obtained if more specialized choices of F are made. For example the following result is easily shown.

(5.4). COROLLARY (McAuley–Tulley). *Let* $p\colon (T, d) \to I^2$ *be a light proper onto mapping. Defining* $F = \{\alpha\colon I \to I^2 \mid (\exists x \in I)\, \alpha(t) = (x, t) \forall t \in I\}$, *$p$ is full over F iff for each* $\beta\colon I \to T$ *with* $p\cdot\beta(t) = (0, t)$ *there is a section* $s\colon I^2 \to T$ *for p extending* β.

Analogues of this theorem can be stated for cells of higher dimension (see [5] and [6]).

As another example, McAuley (in [5]) attempted to eliminate some of the pathology of light open mappings by defining a twist free mapping. A light open onto mapping $p\colon T \to B$ is twist free if for each homeomorphism $h\colon S^1 \to B$ and $x \in p^{-1}(h(1, 0))$, there exists a homeomorphism $H\colon S^1 \to T$ with $p\cdot H = h$ and $H(1, 0) = x$.

A conjecture of McAuley is partially answered by the following.

(5.5). COROLLARY. *If* $p\colon (T, d) \to B$ *is a proper twist free onto mapping and p is full over* $H(S^1, B)$ *then any 2 cell in B can be lifted to T.*

References

[0] J. Dugundji, *Topology*, Boston 1966.
[1] E. E. Floyd, *Some characterizations of interior maps*, Ann. of Math. 51 (1950), pp. 571–575.
[2] J. Hill, *Strongly twist free mappings and the Whyburn conjecture*, to appear.
[3] S. T. Hu, *Homotopy Theory*, New York 1959.
[4] — *Theory of Retracts*, Detroit 1965.
[5] L. F. McAuley, *Lifting disks and certain light open mappings*, Proc. Nat'l. Acad. Sci. (UAS) 53 (1965), pp. 255–260.
[6] — and P. A. Tulley, *Lifting cells for certain light open mappings*, Math. Annalen 175 (1968), pp. 114–120.
[7] H. L. Royden, *Real Analysis*, New York 1963.
[8] G. S. Ungar, *Light fiber maps*, Fund. Math. 62 (1968), pp. 31–45.
[9] G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Coll., Rhode Island 1942.

RUTGERS UNIVERSITY, New Jersey
and
TEXAS CHRISTIAN UNIVERSITY, Texas

Reçu par la Rédaction le 4. 3. 1971

Homeotopy groups of orientable 2-manifolds

by

Jong Pil Lee (Vancouver)

1. Introduction. Let X be a topological space, and let $H(X)$ denote the group of homeomorphisms of X onto itself topologized by the compact open topology. The arc-component of the identity $H_0(X)$ is a normal subgroup of $H(X)$ and $\mathcal{H}(X) = H(X)/H_0(X)$ is the group of the arc-components of $H(X)$, which is called the *homeotopy group of* X. The equivalence relation defined by $H_0(X)$ is called *isotopy*. We can also define the isotopy relation in a subgroup $H'(X)$ of $H(X)$ and the group generated by the isotopy classes will be called the *isotopy group of* $H'(X)$, which is denoted by $\pi_0[H'(X)]$. J will denote the group of integers and J_2 the integers mod 2. In 1914, Tietze [10] showed that the homeotopy group of the 2-sphere is J_2. This was proven again by Kneser in 1926 [7], Baer in 1928 [2], Schreier and Ulam in 1934 [9], and most recently by Fisher in 1960 [4]. In [7] Kneser also obtained a result that the homeotopy group of a disk is J_2. In 1923, Alexander [1] proved that the isotopy group of homeomorphisms of an n-cell onto itself leaving the boundary pointwise fixed is trivial. This result has been a most important tool for further development in this area of study. In 1962, in terms of the winding number of a homeomorphism of an annulus, Gluck [5] proved that the isotopy group of homeomorphisms of a closed annulus onto itself leaving the boundary pointwise fixed is J. He also showed that the homeotopy group of an annulus is $J_2 \times J_2$.

In this paper we compute the homeotopy group and isotopy groups of various subgroups of the homeomorphism group of the manifold obtained from the 2-sphere by removing the interiors of three disjoint subdisks. Further we deal with the orientable 2-manifold with n boundary curves.

2. Preliminaries. In this section we give preliminary results which will be used in the next section.

Basic notations

M_n will denote an orientable 2-manifold with n boundary curves,

$H^+(M_n) = \{h \in H(M_n) |\ h$ is orientation preserving on $M_n\}$,
$H^-(M_n) = \{h \in H(M_n) |\ h$ is orientation reversing on $M_n\}$,
$H_m(M_n) = \{h \in H(M_n) |\ h(C_i) = C_i$ for certain m boundary curves $C_i\}$,
$H^t(M_n) = \{h \in H(M_n) |\ h = e$ on certain t boundary curves$\}$,
$H_m^+(M_n) = H^+(M_n) \cap H_m(M_n)$,

$H_m^t(M_n)$ is similarly defined for certain m and t boundary curves where $m+t \leqslant n$ and the m curves are different from the t curves. S_n will denote the symmetric group on n letters. The notation "$\simeq$" will mean the homotopy relation and $I = [0, 1]$.

LEMMA 2.1. *The space of homeomorphisms of a closed n-cell onto itself which leave the boundary of the n-cell pointwise fixed is contractible* [1], p. 406.

DEFINITION 2.2. An isotopy of a space X is a collection $\{G_t\}$, $t \in I$, of homeomorphisms of X onto itself such that the mapping $G: X \times I \to X$ defined by $G(x, t) = G_t(x)$ is continuous. An isotopy which moves no point on $\mathrm{Bd}(X)$ is called a *B-isotopy*. $h \approx g\,[h \underset{B}{\approx} g]$ will denote that h is isotopic (B-isotopic) to g. The imbeddings f_0, f_1: $X \to Y$ are ambient isotopic if there is a level preserving homeomorphism G: $Y \times I \to Y \times I$ such that $G(y, 0) = (y, 0)$ for all $y \in Y$ and $(f_1(x), 1) = G(f_0(x), 1)$ for all $x \in X$.

LEMMA 2.3. *Let M be a 2-manifold with boundary. Let α and β be two arcs in M such that*

$$\mathrm{Bd}(M) \cap \alpha = \mathrm{Bd}(\alpha) = \mathrm{Bd}(\beta) = \mathrm{Bd}(M) \cap \beta,$$

and which are homotopic keeping the end points fixed. Then they are ambient isotopic by a B-isotopy [3], p. 89.

LEMMA 2.4. *Let M_n be a 2-manifold with n boundary curves and α be an arc connecting the boundary points with $\mathrm{Int}(\alpha) \subset \mathrm{Int}(M_n)$, and let h and g be any homeomorphisms in $H^n(M_n)$ such that the closed arcs $h(\alpha) \circ \alpha^{-1}$ and $g(\alpha) \circ \alpha^{-1}$ belong to the same homotopy class in $\pi_1(M_n, x_0)$ where x_0 is the base point on α. Then $h(\alpha) \simeq g(\alpha)$ with the end points of the arc α held fixed.*

Proof. We note that if $g_1, g_2 \in H^n(M_n)$ and $g_1(\alpha) \circ \alpha^{-1}$ and $g_2(\alpha) \circ \alpha^{-1}$ belong to the same homotopy class in $\pi_1(M_n, x_0)$, then $g_2^{-1} g_1(\alpha) \simeq \alpha$, with the end points of α held fixed. Thus it is sufficient to consider a homeomorphism $h \in H^n(M_n)$ such that $h(\alpha) \circ \alpha^{-1} \simeq 0$ with x_0 as the base point, and the proof is similar to the arguments in [6], p. 42.

LEMMA 2.5. *Let M be a 2-manifold with boundary, and $\{\alpha_i\}_{i \in \Delta}$ be a finite collection of arcs in M such that*

(i) *each α_i connects boundary points with $\mathrm{Int}(\alpha_i) \subset \mathrm{Int}(M)$ for each $i \in \Delta$,*

(ii) *$\alpha_i \cap \alpha_j = \emptyset$ for $i \neq j$ and*

(iii) *cutting M along the α_i's leads to a disk.*

Then if a homeomorphism h has the properties that $h = e$ on $\mathrm{Bd}(M)$ and $h(\alpha_i) \simeq \alpha_i$ for each $i \in I$, then h is B-isotopic to the identity.

Proof. We prove the theorem by an induction on the number of the arcs α_i.

First we prove it for the case $n = 1$. Let α be an arc satisfying the above conditions in a manifold M and h be a homeomorphism of M onto itself such that $h = e$ on $\mathrm{Bd}(M)$ and $h(\alpha) \simeq \alpha$. Since the two arcs $h(\alpha)$ and α are ambient isotopic by a B-isotopy by Lemma 2.3, there is an isotopy G_t: $M \to M$, $0 \leqslant t \leqslant 1$, such that $G_t = e$ on $\mathrm{Bd}(M)$, $G_0 = e$ on M and $G_1^{-1} h = e$ on α. Thus by cutting M along α, we realize that $G_1^{-1} h$ is a homeomorphism of M' such that $G_1^{-1} h = e$ on $\mathrm{Bd}(M')$, where M' is the disk obtained from M by cutting along α. Hence Lemma 2.1 implies that $G_1^{-1} h$ is B-isotopic to the identity on M'. Thus h is also B-isotopic to the identity on M.

Now assuming that the theorem is true for the case $n = k$, we prove it for $n = k+1$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_{k+1}\}$ be a collection of the arcs in a 2-manifold M satisfying the above conditions and h be a homeomorphism of M onto itself such that $h = e$ on $\mathrm{Bd}(M)$ and $h(\alpha_i) \simeq \alpha_i$ for $1 \leqslant i \leqslant k+1$. Then in particular we have $h(\alpha_{k+1}) \simeq \alpha_{k+1}$ keeping the end points held fixed and these two arcs are ambient isotopic by a B-isotopy. There is thus a B-isotopy H_t: $M \to M$, $0 \leqslant t \leqslant 1$, such that $H_0 = e$ on M and $H_1^{-1} h = e$ on α_{k+1}. Now cutting M along α_{k+1}, we see that $H_1^{-1} h$ is a homeomorphism of M' such that $H_1^{-1} h = e$ on $\mathrm{Bd}(M')$ and $H_1^{-1} h(\alpha_i) \simeq \alpha_i$ for $1 \leqslant i \leqslant k$, where M' is a 2-manifold obtained from M by cutting along α_{k+1}. By our assumption $H_1^{-1} h$ is B-isotopic to the identity on M'. Thus h is also B-isotopic to the identity on M and the theorem follows for any integer $n \geqslant 1$.

Let $A = S^1 \times I$ and $H^2(A) = \{h \in H(A) |\ h = e$ on $\mathrm{Bd}(A)\}$. H. Gluck (5) defined the winding number for a homeomorphism $h \in H^2(A)$ as follows. Let η be the isomorphism of $\pi_1(S^1, 0)$ with J which takes the class of the path $f(t) = t$ onto 1. Let α be any path in $S^1 \times I$ from $(0, 0)$ to $(0, 1)$ and P_1: $S^1 \times I \to S^1$ the natural projection. Then $P_1(\alpha)$ is a closed path in S^1 based at 0. Hence $[P_1(\alpha)]$ is an element of $\pi_1(S^1, 0)$ and $\eta([P_1(\alpha)]) = \omega(\alpha)$ is an integer. The integer $\omega(h\alpha) - \omega(\alpha)$ is independent of the path α for any $h \in H^2(A)$.

DEFINITION 2.6. Let h be a homeomorphism in $H^2(A)$ and α a path in A from $(0, 0)$ to $(0, 1)$. Then the integer $W[h; A] = \omega(h\alpha) - \omega(\alpha)$ is called the *winding number of h on A*.

We note that W defines a homomorphism W: $H^2(A) \to J$. But it is shown that the kernel of W is the arc-component of the identity $H_0^2(A)$ and thus W is in fact an isomorphism of $H^2(A)$ onto J [5].

DEFINITION 2.7. Let M be a 2-manifold and A be an annulus in $\mathrm{Int}(M)$. Then there is a homeomorphism h_A of the annulus A onto itself such that $W[h_A; A]=1$ and $h_A=e$ on $\mathrm{Bd}(A)$. This homeomorphism can be extended to M by the identity on $M-A$. We call the *extended homeomorphism an A-homeomorphism* and denote it by h_A also.

THEOREM 2.8. *Let A and A' be two annuli which can be deformed to coincide with each other on an orientable 2-manifold M. Then two A-homeomorphisms h_A and $h_{A'}$, are B-isotopic to each other.*

Proof. Since A can be deformed to be coincide with A', the same side boundary curves C and C' of A and A' respectively are isotopic. Thus there is an isotopy G_t: $M\to M$, $0\leqslant t\leqslant 1$, such that $G_0=e$ on M and $G_1(C)=C'$. Hence $G_t h_A G_t^{-1}$, $0\leqslant t\leqslant 1$, is a B-isotopy between the homeomorphisms h_A and $G_1 h_A G_1^{-1}$. Observing that $G_1 h_A G_1^{-1}$ is supported on an annulus in $\mathrm{Int}(M)$ which has the common interval boundary curve C' with the annulus A', it can be shown that $h_{A'}$, is B-isotopic to $G_1 h_A G_1^{-1}$. Hence h_A is B-isotopic to $h_{A'}$.

LEMMA 2.9. *Let h be a homeomorphism in $H_n^+(M_n)$. Then h can be deformed to a homeomorphism k such that $k=e$ on $\mathrm{Bd}(M_n)$, where the isotopy is taken in $H_n^+(M_n)$.*

Proof. Define an annulus A_i around each of the boundary curves C_i so that each C_i forms a boundary curve of the annulus A_i, $A_i\subset \mathrm{Int}(M_n)\cup\cup C_i$ and $A_i\cap A_j=\emptyset$ for $1\leqslant i\neq j\leqslant n$. Now we construct a homeomorphism g;

$$g=\begin{cases} h & \text{on} \quad \mathrm{Bd}(M_n)\,, \\ e & \text{on} \quad M_n-\bigcup\limits_{i=1}^{n} A_i\,. \end{cases}$$

Then $g|A_i$ is isotopic to the identity on A_i for $1\leqslant i\leqslant n$, since the isotopy is allowed to move on each boundary curve C_i, and hence g is isotopic to the identity on M_n by an isotopy in $H_n^+(M_n)$. Thus it is clear that hg^{-1} is isotopic to the homeomorphism h in $H_n^+(M_n)$ with $hg^{-1}=e$ on $\mathrm{Bd}(M_n)$. Letting $k=hg^{-1}$, the proof is complete.

THEOREM 2.10. *Let M_n be an orientable 2-manifold with n boundary curves and h be a homeomorphism in $H_n(M_n)$. Then h must have the same orientation on all the boundary curves.*

Proof. We assume that h is orientation preserving on a boundary curve C_j and reversing on C_k where $1\leqslant j\neq k\leqslant n$. Let $\overline{M}_n$ be a closed orientable 2-manifold obtained from M_n by filling in the interiors of all the boundary holes, and $\bar{h}$ an extension of the homeomorphism h to $\overline{M}_n$. Let O_j and O_k be the interiors of the disks bounded by the curves C_j and C_k respectively. Then it can be seen that $\bar{h}$ must have different orientations on O_j and O_k, since $\bar{h}$ has different orientations on $\mathrm{Bd}(\overline{O}_j)=C_j$ and $\mathrm{Bd}(\overline{O}_k)=C_k$. Thus we have a contradiction and h must be orientation preserving or reversing on all the boundary curves.

3. Homeotopy groups. In what follows M_3 will denote the manifold obtained from the 2-sphere by removing the interiors of three disjoint subdisks.

We extend the concept of the winding number defined in the previous section to the 2-sphere with three boundary curves. We denote by A_i an annulus in $\mathrm{Int}(M_3)$ enclosing only one corresponding boundary curve C_i and h_{A_i} will denote an A_i-homeomorphism of M_3 supported on A_i for $1\leqslant i\leqslant 3$. By saying that A_i encloses one curve C_i, we mean that A_i can be shrunk onto the boundary curve C_i and thus A_i can be extended up to C_i so that C_i forms a boundary curve of the extended annulus $\overline{A}_i$ while the other boundary remains fixed.

LEMMA 3.1. *Let M_n be a manifold obtained from the 2-sphere by removing the interiors of n disjoint subdisks, and h be a homeomorphism in $H^n(M_n)$. Then h is B-isotopic to a product of A-homeomorphisms of M_n* [8], p. 537.

LEMMA 3.2. *Let M_3 be a manifold defined in Lemma* 3.1 *and h a homeomorphism in $H^3(M_3)$. Then in the product of A-homeomorphisms which is B-isotopic to h, the exponent of the homeomorphism h_{A_i} is unique for $1\leqslant i\leqslant 3$.*

Proof. Without loss of generality we assume that h is B-isotopic to two different products of the following forms;

(A) $$h_{A_1}^{k_1}h_{A_2}^{k_2}h_{A_3}^{k_3}$$

and

(B) $$h_{A_1}^{l_1}h_{A_2}^{l_2}h_{A_3}^{l_3}$$

with $k_1\neq l_1$, where k_i and l_i $(1\leqslant i\leqslant 3)$ are integers. Since the annuli A_i $(1\leqslant i\leqslant 3)$ can be taken to be disjoint from one another, the homeomorphisms h_{A_i} commute with one another.

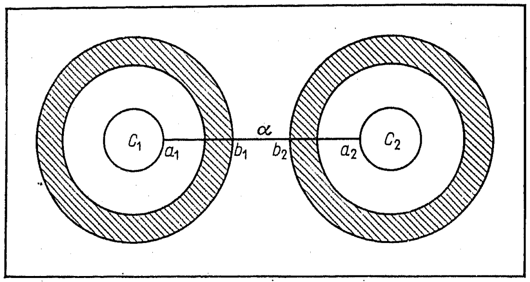
It is enough to consider an arc a connecting two boundary curves C_1 and C_2 with $\mathrm{Int}(a)\subset \mathrm{Int}(M_n)$. Thus from the products (A) and (B), we need to consider only the reduced forms $h_{A_1}^{k_1}h_{A_2}^{k_2}$ and $h_{A_1}^{l_1}h_{A_2}^{l_2}$, since A_3 can be taken to be disjoint from the arc a and thus

$$h_{A_1}^{k_1}h_{A_2}^{k_2}(a)=h_{A_1}^{k_1}h_{A_2}^{k_2}h_{A_3}^{k_3}(a)$$

and

$$h_{A_1}^{l_1}h_{A_2}^{l_2}(a)=h_{A_1}^{l_1}h_{A_2}^{l_2}h_{A_3}^{l_3}(a)\,.$$

We denote $a_i = C_i \cap \alpha$ and $b_i = B_i \cap \alpha$ for $i = 1$ and 2 where B_i is the external boundary curve of A_i. Let $\alpha_1 = [a_1, b_1]$, $\alpha_2 = [b_2, a_2]$ and $\beta = [b_1, b_2]$.



Since

$$[h^{k_1}_{A_1}(\alpha_1)] \cup [h^{k_2}_{A_2}(\alpha_2)] = h^{k_1}_{A_1} h^{k_2}_{A_2}(\alpha_1 \cup \alpha_2)$$

and

$$[h^{l_1}_{A_1}(\alpha_1)] \cup [h^{l_2}_{A_2}(\alpha_2)] = h^{l_1}_{A_1} h^{l_2}_{A_2}(\alpha_1 \cup \alpha_2),$$

we see that

$$h^{k_1}_{A_1} h^{k_1}_{A_2}(\alpha) = [h^{k_1}_{A_1}(\alpha_1)] \circ \beta \circ [h^{k_2}_{A_2}(\alpha_2)]$$

and

$$h^{l_1}_{A_1} h^{l_2}_{A_2}(\alpha) = [h^{l_1}_{A_1}(\alpha_1)] \circ \beta \circ [h^{l_2}_{A_2}(\alpha_2)].$$

Thus

$$(h^{k_1}_{A_1} h^{k_2}_{A_2})(h^{l_1}_{A_1} h^{l_2}_{A_2})^{-1}(\alpha) \simeq [h^{(k_1-l_1)}_{A_1}(\alpha_1)] \circ \beta \circ [h^{(k_2-l_2)}_{A_2}(\alpha_2)].$$

Then

$$\{[h^{(k_1-l_1)}_{A_1}(\alpha_1)] \circ \beta \circ [h^{(k_2-l_2)}_{A_2}(\alpha_2)]\} \circ \alpha^{-1} \simeq \omega^{(k_1-l_1)} \circ \omega^{(k_2-l_2)},$$

where ω_1 and ω_2 are different generators of the homotopy group $\pi_1(M_3, x_0)$ with $x_0 = a_1$. But $\omega_1^{(k_1-l_1)} \circ \omega_2^{(k_2-l_2)} \not\simeq 0$ unless $k_1 - l_1 = 0$ and $k_2 - l_2 = 0$. Thus $k_1 \neq l_1$ implies that

$$h^{k_1}_{A_1} h^{k_2}_{A_2}(\alpha) \not\simeq h^{l_1}_{A_1} h^{l_2}_{A_2}(\alpha)$$

and

$$h^{k_1}_{A_1} h^{k_2}_{A_2} h^{k_3}_{A_3}(\alpha) \not\simeq h^{l_1}_{A_1} h^{l_2}_{A_2} h^{l_3}_{A_3}(\alpha).$$

Then Theorem 2.5 implies that

$$h^{k_1}_{A_1} h^{k_2}_{A_2} h^{k_3}_{A_3} \not\approx h^{l_1}_{A_1} h^{l_2}_{A_2} h^{l_3}_{A_3}$$

which is a contradiction. Thus $k_1 = l_1$ and the exponent of h_{A_1} must be unique in the product. Similarly it can be shown that the exponents of the other homeomorphisms h_{A_2} and h_{A_3} are also unique.

DEFINITION 3.3. Let h be a homeomorphism in $H^3(M_3)$ and h' a product of A-homeomorphisms which is B-isotopic to h. The *winding number of* h around each of the boundary holes C_i is defined to be the exponent of the homeomorphism h_{A_i} in the product h'. We denote by $W[h; C_i]$ the winding number of h around the boundary hole C_i.

THEOREM 3.4. *The homotopy group* $\mathcal{K}(M_3)$ *is* $S_3 \times J_2$.

Proof. We first observe that every $h \in H_3^+(M_3)$ is isotopic to the identity by an isotopy in $H_3^+(M_3)$. By Lemma 2.9 h can be deformed to a homeomorphism $g \in H^3(M_3)$ where g is B-isotopic to a homeomorphism of the form $h^{k_1}_{A_1} h^{k_2}_{A_2} h^{k_3}_{A_3}$ for some integers k_i $(1 \leqslant i \leqslant 3)$. But each $h^{k_i}_{A_i}$ can be deformed to the identity by rotating the boundary curve C_i through 0 to $2(-k_i)\pi$. Hence g (and thus h) is isotopic to the identity in $H_3^+(M_3)$ and the homeotopy group is $H(M_3)/H_3^+(M_3)$.

We note that $\dfrac{H^+(M_3)}{H_3^+(M_3)} \cong S_3$ and

$$\frac{H_3(M_3)}{H_3^+(M_3)} \cong \frac{H(M_3)/H_3^+(M_3)}{H^+(M_3)/H_3^+(M_3)} \cong J_2,$$

where $\dfrac{H^+(M_3)}{H_3^+(M_3)}$ and $\dfrac{H_3(M_3)}{H_3^+(M_3)}$ are normal subgroups of $\dfrac{H(M_3)}{H_3^+(M_3)}$. Let φ be the canonical homomorphism from $\dfrac{H(M_3)}{H_3^+(M_3)}$ onto $\dfrac{H(M_3)/H_3^+(M_3)}{H^+(M_3)/H_3^+(M_3)}$. Then φ induces an isomorphism of $\dfrac{H_3(M_3)}{H_3^+(M_3)}$ onto its range, and thus

$$\mathcal{K}(M_3) = \frac{H(M_3)}{H_3^+(M_3)} \cong \frac{H^+(M_3)}{H_3^+(M_3)} \times \frac{H_3(M_3)}{H_3^+(M_3)} \cong S_3 \times J_2.$$

THEOREM 3.5. *The isotopy group of* $H_1(M_3)$ *is* $J_2 \times J_2$, *where* $H_1(M_3) = \{h \in H(M_3) \mid h(C_1) = C_1\}$.

Proof. We note that $\pi_0[H_3^+(M_3)] = [e]$ in $\pi_0[H_1(M_3)]$, since as in the proof of Theorem 3.4, every $h \in H_3^+(M_3)$ is isotopic to the identity in $H_3^+(M_3)$ and $H_3^+(M_3) \subset H_1(M_3) \subset H(M_3)$. Thus it is enough to consider the isotopy classes of the subspace $H_1(M_3) - H_3^+(M_3)$. We divide

$$H_1(M_3) - H_3^+(M_3) = K_1 \cup K_2 \cup K_3,$$

where

$$K_1 = \{h \in H_1^+(M_3) \mid h(C_2) = C_3\},$$

$$K_2 = H_3^-(M_3),$$
$$K_3 = \{h \in H_1^-(M_3) |\ h(C_2) = C_3\}.$$

Then every homeomorphism in K_3 is generated by homeomorphisms in K_1 and K_2. Hence the isotopy classes of $H_1(M_3) - H_3^+(M_3)$ are generated by the classes of K_1 and K_2, each of which generates a group J_2.

Now we show that the isotopy group $\pi_0[H_1(M_3)]$ is the direct product of the groups generated by the isotopy classes of K_1 and K_2. Let G_i be the group generated by the isotopy class of K_i for $i = 1$ and 2. Then by the above arguments,

$$\pi_0[H_1(M_3)] = G_1 \circ G_2 = \{g_1 g_2 |\ g_i \in G_i,\ i = 1, 2\}.$$

Further observing that $G_1 \cap G_2 = \{[e]\}$ and each G_i is a normal subgroup of $\pi_0[H_1(M_3)]$, we have $\pi_0[H_1(M_3)] \cong G_1 \times G_2$ and thus the isotopy group is $J_2 \times J_2$.

THEOREM 3.6. *The isotopy group of $H^3(M_3)$ is $J \times J \times J$.*

Proof. Since all the possible annuli in $\mathrm{Int}(M_3)$ can be regarded as the annuli A_i $(1 \leqslant i \leqslant 3)$ enclosing only one corresponding boundary curve C_i with $A_i \cap A_j = \emptyset$ for $1 \leqslant i \neq j \leqslant 3$, Lemma 3.1 implies that every $h \in H^3(M_3)$ is B-isotopic to a homeomorphism of the form $h_{A_1}^{k_1} h_{A_2}^{k_2} h_{A_3}^{k_3}$ for some integers k_i $(1 \leqslant i \leqslant 3)$. Thus we have $\pi_0[H^3(M_3)] \cong \pi_0[\{h_{A_1}^{k_1} h_{A_2}^{k_2} h_{A_3}^{k_3}\}]$ where by Lemma 3.2 any two homeomorphisms of the form $h_{A_1}^{k_1} h_{A_2}^{k_2} h_{A_3}^{k_3}$ having different exponents of h_{A_i} are not B-isotopic to each other. But each h_{A_i} generates the isotopy group J classified by the winding numbers $W[h_{A_i}^{k_i}; A_i]$. Thus the isotopy group $\pi_0[h_{A_1}^{k_1} h_{A_2}^{k_2} h_{A_3}^{k_3}]$ is $J \times J \times J$, and the theorem follows.

THEOREM 3.7. *The isotopy group of $H^2(M_3)$ is $J \times J$, where*

$$H^2(M_3) = \{h \in H(M_3) |\ h = e \text{ on } C_1 \cup C_2\}.$$

Proof. We first note that $H^2(M_3) \subset H^+(M_3)$ by Theorem 2.10 and every $h \in H^2(M_3)$ can be deformed to a homeomorphism in $H^3(M_3)$ by an isotopy in $H^2(M_3)$. But the isotopy in $H^2(M_3)$ is allowed to move on the boundary curve C_3, and thus every $h \in H^2(M_3)$ is isotopic to a homeomorphism of the form $h_{A_1}^{k_1} h_{A_2}^{k_2}$ for some integers k_1 and k_2. Hence

$$\pi_0[H^2(M_3)] \cong \pi_0[\{h_{A_1}^{k_1} h_{A_2}^{k_2} |\ k_i \in J\}],$$

where any two homeomorphisms of the form $h_{A_1}^{k_1} h_{A_2}^{k_2}$ having different exponents of h_{A_i} are not isotopic to each other in $H^2(M_3)$. Thus we see that $\pi_0[\{h_{A_1}^{k_1} h_{A_2}^{k_2}\}] = J \times J$ and the theorem follows.

Now we consider the homeotopy and isotopy groups of the orientable 2-manifold with n boundary curves.

THEOREM 3.8. *Let M_n be an orientable 2-manifold with n boundary curves. Then $\pi_0[H_m(M_n)]$ is homomorphic to $S_{n-m} \times J_2$ with the kernel $\pi_0[H_n^+(M_n)]$, where $0 \leqslant m \leqslant n$.*

Proof. Let φ be the canonical homomorphism from $\dfrac{H_m(M_n)}{H_{n0}(M_n)}$ onto $\dfrac{H_m(M_n)}{H_n^+(M_n)}$, where $H_{n0}(M_n)$ is the identity component of $H_n(M_n)$ and $\dfrac{H_m(M_n)}{H_{n0}(M_n)} = \pi_0[H_m(M_n)]$. Then it can be seen that the kernel of φ is $\dfrac{H_n^+(M_n)}{H_{n0}(M_n)} = \pi_0[H_n^+(M_n)]$.

Now we note that $\dfrac{H_m^+(M_n)}{H_n^+(M_n)}$ and $\dfrac{H_n(M_n)}{H_n^+(M_n)}$ are normal subgroups of $\dfrac{H_m(M_n)}{H_n^+(M_n)}$, where

$$\frac{H_m^+(M_n)}{H_n^+(M_n)} \cong S_{n-m} \quad \text{and} \quad \frac{H_n(M_n)}{H_n^+(M_n)} \cong \frac{H_m(M_n)/H_n^+(M_n)}{H_m^+(M_n)/H_n^+(M_n)} \cong J_2.$$

Let ψ be the canonical homomorphism from $\dfrac{H_m(M_n)}{H_n^+(M_n)}$ onto $\dfrac{H_m(M_n)/H_n^+(M_n)}{H_m^+(M_n)/H_n^+(M_n)}$. Then ψ induces an isomorphism of $\dfrac{H_n(M_n)}{H_n^+(M_n)}$ onto its range. Thus we obtain

$$\frac{H_m(M_n)}{H_n^+(M_n)} \cong \frac{H_m^+(M_n)}{H_n^+(M_n)} \times \frac{H_n(M_n)}{H_n^+(M_n)} \cong S_n \times J_2$$

and the proof is complete.

THEOREM 3.9. *Let M_n be an orientable 2-manifold with n boundary curves. Then the isotopy group of $H_m^t(M_n)$ is homomorphic to the symmetric group $S_{n-(m+t)}$ with the kernel $\pi_0[H_{n-t}^t(M_n)]$, where $m+t \leqslant n$.*

Proof. Noting that $H_{n-t}^t(M_n) \subset H_m^t(M_n) \subset H^+(M_n)$ by Theorem 2.10 and $\dfrac{H_m^t(M_n)}{H_{n-t}^t(M_n)} \cong S_{n-(m+t)}$, this theorem can also be proved by similar arguments as in the proof of Theorem 3.8.

I would like to express my sincere gratitude to Professor J. V. Whittaker for his generous assistance.

References

[1] J. W. Alexander, *On the deformation of an n-cell*, Proc. Nat. Acad. Sci. 9 (1923), pp. 406–407.

[2] R. Baer, *Isotopie von Kurven auf orientbaren, geschlossenen Flächen und ihr*

Zusammenhang mit der topologischen Deformation der Flächen, J. Reine Angew. Math. 159 (1928), pp. 101–116.

[3] D. B. A. Epstein, *Curves on 2-manifolds and isotopies*, Acta Math. 115 (1966), pp. 83–107.

[4] G. M. Fisher, *On the group of all homeomorphisms of a manifold*, Trans. Amer. Math. Soc. 97 (1960), pp. 193–212.

[5] H. Gluck, *The embedding of two-spheres in the four-sphere*, Trans. Amer. Math. Soc. 104 (1962), pp. 308–333.

[6] S. T. Hu, *Homotopy Theory*, New York 1959.

[7] H. Kneser, *Die Deformationssätze der einfach zusammenhängenden Flächen*, Math. Z. 25 (1926), pp. 362–372.

[8] W. B. Lickorish, *A representation of orientable combinatorial 3-manifolds* Ann. Math. 76 (1962), pp. 531–540.

[9] J. Schreier and S. Ulam, *Über topologische Abbildungen der euklidischen Sphären*, Fund. Math. 23 (1934), pp. 102–118.

[10] H. Tietze, *Über stetige Abbildungen einer Quadratfläche auf sich selbst*, Rend. Circ. Mat. Palermo 38 (1914), pp. 247–304.

THE UNIVERSITY OF BRITISH COLUMBIA
Vancouver

Reçu par la Rédaction le 31. 3. 1971

Uniformly movable compact spaces and their algebraic properties

by

M. Moszyńska (Warszawa)

Contents

Let us consider a pointed compact Hausdorff space (X, x_0) and an ANR-system $(\underline{X}, \boldsymbol{x}_0)$ associated with (X, x_0) (see [3]). Let $\pi_n(\boldsymbol{X}, \boldsymbol{x}_0)$ be an inverse system of nth homotopy groups. Its inverse limit does not depend on the choice of $(\boldsymbol{X}, \boldsymbol{x}_0)$ (see 6.3) and here is referred to as the nth limit homotopy group of (X, x_0) (in symbols $\pi_n^*(X, x_0)$).

Consider two pairs, (X, x_0) and (Y, y_0), and the associated inverse systems $(\boldsymbol{X}, \boldsymbol{x}_0)$ and $(\boldsymbol{Y}, \boldsymbol{y}_0)$. To any map $\boldsymbol{f}\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$ (in the sense of [3]) and a natural number n we can assign the induced morphism $\boldsymbol{f}_n\colon \pi_n(\boldsymbol{X}, \boldsymbol{x}_0) \to \pi_n(\boldsymbol{Y}, \boldsymbol{y}_0)$ and its inverse limit, $\varprojlim \boldsymbol{f}_n = f_n^*\colon \pi_n^*(X, x_0) \to \pi_n^*(Y, y_0)$.

In general, the algebraic properties of $\varprojlim \boldsymbol{f}_n$ do not determine the algebraic properties of $\boldsymbol{f}_n$. For instance, the implication

$$(*) \qquad \varprojlim \boldsymbol{f}_n \text{ is a bimorphism} \Rightarrow \boldsymbol{f}_n \text{ is a bimorphism}$$

in general fails (see § 6).

The purpose of this paper is to distinguish a class of spaces which satisfies (*). This leads to the notion of uniform movability of an inverse system in an arbitrary category, in particular in the category of ANR's or in the category of groups (§§ 3, 4). A uniformly movable inverse system