)\colon U_y\in\mathcal{U}_y\}$ for some $y\in Y$, x being an arbitrary point of $*X-X$. This implies $*f(x)=y$, for if $*f(x)=y'\neq y$, then, for $U_{y'}$ chosen in such a manner that $y\notin \mathrm{Cl}\,U_{y'}$, there exists a U_x such that $*f(\mathrm{Cl}\,U_x)\subset\mathrm{Cl}\,U_{y'}$, whence $f(U_x\cap X)\subset\mathrm{Int}\,\mathrm{Cl}\,U_{y'}$, f being an open map; in consequence $y\notin \mathrm{Cl}f(U_x\cap X)$; thus $y\notin\bigcap\{\mathrm{Cl}\,U\colon U\in\mathcal{U}(f,\mathcal{U}_x)\}$ since $f(U_x\cap X)\in\mathcal{U}(f,\mathcal{U}_x)$, contrary to 2.2 and the assumption on x and y. Now, let A be a r.c. set for which $y\notin A$. There exists a neighbourhood U_x of x such that $*f(\mathrm{Cl}\,U_x)\subset\mathrm{Cl}\,U_y=\mathrm{Cl}(Y-A)$, and $f(U_x\cap X)\subset\mathrm{Int}\,\mathrm{Cl}(Y-A)=Y-A$, since f is open and A regularly closed. Then $U_x\cap X\cap f^{-1}(A)=\emptyset$, whence $U_x\cap f^{-1}(A)=\emptyset$ and $x\notin\mathrm{Cl}_*f^{-1}(A)$ and f is $*$-free.

In analogy to [10], call a map $f\colon X\to Y$ a *Urysohn map* whenever for each pair of distinct points y and y' of Y there exist open neighbourhoods U_y and $U_{y'}$ for which $\mathrm{Int}\mathrm{Cl}f^{-1}(\mathrm{Cl}U_y)\cap\mathrm{Int}\mathrm{Cl}f^{-1}(\mathrm{Cl}U_{y'})=\emptyset$. In [10] continuous Urysohn maps were studied in connection with ex-

tending maps on the Katětov extension. The Katětov extension τX of a Hausdorff space X is known from [7] to possess the following properties: 1° $\mathrm{Cl}_\tau A=A$ for each closed and nowhere-dense set $A\subset X$ and 2° $(\tau X-X)\cap\mathrm{Cl}_\tau U\cap\mathrm{Cl}_\tau U'=\emptyset$ for each pair of disjoint open sets U and U' of X. It follows from 1° and 2° that a Urysohn map is τ-free; thus 3.1 implies

COROLLARY 3. *Each* θ*-continuous Urysohn map* $f\colon X\to Y$ *into an* H*-closed space* Y *possesses a unique* θ*-continuous extension* $\tau f\colon\tau X\to Y$.

This is a generalization of a theorem of [10], concerning continuous Urysohn maps.

III. Weakly continuous maps

1. Defectiveness of θ**-continuous maps.** The troubles in giving a full description of θ-continuously extendable maps are caused by a certain oddity of θ-continuity. A θ-continuous map $f\colon X\to Y$ is called *defective at a point* x of X whenever there exists such an open neighbourhood U_y of $y=f(x)$ that $f(U_x-\{x\})\subset\mathrm{Cl}\,U_y-U_y$ for a certain (and in consequence for each sufficiently small) open neighbourhood U_x of x. The point $*$ in Example 1 is such a defective point of the map $*i$. This possible deficiency of θ-continuous maps leads to bad categorial properties of the category $\theta\boldsymbol{H}$ of all Hausdorff spaces and all their θ-continuous maps. One of them can be seen in Example 1: in the decomposition $X\to f(X)\subset Y$ of a defective θ-continuous map $f\colon X\to Y$ the inner factor $X\to f(X)$ need not be θ-continuous and $f(X)$ need not be H-closed, although X is H-closed. Further, the value of a defective θ-continuous map $f\colon X\to Y$ at a point $x\in X$ need not be determined by the values of f on $X-\{x\}$, and a categorial consequence of this fact is the following theorem:

THEOREM 1.1. *Epimorphisms of* $\theta\boldsymbol{H}$ *coincide with onto maps.*

Proof of $\Rightarrow$. To prove that an epimorphism in $\theta\boldsymbol{H}$ is onto it suffices to show that for each θ-continuous map $f\colon X\to Y$ for which $f(X)\nsubseteq Y$, there exist distinct maps $f',f''\colon Y\to Z$ such that $f'\circ f=f''\circ f$.

Suppose $y_0\in Y-f(X)$. Denote by S the set $\{\pm 1/n\colon n\in N\}\cup\{0\}$, with topology from the real line and let $Z_0=S\times(Y-(y_0\})$ with the product topology and let $Z=Z_0\cup\{y_0',y_0''\}$ with topology generated by all open sets of Z_0 and sets of the form

$$U'(U_{y_0})=\{y_0'\}\cup\{-1/n\colon n\in N\}\times(U_{y_0}-\{y_0\})$$

and

$$U''(U_{y_0})=\{y_0''\}\cup\{+1/n\colon n\in N\}\times(U_{y_0}-\{y_0\})$$

for each $U_{y_0} \in \mathfrak{U}_{y_0}$. Then Z is a Hausdorff space and f' and f'' are defined by the formulas

$$f'(y) = \begin{cases} (0, y) & \text{when } y \neq y_0 \\ y_0' & \text{when } y = y_0 \end{cases} \quad \text{and} \quad f''(y) = \begin{cases} (0, y) & \text{when } y \neq y_0, \\ y_0'' & \text{when } y = y_0. \end{cases}$$

The maps f' and f'' are θ-continuous. It is clear for points $y \neq y_0$ of Y since f' and f'' restricted to $Y - \{y_0\}$ are embeddings, and to prove it for $y = y_0$ observe that $\mathrm{Cl}_Z U'(U_{y_0}) = \{y_0'\} \cup (\{0\} \cup \{-1/n\colon\ n \in N\}) \times \times (\mathrm{Cl}\, U_{y_0} - (y_0\})$; thus $f'(\mathrm{Cl}\, U_{y_0}) \subset \mathrm{Cl}_Z U'(U_{y_0})$ and analogous formulas are valid for f''. Moreover, $f' \circ f = f'' \circ f$ since $f'(y) = f''(y)$ for each $y \neq y_0$ and all the more for each $y \in f(X)$. But f' and f'' being distinct maps, this proves that f is not an epimorphism.

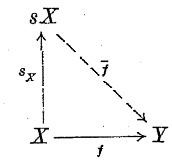
Proof of $\Leftarrow$. Since there is a forgetful functor $\theta\boldsymbol{H} \to \mathfrak{S}$ into the category $\mathfrak{S}$ of all sets and all their maps and each onto map is an epimorphism in $\mathfrak{S}$, the same is valid in $\theta\boldsymbol{H}$.

Remark. Observe that Z is H-closed whenever Y is H-closed. It follows that, given the Katětov extension τX of a non-H-closed Hausdorff space X, there always exists a map $f\colon X \to Y$ into an H-closed space Y which possesses two different θ-continuous extensions $f', f''\colon \tau X \to Y$, both being defective. Thus the Katětov extension is not a reflection in the sense of [5] of $\theta\boldsymbol{H}$ into the category $\theta\boldsymbol{HCl}$ of all H-closed spaces and all their θ-continuous maps. To avoid the categorial triviality of θ-continuous maps a restriction to maps without the deficiency in question is necessary.

2. θ-continuous maps without defects. Let $f\colon X \to Y$ be a θ-continuous map. The non-defectiveness of f depends, roughly speaking, on the following property: for each $x \in X$ and each open neighbourhood U_y of $y = f(x)$ there exists an open neighbourhood U_x of x such that $f(\mathrm{Cl}\, U_x) \subset \mathrm{Cl}\, U_y$ and that $(U_x - \{x\}) \cap f^{-1}(U_y)$ contains at least one point, or a set which is either open, or dense, or open and dense in U_x. In the first case f is called a *non-defective map*, in the last case a *weakly continuous map*. It is clear that each continuous map is weakly continuous and that the converse is not true. Although the difference between those two extreme notions of θ-continuity without defects seems to be large, they are of equal importance from the categorial viewpoint as shown in Theorems 3.2 and 3.3 of the next paragraph. Especially the notion of weak continuity proves very useful in extension theory.

3. Weakly continuous maps and s-maps. Recall that topological spaces $(X, \mathcal{T})$ and $(X, \mathcal{T}')$ are said to be *r.o.-equivalent* (for details see [9]) whenever the families of r.o. sets of both $\mathcal{T}$ and $\mathcal{T}'$ are identical. If in addition $\mathcal{T} \subset \mathcal{T}'$, the set-theoretical identity $(X, \mathcal{T}) \to (X, \mathcal{T}')$ is called an *r.o.-expansion*. The *standard r.o.-expansion* is the r.o.-expansion $s_X\colon (X, \mathcal{T}) \to (X, s\mathcal{T})$ where $s\mathcal{T}$ is generated by the base $\mathcal{B} = \{U\colon V \subset U \subset \mathrm{Int}\,\mathrm{Cl}\, V$ for some $V \in \mathcal{T}\}$. Observe that for each point x of X the family $\{\{x\} \cup (U_x - A)\colon U_x$ is an open neighbourhood of x from $\mathcal{T}$ and A is closed and nowhere-dense in $\mathcal{T}\}$ is a neighbourhood basis of x in $s\mathcal{T}$ (now it is easy to check, using a criterion given in [9], that s_X is, in fact, an r.o.-expansion). Denote the space $(X, s\mathcal{T})$ by sX. There is a close connection between weakly continuous and continuous maps, given in the following theorem.

THEOREM 3.1. *A map $f\colon X \to Y$ is weakly continuous iff the map $\bar{f}$ in the diagram*



is continuous.

Proof. This is easily proved by using the described basic neighbourhoods of points in sX together with θ-continuity of the r.o.-expansion s_X.

Theorem 3.1 implies

COROLLARY 1. *Let $f\colon X \to Y$ be a weakly continuous map. Then the inner factor $X \to f(X)$ of the decomposition $X \to f(X) \subset Y$ is also weakly continuous.*

COROLLARY 2. *The image of an H-closed space under a weakly continuous map into a Hausdorff space is H-closed.*

Proof. Recall that a space is H-closed iff its standard r.o.-expansion is H-closed because of their r.o.-equivalence and apply 3.1.

However, weakly continuous maps do not form a category, even when only H-closed spaces are in question, as shown in

EXAMPLE 2. Let Y_0 be the space defined in Example 1. Let Y_1 denote the subset of Y_0 consisting of points with non-negative first coordinates with topology induced from Y_0 and let μY_1 denote the set Y_1 with topology generated by r.o. sets of Y_1. Let $*N$ be the space used in Example 1. Then the map $i\colon *N \to \mu Y_1$ defined by $i(n) = (0, 1/n)$ and $i(*) = (1, 0)$ is continuous, since the basic neighbourhoods of the point $(1, 0)$ in μX_1 have the form $\mu U^k_{(1,0)} = U^k_{(1,0)} \cup \{(0, 1/m)\colon\ m \geqslant k\}$, and all the more the map i is weakly continuous. The map $j\colon \mu Y_1 \to Y_0$, defined as the inclusion of the set μY_1 into Y_0, is weakly continuous. But the composition $j \circ i\colon *N \to Y_0$ is not weakly continuous since $j \circ i$ is defective at $*$.

The question arises whether there exists a greatest category of weakly continuous maps. In the sequel such a category will be defined.

A map $f\colon X \to Y$ is called an *s-map* whenever the map sf in the diagram

$$(\mathrm{s})\qquad \begin{array}{ccc} sX & \xrightarrow{\;sf\;} & sY \\ {\scriptstyle s_X}\uparrow & & \uparrow{\scriptstyle s_Y} \\ X & \xrightarrow[\;f\;]{} & Y \end{array}$$

is continuous.

This formal definition can easily be translated into the following $\varepsilon-\delta$ description: a map $f\colon X \to Y$ is an s-map iff for each $x \in X$ and each open neighbourhood U_y of $y = f(x)$ and each closed nowhere-dense set $A \subset Y$ there exists an open neighbourhood U_x of x and a closed nowhere-dense set $A' \subset X$ such that $f(\mathrm{Cl}\,U_x) \subset \mathrm{Cl}\,U_y$ and $f(U_x - A') \subset U_y - A$.

Each s-map is weakly continuous by 3.1 and s-maps form a category abbreviated by $\boldsymbol{S}$, of weakly continuous maps. This category contains all standard expansions since $s(sX) = sX$. The importance of s-maps, from the categorial viewpoint, is established by

THEOREM 3.2. *A category $\boldsymbol{K}$ of topological spaces and their weakly continuous maps, containing with each object X the standard expansion s_X, is a subcategory of $\boldsymbol{S}$.*

Proof. Let $f\colon X \to Y$ be a map from $\boldsymbol{K}$. Then the map $s_Y \circ f$ of diagram (s) is a map from $\boldsymbol{K}$; thus it is weakly continuous. In view of 3.1 it follows that $\overline{s_Y \circ f} = sf$ is continuous and so f is proved to be an s-map and thus $\boldsymbol{K} \subset \boldsymbol{S}$.

In other words, $\boldsymbol{S}$ is the greatest category consisting of weakly continuous maps. It is remarkable that a similar result is true for the class of all non-defective maps.

THEOREM 3.3. *Let $\boldsymbol{K}$ be a category consisting of topological spaces and their non-defective θ-continuous maps and containing the category $\boldsymbol{S}$. Then $\boldsymbol{K} = \boldsymbol{S}$.*

Proof. Suppose $f\colon X \to Y$ is a map from $\boldsymbol{K}$, which is not in $\boldsymbol{S}$. Then there exist a point $x \in X$ and a closed nowhere-dense set A in Y such that $A \subset \mathrm{Cl}\,U_y$ for some open neighbourhood U_y of $y = f(x)$ and

$$(28)\qquad U_x \cap \mathrm{Int}\,\mathrm{Cl} f^{-1}(A - \{y\}) \neq \emptyset \quad \text{for each } U_x \in \mathfrak{U}_x .$$

Let X' be the space obtained from X by adding sets of the form $\{x\} \cup U_x \cap \mathrm{Int}\,\mathrm{Cl} f^{-1}(A - \{y\}) \cap f^{-1}(A - \{y\})$ as open sets for each $U_x \in \mathfrak{U}_x$. Denote by i' the set-theoretical identity $X' \to X$. Then for each basic open set of X' of the form $U' = U \cap \big(\{x\} \cup U_x \cap \mathrm{Int}\,\mathrm{Cl} f^{-1}(A - \{y\}) \cap f^{-1}(A - \{y\})\big)$, where U is an arbitrary open set of X, one gets:

$$(29)\qquad \begin{aligned} \mathrm{Int}\,\mathrm{Cl}\, i'(U') &\supset \mathrm{Int}\,\mathrm{Cl}\big(U \cap U_x \cap \mathrm{Int}\,\mathrm{Cl} f^{-1}(A - \{y\}) \cap f^{-1}(A - \{y\})\big) \\ &\supset \mathrm{Int}\big(U \cap U_x \cap \mathrm{Int}\,\mathrm{Cl} f^{-1}(A - \{y\}) \cap \mathrm{Cl} f^{-1}(A - \{y\})\big) \\ &= U \cap U_x \cap \mathrm{Int}\,\mathrm{Cl} f^{-1}(A - \{y\}) , \end{aligned}$$

the second inclusion being true since $f^{-1}(A - \{y\})$ is dense in $U \cap U_x \cap \mathrm{Int}\,\mathrm{Cl} f^{-1}(A - \{y\})$. Suppose $U' \neq \emptyset$. When $x \in U'$, it follows from the definition of U' that $x \in U$; thus (29) yields $\mathrm{Int}\,\mathrm{Cl}\, i'(U') \supset V_x \cap \mathrm{Int}\,\mathrm{Cl} f^{-1}(A - \{y\})$ for $V_x = U \cap U_x$, whence on account of (28)

$$(30)\qquad \mathrm{Int}\,\mathrm{Cl}\, i'(U') \neq \emptyset .$$

In the case where $x \notin U'$, it follows from the definition of U' and (29) that (30) is true. Since (30) is true for each non-void open set U' of X', it follows that $\mathrm{Int}\, i'^{-1}(A') = \emptyset$ for each closed nowhere-dense $A' \subset X$ and, in consequence, i' is an s-map, since, in addition, it is continuous. Then, since s_Y is also a map from $\boldsymbol{S}$ and $\boldsymbol{S} \subset \boldsymbol{K}$ by assumption, the map $X' \xrightarrow{i'} X \xrightarrow{f} Y \xrightarrow{s_Y} sY$ is a map from $\boldsymbol{K}$. But for the (open in sY) neighbourhood $U'_y = \{y\} \cup (U_y - A)$ of the point y and the (open in X') neighbourhood $U'_x = \{x\} \cup U_x \cap \mathrm{Int}\,\mathrm{Cl} f^{-1}(A - \{y\}) \cap f^{-1}(A - \{y\})$ of X, where U_x is chosen to satisfy the formula $f(\mathrm{Cl}\,U_x) \subset \mathrm{Cl}\,U_y$, there is

$$(s_Y \circ f \circ i')(U'_x - \{x\}) \subset A - \{y\} \subset \mathrm{Cl}\,U'_y - U'_y$$

and so $\boldsymbol{K}$ contains a defective θ-continuous map contrary to the assumption and the theorem is proved.

IV. Weakly continuous extensions of maps

1. A Tajmanov-type theorem. An extension $*X$ of X is called a *Katětov-type extension* (¹) whenever

1° X is an open set in $*X$,

2° $*X - X$ is discrete in the induced topology,

3° each nowhere-dense closed subset of X is closed in $*X$.

Each extension $*X$ of X can be modified to a Katětov-type extension $k{*X}$ by adding to the topology of $*X$ the set X and all sets of the form $\{x\} \cup (U_x \cap X - A)$, where $x \in *X - X$ and $U_x \in \mathfrak{U}_x$ and A is a closed nowhere-dense subset of X. Then $*X$ is θ-homeomorphic to $k{*X}$, since

(¹) The notion of a Katětov-type extension defined in [9] is equivalent to that given here. However, the characterization of a Katětov-type extension in [9] by means of properties 1° and 2° only is not complete.

the identity map $*X \to k*X$ is θ-continuous. Therefore in the sequel only Katětov-type extensions are considered. Henceforth $*$-free maps are understood to be $*$-free with respect to all closed sets.

For use in the sequel first note that

THEOREM 1.1. *If $f\colon X \to Y$ is a weakly continuous map, then*

1° $f(\operatorname{Cl}\operatorname{Int}\operatorname{Cl}f^{-1}(A)) \subset \operatorname{Cl}A$ *for each* $A \subset Y$, *and*

2° $f(\operatorname{Cl}U) \subset \operatorname{Cl}f(U)$ *for each open* $U \subset X$.

Proof. Suppose $y = f(x) \notin \operatorname{Cl}A$. For $U_y = Y - \operatorname{Cl}A$ take U_x, on account of weak continuity, in such a way that $f(U_x - A') \subset U_y$ for some closed nowhere-dense $A' \subset X$. Then $f(U_x - A') \cap \operatorname{Cl}A = \emptyset$ and all the more $(U_x - A') \cap f^{-1}(A) = \emptyset$. But $U_x - A'$ being open, it follows that $(U_x - A') \cap \operatorname{Cl}f^{-1}(A) = \emptyset$; thus $\operatorname{Cl}(U_x - A') \cap \operatorname{Int}\operatorname{Cl}f^{-1}(A) = \emptyset$ and since $\operatorname{Cl}(U_x - A') = \operatorname{Cl}U_x$, the set A' being nowhere-dense, it follows that $U_x \cap \operatorname{Int}\operatorname{Cl}f^{-1}(A) = \emptyset$. Thus $x \notin \operatorname{Cl}\operatorname{Int}\operatorname{Cl}f^{-1}(A)$ when $f(x) \notin \operatorname{Cl}A$ and 1° is proved. To prove 2° observe that $\operatorname{Cl}U = \operatorname{Cl}\operatorname{Int}\operatorname{Cl}U \subset \operatorname{Cl}\operatorname{Int}\operatorname{Cl}f^{-1}(f(U))$ and apply 1° for $A = f(U)$.

THEOREM 1.2. *Let $*X$ be a Katětov-type extension of X. Then a weakly continuous map $f\colon X \to Y$ possesses a weakly continuous extension $*f\colon *X \to Y$ iff f is $*$-proper. The extension of f is unique when Y is a Hausdorff space.*

Proof of $\Rightarrow$. Suppose $*f$ is a weakly continuous extension of f on $*X$. To prove that f is $*$-proper it suffices, in view of 1.1. I, to show that f is $*$-free with respect to all closed sets. Accordingly, let $x \in *X - X$ and suppose that $x \in \bigcap\{\operatorname{Cl}_* f^{-1}(\operatorname{Cl}U_y)\colon U_y \in \mathfrak{U}_y\}$ for some $y \in Y$. Then

(31) $$*f(x) = y\,.$$

To prove this observe that

$$\operatorname{Cl}_* f^{-1}(\operatorname{Cl}U_y) = \operatorname{Cl}_* \operatorname{Cl}f^{-1}(\operatorname{Cl}U_y) = \operatorname{Cl}_*\big(\operatorname{Int}f^{-1}(\operatorname{Cl}U_y) \cup A\big)\,,$$

where $A = \operatorname{Cl}f^{-1}(\operatorname{Cl}U_y) - \operatorname{Int}f^{-1}(\operatorname{Cl}U_y)$ is a closed nowhere-dense subset of X; thus

(32) $$(*X - X) \cap \operatorname{Cl}_* f^{-1}(\operatorname{Cl}U_y) = (*X - X) \cap \operatorname{Cl}_* \operatorname{Int}f^{-1}(\operatorname{Cl}U_y)$$

since $\operatorname{Cl}_* A = A \subset X$ holds for the Katětov-type extension $*X$. But, by assumption, $x \in (*X - X) \cap \operatorname{Cl}_* f^{-1}(\operatorname{Cl}U_y)$ for each $U_y \in \mathfrak{U}_y$; thus (32) yields $x \in \operatorname{Cl}_* \operatorname{Int}f^{-1}(\operatorname{Cl}U_y)$ for each $U_y \in \mathfrak{U}_y$, and all the more

(33) $$x \in \operatorname{Cl}_* \operatorname{Int}_* {*f}^{-1}(\operatorname{Cl}U_y) \quad \text{for each } U_y \in \mathfrak{U}_y\,.$$

Now (33) implies, on account of formula 1° from 1.1, that

$$*f(x) \in \bigcap\{\operatorname{Cl}U_y\colon U_y \in \mathfrak{U}_y\} = \{y\}$$

and so (31) is proved.

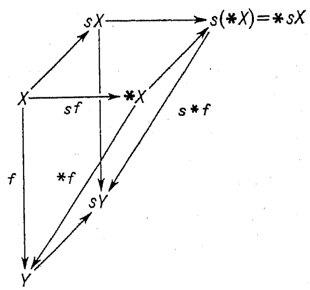
Let A be a closed subset of Y not containing the point y. For the open neighbourhood $U_y = Y - A$ of y there exist an open neighbourhood U_x of x and a nowhere-dense closed set A' in $*X$ such that $*f(U_x - A') \subset U_y$, since $*f$ is weakly continuous by assumption. From 2° of the definition of a Katětov-type extension it follows that there exists a neighbourhood U'_x such that $U'_x \cap (*X - X) = \{x\}$, and since in view of 1° $A' \cap X$ is nowhere-dense and closed in X, a neighbourhood U''_x can be chosen by 3° in such a way that $U''_x \cap A' \cap X = \emptyset$. Take $V_x = U_x \cap U'_x \cap U''_x$. Then

$$\begin{aligned} *f(V_x) &= *f(V_x \cap (*X - X) \cup V_x \cap X) \subset *f(x) \cup f(U_x \cap U''_x \cap X) \\ &\subset \{y\} \cup f(U_x - A') \subset U_y = Y - A\,, \end{aligned}$$

so $V_x \cap {*f}^{-1}(A) = \emptyset$ and all the more $V_x \cap f^{-1}(A) = \emptyset$, which implies that $x \notin \operatorname{Cl}_* f^{-1}(A)$, and so f is proved to be $*$-free with respect to all closed sets.

Proof of $\Leftarrow$. Since f is assumed to be $*$-proper, there exists by 3.1. II a (unique, if Y is a Hausdorff space) θ-continuous extension $*f\colon *X \to Y$. To prove the theorem, it suffices to check that $*f$ is even weakly continuous. This is easy to see for $x \in X$, since f is assumed to be weakly continuous and $*X$ is a Katětov-type extension. Given a point $x \in *X - X$ and an open neighbourhood U_y of $y = *f(x)$, take the neighbourhood U_x of x for which $*f(\operatorname{Cl}U_x) \subset \operatorname{Cl}U_y$ and $f(U_x \cap X) \subset U_y$, which exists, as pointed out in the remark following the proof of 3.1.II, since f is $*$-free with respect to all closed subsets. Then $U'_x = \{x\} \cup (U_x - (*X - X))$ is an open neighbourhood of x, since $*X$ is a Katětov-type extension, and $*f(U'_x) = *f(x) \cup f(U_x \cap X) \subset U_y$ and so $*f$ is proved to be weakly continuous (in fact, even continuous) at points of $*X - X$ also, and this ends the proof.

Remark. Observe that the proved theorem remains true when the weak continuity of f and $*f$ is replaced by the assumption on both to be s-maps. To see this, note that for each Katětov-type extension $*X$, the space $s(*X)$ is homeomorphic with the space $*sX$ obtained from $*X$ by adding all open sets of sX as open sets to the topology of $*X$. Now, the map $s*f$ in the diagram



is continuous iff sf is continuous, the "if" implication following from the proof of the last theorem, the "only if" being trivial. Thus f is an s-map iff $*f$ is an s-map.

2. The case of τX. The Katětov extension τX of a Hausdorff space X is known from [9] to be an H-closed Katětov-type extension characterized by the following property: for each pair of disjoint open sets U and U' of X there is $(\mathrm{Cl}_\tau U \cap \mathrm{Cl}_\tau U') \cap (\tau X - X) = \emptyset$. The last property implies that for each point $x \in \tau X - X$, if $x \in \mathrm{Cl}_\tau U$ for U being an open set of X, then $\{x\} \cup U$ is an open neighbourhood of x in τX. Use Theorem 1.1 to give a description of maps possessing a weakly continuous τ-extension.

THEOREM 2.1. *For a weakly continuous map $f\colon X \to Y$ of a Hausdorff space X into an H-closed space Y the following conditions are equivalent*:

(i) *f is τ-free,*

(ii) *$\bigcap \{\mathrm{Cl} f(U_x \cap X)\colon U_x \in \mathfrak{U}_x\}$ is a one-point set for each $x \in \tau X - X$,*

(iii) *for each $y \in Y$ and each open neighbourhood U_y of y there exists an open neighbourhood V_y of y such that $\mathrm{Int}\,\mathrm{Cl} f^{-1}(\mathrm{Cl} V_y) \subset \mathrm{Cl}\,\mathrm{Int} f^{-1}(U_y)$.*

Proof of (i) ⇒ ii. Suppose f is τ-free. From 1.2 it follows that there exists a weakly continuous $\tau f\colon \tau X \to Y$. Now since $\mathrm{Cl}_\tau(U \cap X) = \mathrm{Cl}_\tau U$ for each open $U \subset \tau X$ and $U \cap X$ is open in τX, it follows from 2° of 1.1 that for each $x \in \tau X - X$ there is

$$\begin{aligned}\tau f(x) &= \tau f\big(\bigcap \{\mathrm{Cl}\, U_x\colon U_x \in \mathfrak{U}_x\}\big)\\ &\subset \bigcap \{\tau f\big(\mathrm{Cl}(U_x \cap X)\big)\colon U_x \in \mathfrak{U}_x\}\\ &\subset \bigcap \{\mathrm{Cl} f(U_x \cap X)\colon U_x \in \mathfrak{U}_x\}.\end{aligned}$$

Then 2.3.II implies (ii) since

$$\bigcap \{\mathrm{Cl} f(U_x \cap X)\colon U_x \in \mathfrak{U}_x\} \subset \bigcap \{\mathrm{Cl}\, U\colon U \in \mathfrak{U}(f, \mathfrak{U}_x)\}$$

and the latter set consists of one point.

Proof of (ii) ⇒ (iii). Suppose, on the contrary, that there exist a point $y \in Y$ and an open neighbourhood U_y such that for each V_y the set $U(V_y) = \mathrm{Int}\,\mathrm{Cl} f^{-1}(\mathrm{Cl} V_y) - \mathrm{Cl}\,\mathrm{Int} f^{-1}(U_y)$ is non-empty. It is easy to check that $\{U(V_y)\colon V_y \in \mathfrak{U}_y\}$ is a centred family consisting of open sets of X. Thus there exists an adherence point $x \in \bigcap \{\mathrm{Cl}_\tau U(V_y)\colon V_y \in \mathfrak{U}_y\}$. Since

$$\begin{aligned}\mathrm{Cl}\, U(V_y) &\subset \mathrm{Cl}\,\mathrm{Int}\,\mathrm{Cl} f^{-1}(\mathrm{Cl} V_y) - \mathrm{Int}\,\mathrm{Cl}\,\mathrm{Int} f^{-1}(U_y)\\ &= \mathrm{Cl}\,\mathrm{Int}\,\mathrm{Cl} f^{-1}(\mathrm{Cl} V_y) \cap \mathrm{Cl}\,\mathrm{Int}\,\mathrm{Cl} f^{-1}(Y - U_y)\end{aligned}$$

it follows on account of formula 1° from 1.1 that

$$\begin{aligned}f\big(\bigcap \{X \cap \mathrm{Cl}_\tau U(V_y)\colon V_y \in \mathfrak{U}_y\}\big) &\subset \bigcap \{f\big(\mathrm{Cl}\, U(V_y)\big)\colon V_y \in \mathfrak{U}_y\}\\ &\subset \bigcap \{f\big(\mathrm{Cl}\,\mathrm{Int}\,\mathrm{Cl} f^{-1}(\mathrm{Cl} V_y)\big)\colon V_y \in \mathfrak{U}_y\} \cap f\big(\mathrm{Cl}\,\mathrm{Int}\,\mathrm{Cl} f^{-1}(Y - U_y)\big)\\ &\subset \bigcap \{\mathrm{Cl} V_y\colon V_y \in \mathfrak{U}_y\} \cap (Y - U_y) = \{y\} \cap (Y - U_y) = \emptyset;\end{aligned}$$

thus the adherence point x is an element of $\tau X - X$. Since $x \in \mathrm{Cl}_\tau U(V_y)$ for each $V_y \in \mathfrak{U}_y$, it follows in consequence of the structure of τX that $\{x\} \cup U(V_y)$ is an open neighbourhood of x for each $U(V_y)$. Since

$$\bigcap \{\mathrm{Cl} f(U_x \cap X)\colon U_x \in \mathfrak{U}_x\} \subset \bigcap \{\mathrm{Cl} f\big(U(V_y)\big)\colon V_y \in \mathfrak{U}_y\}$$

and

$$f\big(U(V_y)\big) \subset \mathrm{Cl} V_y - U_y,$$

it follows that

$$\bigcap \{\mathrm{Cl} f(U_x \cap X)\colon U_x \in \mathfrak{U}_x\} \subset \bigcap \{\mathrm{Cl} V_y - U_y\colon V_y \in \mathfrak{U}_y\} = \emptyset,$$

contrary to (ii). Thus f not satisfying (iii) leads to a contradiction.

Proof of (iii) ⇒ (i). To prove that f is τ-free let y be a point of Y not belonging to a closed set $A \subset Y$, and suppose that $x \in \bigcap \{\mathrm{Cl}_\tau f^{-1}(\mathrm{Cl}\, U_y)\colon U_y \in \mathfrak{U}_y\}$ for some $x \in \tau X - X$. It remains to prove

$$x \notin \mathrm{Cl}_\tau f^{-1}(A). \tag{34}$$

Take $U_y = Y - A$. Then for V_y chosen according to (iii) we have

$$x \in (\tau X - X) \cap \mathrm{Cl}_\tau f^{-1}(\mathrm{Cl} V_y) = (\tau X - X) \cap \mathrm{Cl}_\tau \mathrm{Int} f^{-1}(U_y),$$

the equality being a consequence of the choice of V_y: the set $\mathrm{Cl}_\tau f^{-1}(\mathrm{Cl} V_y)$ differs from $\mathrm{Int} f^{-1}(U_y)$ by a nowhere-dense closed set, and of the properties of τX. Thus $U_x = \{x\} \cup \mathrm{Int} f^{-1}(U_y)$ is an open neighbourhood of x such that $U_x \cap f^{-1}(A) = \mathrm{Int} f^{-1}(Uy) \cap f^{-1}(Y - U_y) = \emptyset$ and (34) is proved.

Remark. In virtue of 1.2 the property of $f\colon X \to Y$ to be τ-free is equivalent to the existence of a weakly continuous τ-extension when f is weakly continuous and Y is H-closed. So the above theorem is the announced solution of the question how to extend weakly continuous maps on τX. In the case where f is a continuous map, the extension $*f$ described in 1.2 becomes even continuous — a consequence of the structure of a Katětov-type extension. For continuous f, conditions (ii) and (iii) have been proved to be equivalent with the existence of a continuous extension τf in [8] and [9], respectively. Theorem 2.1 is a common generalization of those results.

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A short proof of Hausdorff's theorem on extending metrics

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The aim of this paper is to give a short proof of the following theorem:

THEOREM. *Let X be a metrizable topological space and let $A\subset X$ be a closed set. Then, for every metric ϱ on A which induces the relative topology on A, there is a metric $\bar{\varrho}$ on X which is an extension of ϱ and is compatible with the topology of X. If moreover X is complete-metrizable and ϱ is a complete metric on A, then the extension $\bar{\varrho}$ above can be obtained to be a complete metric on X.*

The first part of the theorem was proved by F. Hausdorff [5] in 1930 (cf. also [6]) and independently by R. H. Bing ([3], Theorem 5) in 1947, whereas the remark concerning the "complete" case was made by P. Bacon [2] in 1968. Let us note that R. Arens ([1], Theorem 3.3) gave in 1952 a relatively simple proof of Hausdorff's theorem; his arguments were based on a close examination of "Dugundji's retraction".

The proof we are going to present involves (besides other well-known facts) the use of the following lemma of V. L. Klee ([7], pp. 36).

LEMMA. *Let E and F be normed linear spaces and let $K\subset E\times\{0\}$ and $L\subset\{0\}\times F$ be closed subsets of $E\times F$. Then, for every homeomorphism $f\colon K\xrightarrow{\text{onto}}L$, there is an extension of f to a homeomorphism $\bar{f}\colon E\times F\xrightarrow{\text{onto}}E\times F$.*

Proof. Denote by p_E and p_F the natural projections of $E\times F$ onto E and F, respectively. Since F is an ANR($\mathfrak{M}$) ([4], Theorem 4.1), the function $p_F\circ f\colon K\to F$ can be extended to $\lambda\colon E\times\{0\}\to F$. We put $f_1(\alpha,\beta)=\big(\alpha,\beta+\lambda(\alpha,0)\big)$, $(\alpha,\beta)\in E\times F$; f_1 is then a homeomorphism of $E\times F$ onto itself satisfying $f_1(\alpha,\beta)=\big(\alpha,p_F\circ f(\alpha,\beta)\big)$ for $(\alpha,\beta)\in K$. Similarly, there is a homeomorphism $f_2\colon E\times F\xrightarrow{\text{onto}}E\times F$ such that $f_2(\alpha,\beta)=\big(p_E\circ f^{-1}(\alpha,\beta),\beta\big)$ for $(\alpha,\beta)\in L$. We then have $f_2\circ f(\alpha,\beta)=f_1(\alpha,\beta)$ for $(\alpha,\beta)\in K$, whence $\bar{f}=f_2^{-1}\circ f_1$ is the desired extension of f.

Now we pass to the proof of the theorem; Bacon's remark will be considered in brackets.