6.6. COROLLARY. *If* $(\mathcal{X}, x_0)$, $(\mathcal{Y}, y_0)$ *are uniformly movable pointed compact Hausdorff spaces, and* $(\boldsymbol{X}, \boldsymbol{x}_0)$, $(\boldsymbol{Y}, \boldsymbol{y}_0)$ — *the associated* ANR-*systems, then for any map* $\boldsymbol{f}\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$

(1) f_n^* *is a monomorphism in* $\mathcal{G} \Rightarrow \boldsymbol{f}_n$ *is a monomorphism in* $\hat{\mathcal{G}}^*$,

(2) f_n^* *is an epimorphism in* $\mathcal{G} \Rightarrow \boldsymbol{f}_n$ *is an epimorphism in* $\hat{\mathcal{G}}^*$,

(3) f_n^* *is a bimorphism in* $\mathcal{G} \Rightarrow \boldsymbol{f}_n$ *is a bimorphism in* $\hat{\mathcal{G}}^*$.

6.7. Remark. When the paper was in press, the question 5.5 was answered by S. Spież [8]. He proved that every movable compactum is uniformly movable.

References

[1] K. Borsuk, *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968), pp. 223–254.

[2] — *On movable compacta*, Fund. Math. 66 (1969), pp. 137–146.

[3] S. Mardešić and J. Segal, *Shapes of compacta and* ANR-*systems*, Fund. Math. 72 (1971), pp. 41–59.

[4] — and — *Equivalence of the Borsuk and the* ANR-*system approach to shapes*, Fund. Math. 72 (1971), pp. 61–68.

[5] — and — *Movable compacta and* ANR-*systems*, Bull. Ac. Pol. Sc. 18 (1970), pp. 649–654.

[6] M. Moszyńska, *On shape and fundamental deformation retracts I*, Fund. Math. 75 (1972), pp. 145–167.

[7] H. Schubert, *Kategorien*, Berlin–Heidelberg–New York 1970.

[8] S. Spież, *Movability and uniform movability*, to appear in Colloq. Math.

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An atomic map onto an arbitrary metric continuum

by

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A continuum means in this paper a compact connected Hausdorff space. A continuous map $f\colon X \xrightarrow{\text{onto}} Y$ is said to be *atomic* if for each subcontinuum K of X such that $f(K)$ is non-degenerate we have $f^{-1}(f(K)) = K$. The notion of an atomic (continuous and open) map was originally introduced by Anderson [2] and was applied by Anderson and Choquet [3], and then by Cook [4], to the constructions of some singular continua. In 1966 Mahavier [8] and Thomas [10] showed independently that there is no atomic map from an irreducible, metric continuum onto an arc such that the preimage of each point is a non-degenerate, hereditarily decomposable, chainable continuum. In 1970 Mahavier [9] showed that if K is a metric continuum, then there is an atomic map from a separable, first countable, irreducible continuum onto an arc such that the preimage of each point is homeomorphic to K. In this note we show that if X is a metric continuum and K_x, $x \in X$, are metric continua, then there is an atomic map f from a separable, first countable Hausdorff continuum onto X such that the preimage under f of any point x of X is homeomorphic to K_x. If, in addition, X is irreducible, then the continuum in question proves to be irreducible, and so the construction given here is a generalization of that of Mahavier. A similar construction is given also in a paper of Fedorčuk [6], who applied it to the proof of the existence of a compact Hausdorff space having the dimension dim less than the dimension ind. However, Fedorčuk's construction is incomparable with that of the present paper: although it satisfies some special conditions, the map is not atomic, and X and K_x, for $x \in X$, are rather special spaces, such as an n-sphere or an n-torus, and are locally connected continua in the most general case.

Let X be an arbitrary metric continuum. For each $x \in X$, let M_x be a metric continuum and let $T_x\colon M_x \xrightarrow{\text{onto}} X$ be a continuous map. Let $S = \bigcup \{\{x\} \times T_x^{-1}(x)\colon x \in X\}$. For each $x \in X$ and an open subset U of M_x which intersects $T_x^{-1}(x)$, let $R(x, U)$ denote the subset of S to which

(t, P) belongs iff either $t = x$ and P is in $U \cap T_x^{-1}(x)$ or P is in $T_t^{-1}(t)$ and $T_x^{-1}(t) \subset U$. The collection of all such subsets of S generates a topology in S. Let π denote a map (projection) of S onto X such that $\pi^{-1}(x) = T_x^{-1}(x)$ for each $x \in X$. Let d be a metric on X and let $K(t, \varepsilon) = \{t' \in X \colon d(t, t') < \varepsilon\}$.

LEMMA 1. *If (t, P) is a point of S and $\varepsilon > 0$, then there is an $R(t, U)$ such that $(t, P) \in R(t, U) \subset \pi^{-1}(K(t, \varepsilon))$.*

Proof. If $U = T_t^{-1}(K(t, \varepsilon))$, then from the definition we have $(t, P) \in R(t, U) \subset \pi^{-1}(K(t, \varepsilon))$.

LEMMA 2. *If $(t, P) \in R(x, U)$ and $x \neq t$, then there is an $\varepsilon > 0$ such that $\pi^{-1}(K(t, \varepsilon)) \subset R(x, U)$.*

Proof. Let $t \neq x$ and $(t, P) \in R(x, U)$. Then $T_x^{-1}(t) \subset U$ and the set $V = \{t' \in X \colon T_x^{-1}(t') \subset U\}$ is non-void and open (since the map T_x is closed, M_x being compact, X Hausdorff and T_x continuous). So there is an $\varepsilon > 0$ such that $T_x^{-1}(K(t, \varepsilon)) \subset U$. From the definition of $R(x, U)$ it follows that $\pi^{-1}(K(t, \varepsilon)) \subset R(x, U)$.

COROLLARY 1. *The map π is continuous.*

COROLLARY 2. *The collection of all subset of S of the form $R(x, U)$ is a basis for the topology in S.*

Proof. It suffices to prove that if the point (t, P) is in both $R(x, U)$ and $R(y, V)$, then there is an $R(z, W)$ containing (t, P) and lying in $R(x, U) \cap R(y, V)$. If $x \neq t$ and $y \neq t$, then from Lemma 2 we infer that there is an $\varepsilon > 0$ such that $\pi^{-1}(K(t, \varepsilon)) \subset R(x, U) \cap R(y, V)$. By Lemma 1, there is an $R(a, W)$ containing (t, P) and lying in $\pi^{-1}(K(t, \varepsilon))$. If $x = t$ and $y = t$, then $(t, P) \in R(t, U \cap V) \subset R(t, U) \cap R(t, V)$. If $x \neq t$ and $y = t$, then from Lemma 2 we infer that there is an $\varepsilon > 0$ such that $\pi^{-1}(K(t, \varepsilon)) \subset R(x, U)$. By Lemma 1, there is an $R(t, W)$ such that $(t, P) \in R(t, W) \subset \pi^{-1}(K(t, \varepsilon))$. This implies that $(t, P) \subset R(t, W \cap V) \subset R(x, U) \cap R(y, V)$.

THEOREM 1. *S is a Hausdorff space.*

Proof. Let (a, P) and (b, Q) be two points of S. Suppose $a \neq b$. Let $\varepsilon = \frac{1}{3} d(a, b)$. There are, by Lemma 1, R_1 and R_2 such that $(a, P) \in R_1 \subset \pi^{-1}(K(a, \varepsilon))$ and $(b, Q) \in R_2 \subset \pi^{-1}(K(b, \varepsilon))$, and thus $R_1 \cap R_2 = 0$. If $a = b$, then $P \neq Q$ and there are mutually disjoint open subsets U and V containing P and Q, respectively. Then $R(a, U) \cap R(a, V) = 0$.

THEOREM 2. *S is a first countable space.*

Proof. Let (a, P) denote a point of S and let $\{U_i \colon i = 1, 2, \ldots\}$ be a countable base in M_a at the point P. Suppose (a, P) is in $R(x, V)$. If $a \neq x$, then there is an $\varepsilon > 0$ such that $\pi^{-1}(K(a, \varepsilon)) \subset R(x, V)$ and an $n > 0$ such that $U_n \subset T_a^{-1}(K(a, \varepsilon))$. This implies that $R(a, U_n) \subset R(x, V)$. If $a = x$, then P is in V. Hence there is an $n > 0$ such that $U_n \subset V$. Then $R(a, U_n) \subset R(a, V)$. Thus $\{R(a, U_i) \colon i = 1, 2, \ldots\}$ is a countable base in S at (a, P).

THEOREM 3. *If, for each $x \in X$, $T_x^{-1}(t)$ are one-point sets for $t \neq x$, then S is a compact space.*

Proof. Let G be a covering of S consisting of sets from the basis. We first show that for each $x \in X$ we have $\varepsilon_x > 0$ and a finite subfamily of G which covers $\pi^{-1}(K(x, \varepsilon_x))$. Let $x \in X$. If there are a point P in $T_x^{-1}(x)$, $t \neq x$, and $R(t, U)$ in G containing (x, P), then, by Lemma 2, there is an $\varepsilon_x > 0$ such that $\pi^{-1}(K(x, \varepsilon_x)) \subset R(t, U)$. Otherwise, for each point P in $T_x^{-1}(x)$, there is an open subset U_P of M_x such that $R(x, U_P)$ is an element of G containing (x, P). Since $T_x^{-1}(x)$ is compact, there is a finite subset H of $T_x^{-1}(x)$ such that $T_x^{-1}(x) \subset \bigcup \{U_P \colon P \in H\}$. There is an $\varepsilon_x > 0$ such that $T_x^{-1}(K(x, \varepsilon_x)) \subset \bigcup \{U_P \colon P \in H\}$. If $t \in K(x, \varepsilon_x)$, then $T_x^{-1}(t) \subset U_P$ for some point P in H. Hence $\pi^{-1}(t) \subset R(x, U_P)$, and therefore $\pi^{-1}(K(x, \varepsilon_x)) \subset \bigcup \{R(x, U_P) \colon P \in H\}$. Note that X is compact, and therefore a finite family of $K(x, \varepsilon_x)$ covers X. But we have proved that each $\pi^{-1}(K(x, \varepsilon_x))$ can be covered by a finite subfamily of G. This leads to the compactness of S.

THEOREM 4. *If, for each $x \in X$, $T_x^{-1}(x)$ is connected and $T_x^{-1}(t)$ are one-point sets for $t \neq x$, then S is connected.*

Proof. Since $\pi^{-1}(x)$ is homeomorphic to $T_x^{-1}(x)$ for each $x \in X$, the map π is monotone. It is known (e.g. from Kuratowski's book [7], p. 123) that a continuous map f from a compact Hausdorff space M onto a Hausdorff space N is monotone iff the preimage under f of any subcontinuum of N is connected. Since π is a continuous map, S is compact (in virtue of Theorem 3), X is a connected Hausdorff space and $S = \pi^{-1}(X)$, S is connected.

LEMMA 3. *If $T_x^{-1}(t)$ are one-point sets for $t \neq x$, $H \subset X - \{x\}$ and $P \in T_x^{-1}(x)$ is a limit point of $T_x^{-1}(H)$, then if $(x, P) \in R(t, U)$, then there is an $a \in H$ such that $\pi^{-1}(a) \subset R(t, U)$.*

Proof. If $t \neq x$, then, by Lemma 2, there is an $\varepsilon > 0$ such that $\pi^{-1}(K(x, \varepsilon)) \subset R(t, U)$. By hypothesis, there is an $a \in H$ such that $a \in K(x, \varepsilon)$. Hence $\pi^{-1}(a) \subset \pi^{-1}(K(x, \varepsilon)) \subset R(t, U)$. If $t = x$, then P is in U. By hypothesis, there is an $a \in H$ such that $T_x^{-1}(a) \subset U$, whence $\pi^{-1}(a) \subset R(t, U)$.

THEOREM 5. *If for $x \in X$, $T_x^{-1}(x)$ are sets with a void interior in M_x and $T_x^{-1}(t)$ are one-point sets for $t \neq x$, then S is separable.*

Proof. Let $\{x_i \colon i = 1, 2, \ldots\}$ be a countable dense subset of X. Let us choose P_i in each $T_{x_i}^{-1}(x_i)$. We show that each set $\{(x_i, P_i) \colon i = 1, 2, \ldots\}$ is a dense subset of S. Let $R(x, U)$ be given. Since $U \cap T_x^{-1}(x) \neq 0$, let $Q \in U \cap T_x^{-1}(x)$. By hypothesis, Q is a limit point of $T_x^{-1}(X - \{x\})$. Hence, by Lemma 3, there is an $a \in X - \{x\}$ such that $\pi^{-1}(a) \subset R(x, U)$. By

Lemma 2, there is an $\varepsilon > 0$ such that $\pi^{-1}(K(a,\varepsilon)) \subset R(x, U)$. But there is an x_i such that $x_i \in K(a,\varepsilon)$. Hence $(x_i, P_i) \in R(x, U)$. Note that the axiom of choice has been used in the proof.

THEOREM 6. *If for $x \in X$, $T_x^{-1}(x)$ are sets with a void interior in M_x and $T_x^{-1}(t)$ are one-point sets for $t \neq x$, then the map π is irreducible.*

Proof. In the proof of the preceding theorem it was shown that for each set $R(t, U)$ there are $a \in X$ and $\varepsilon > 0$ such that $\pi^{-1}(K(a,\varepsilon)) \subset R(t, U)$. This implies that there exist no closed subsets Z of S different from S and such that $\pi(Z) = X$. This means that π is irreducible.

Note. If, in addition, X is irreducible, than S is irreducible, in virtue of the irreducibility of π.

LEMMA 4. *Let f be an atomic map from a continuum X onto a non-degenerate continuum Y. If K is a subcontinuum of X such that $f(K)$ is non-degenerate, then for each $y \in f(K)$ the set $f^{-1}(y)$ has a void interior in K.*

Proof. It is shown in [5] that if f is an atomic map from a continuum X onto a non-degenerate Y, then a preimage under f of any point of Y is the set with a void interior in X. It is easy to check that the partial map $f|f^{-1}(f(K))$, where $f(K)$ is non-degenerate, is atomic if f is atomic. Hence the previous conclusion on $f^{-1}(y)$ is true for $f|K$, K and $f(K)$ instead of f, X and Y. This ends the proof.

THEOREM 7. *If, for $x \in X$, $T_x^{-1}(x)$ is connected, T_x is atomic and $T_x^{-1}(t)$ are one-point sets for $t \neq x$, then π is atomic.*

Proof. Let C be a subcontinuum of S such that $\pi(C)$ is non-degenerate. Suppose that $(t, P) \in \pi^{-1}(\pi(C))$. Then $t \in \pi(C)$. Since each atomic map is monotone (see a note by the present author and Horbanowicz [5]), we have $T_t^{-1}(\pi(C))$ is a subcontinuum of M_t and, by Lemma 4, we infer that $T_t^{-1}(t)$ is a set with a void interior in $T_t^{-1}(\pi(C))$. By Lemma 3, for each $R(s, U)$ containing (t, P) we have $a \in \pi(C)$ such that $\pi^{-1}(a) \subset R(s, U)$. This implies that $C \cap R(s, U) \neq 0$. Since C is closed, we have $(t, P) \in C$ and, in virtue of $C \subset \pi^{-1}(\pi(C))$, we get $\pi^{-1}(\pi(C)) = C$.

MAIN THEOREM. *Let X be a metric continuum. Let K_x, for each $x \in X$, be metric continua. Then there are a separable, first countable continuum S and an atomic irreducible map $\pi\colon S \xrightarrow{\text{onto}} X$ such that $\pi^{-1}(x) = K_x$ for each $x \in X$; if, in addition, X is irreducible, then S irreducible.*

Proof. The preceding theorems allow us to construct an atomic irreducible map $\pi\colon S \xrightarrow{\text{onto}} X$, S satisfying all the required conditions, under some hypotheses concerning the existence, for each $x \in X$, of maps $T_x\colon M_x \xrightarrow{\text{onto}} X$ such that (1) T_x is atomic, (2) $T_x^{-1}(x)$ is a set with a void interior in M_x, and (3) $T_x^{-1}(t)$ are one-point sets for $t \neq x$. Now we shall show that these hypotheses may be satisfied even with additional conditions which ensure that, for each $x \in X$, $T_x^{-1}(x)$ is a given continuum K_x. It is shown in [1] that if K is a metric continuum and Z is a locally compact, non-compact metric space with a countable base, then there is a compact metric space M containing a dense subset Z' homeomorphic to Z and such that $M - Z'$ is homeomorphic to K; furthermore, the compactification M of $M - Z'$ has the following additional property: (*) if C is a subcontinuum of M which intersects both Z' and $M - Z'$, then C contains $M - Z'$. To get the required atomic map we take $X - \{x\}$ for Z. Then X is a one-point compactification of Z. Take K for K_x. The continuum M_x is M for K and Z defined above. So we get another compactification of $X - \{x\}$, the remainder of which is K_x. Thus there exists a map $T_x\colon M_x \xrightarrow{\text{onto}} X$ which leaves the points of $X - \{x\}$ fixed and maps the remainder K_x onto the remainder $\{x\}$. The maps T_x are the required maps. The atomicity of these maps follows immediately from the property (*) of the compactification. The additional assertion of the theorem, namely the irreducibility of S, is a consequence of the assertion formulated in the Note following Theorem 6.

Note. Theorems 3–7 are valid under more general conditions. Namely, the condition that, for $t \neq x$, $T_x^{-1}(t)$ are one-point sets, can be replaced by weaker conditions that $\lim_{t \to x} [\operatorname{diam} T_x^{-1}(t)] = 0$.

References

[1] J. M. Aarts and P. van Emde Boas, *Continua as remainders in compact extensions*, Niew. Archief voor Wiskunde 15 (1967), pp. 34–37.

[2] R. D. Anderson, *Atomic decomposition of continua*, Duke Math. J. 24 (1956), pp. 507–514.

[3] — and G. Choquet, *A plane continuum no two of whose non-degenerate subcontinua are homeomorphic: An application of inverse limits*, Proc. Amer. Math. Soc. 10 (1959), pp. 347–353.

[4] H. Cook, *Continua which admit only the identity mapping onto non-degenerate subcontinua*, Fund. Math. 60 (1967), pp. 241–249.

[5] A. Emeryk and Z. Horbanowicz, *On atomic mappings*, submitted to Colloq. Math.

[6] В. Федорчук, *О бикомпактах с несовпадающими размерностями*, ДАН СССР 182 (1968), pp. 275–278.

[7] K. Kuratowski, *Topologie II*, Warszawa 1952.

[8] W. S. Mahavier, *Upper semi-continuous decompositions of irreducible continua*, Fund. Math. 60 (1967), pp. 53–57.

[9] — *Atomic mappings on irreducible Hausdorff continua*, Fund. Math. 69 (1970), pp. 147–151.

[10] E. S. Thomas, *Monotone decompositions of irreducible continua*, Rozprawy Mat. 50 (1966), pp. 1–73.

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