unit ball $B_1(0) = \{x \in l^1: \|x\| < 1\}$. Since the norm $\|\cdot\|: l^1 \to R$ is $\mathfrak{J}$-continuous, $B_1(0)$ is $\mathfrak{J}$-open and hence $\sigma(l^1, l^\infty)$-sequentially open. However, the norm $\|\cdot\|: l^1 \to R$ is not $\sigma(l^1, l^\infty)$-continuous. Consequently, $B_1(0)$ is not $\sigma(l^1, l^\infty)$-open. Thus $(l^1, \sigma(l^1, l^\infty))$ is not sequential.

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An atriodic tree-like continuum with positive span

by

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1. Introduction. In 1964 A. Lelek defined the span of a metric space, and he proved that every chainable continuum has span zero [5], section 5. In this paper we construct an example of an atriodic tree-like continuum with positive span. The continuum is obtained as an inverse limit on simple triods using only one bonding map. The question of the existence of an atriodic tree-like continuum which is not chainable was mentioned by Bing [2], p. 45, and Anderson [1] claimed in an abstract that such an example indeed exists.

Throughout this paper the term space refers to metric space and the term mapping to continuous function. The projection of a product space onto its ith coordinate space will be denoted by π_i.

Suppose X and Y are spaces, d is a metric for Y, and f is a mapping of X into Y. The *span of* f, denoted by σf, is the least upper bound of the set of numbers ε for which there is a connected subset Z of $X \times X$ such that $\pi_1(Z) = \pi_2(Z)$ and $d(f(x), f(y)) \geqslant \varepsilon$ for each (x, y) in Z. (Of course σf may be infinite). The span of X, denoted by σX, as defined by Lelek, [5], is the span of the identity mapping on X.

Suppose $X_1, X_2, \ldots$ is a sequence of compact spaces and $f_1, f_2, \ldots$ is a sequence of mappings such that $f_i\colon X_{i+1} \to X_i$. The inverse limit of the inverse limit sequence $\{X_i, f_i\}$ is the subset X of $\prod_{i>0} X_i$ such that $(x_1, x_2, \ldots)$ is in X if and only if $f_i(x_{i+1}) = x_i$ for each i. We consider $\prod_{i>0} (X_i, d_i)$ metrized by

$$d(x, y) = \sum_{i>0} 2^{-i} d_i(x_i, y_i)\,.$$

2. The mapping f and the continuum M. Let T denote the simple triod $\{(\varrho, \theta) |\ 0 \leqslant \varrho \leqslant 1 \text{ and } \theta = 0,\ \theta = \tfrac{1}{2}\pi \text{ or } \theta = \pi\}$ (in polar coordinates in the plane). Define $f\colon T \to T$ as follows:

$$f(x, \tfrac{1}{2}\pi) = \begin{cases} (1-4x, \pi) & \text{if } 0 \leqslant x \leqslant \tfrac{1}{4}, \\ (4x-1, \tfrac{1}{2}\pi) & \text{if } \tfrac{1}{4} \leqslant x \leqslant \tfrac{1}{2}, \\ (3-4x, \tfrac{1}{2}\pi) & \text{if } \tfrac{1}{2} \leqslant x \leqslant \tfrac{3}{4}, \\ (4x-3, 0) & \text{if } \tfrac{3}{4} \leqslant x \leqslant 1\,. \end{cases}$$

$$f(x,\pi)=\begin{cases}(1-3x,\pi) & \text{if } 0\leqslant x\leqslant\frac{1}{3},\\ (3x-1,\frac{1}{2}\pi) & \text{if } \frac{1}{3}\leqslant x\leqslant\frac{1}{2},\\ (2-3x,\frac{1}{2}\pi) & \text{if } \frac{1}{2}\leqslant x\leqslant\frac{2}{3},\\ (3x-2,0) & \text{if } \frac{2}{3}\leqslant x\leqslant 1.\end{cases}$$

$$f(x,0)=\begin{cases}(1-2x,\pi) & \text{if } 0\leqslant x\leqslant\frac{1}{2},\\ (2x-1,0) & \text{if } \frac{1}{2}\leqslant x\leqslant 1.\end{cases}$$

For each n let $T_n=T$ and $f_n=f$. Then denote by M the inverse limit of the inverse limit sequence $\{T_n, f_n\}$.

Before we show that the simple triod-like continuum M is atriodic and has positive span, we adopt some notational conventions which will be used in this paper.

Denote by O the point $(0,0)=(0,\frac{1}{2}\pi)=(0,\pi)$, by A the point $(1,\frac{1}{2}\pi)$, by B the point $(1,\pi)$, and by C the point $(1,0)$. Thus the arc OA of T is $\{(\varrho,\theta)\in T|\ \theta=\frac{1}{2}\pi\}$ while $OB=\{(\varrho,\theta)\in T|\ \theta=\pi\}$ and $OC=\{(\varrho,\theta)\in T|\ \theta=0\}$. If each of p and q is a positive integer and $p\leqslant q$, we will denote by $\frac{pA}{q}$ the point $\left(\frac{p}{q},\frac{\pi}{2}\right)$ while $\frac{pB}{q}$ will denote $\left(\frac{p}{q},\pi\right)$ and $\frac{pC}{q}$ will denote $\left(\frac{p}{q},0\right)$.

3. The continuum M is atriodic. In [4], Theorem 3, it is proved that if every proper subcontinuum of a compact continuum K is chainable, then K is atriodic.

THEOREM 1. *Every proper subcontinuum of M is chainable, and thus M is atriodic.*

Proof. Suppose H is a proper subcontinuum of M. Then there is an integer N such that if $n\geqslant N$ then $\pi_n(H)$ is not T_n. We consider two cases.

Case I. If i is a positive integer there exists an integer $j\geqslant i$ such that O is not a point of $\pi_j(H)$. Suppose $\varepsilon>0$. Then there is an integer t such that if $k\geqslant t$, π_k is an ε-mapping of M onto T_k. There exists an integer $j\geqslant t$ such that O is not $\pi_j(H)$. Thus $\pi_j(H)$ is an arc. Therefore, if $\varepsilon>0$ there is an ε-mapping of H to an arc, so H is chainable.

Case II. There is an integer k such that if $j\geqslant k$ then O is in $\pi_j(H)$. Suppose $j\geqslant k$ and $j\geqslant N$. Then O is in $\pi_{j+3}(H)$, so O and $B=f(O)$ are in $\pi_{j+2}(H)$. Thus, $[OB]$ is a subset of $\pi_{j+2}(H)$. Since $f([OB])=[OB]\cup[OC]\cup\left[O\frac{A}{2}\right]$ and $[OA]$ is a subset of $f\left(\left[O\frac{A}{2}\right]\right)$, $f^2([OB])=T$. Therefore, $\pi_j(H)=T_j$, but $j\geqslant N$.

4. The continuum M has positive span. In this section we show that the continuum M has positive span. If x and y are points of T, the arc xy of T will be denoted by xy.

THEOREM 2. *There exists a sequence $Z_1, Z_2, \ldots$ of subcontinua of $T\times T$ such that for each n $\pi_1(Z_n)=\pi_2(Z_n)=T$, $f\times f(Z_{n+1})=Z_n$, $Z_n=Z_n^{-1}$, and if (p,q) is in Z_1, then $d(p,q)\geqslant\frac{1}{2}$. Thus $\sigma f^n\geqslant\frac{1}{2}$.*

Proof. Let

$$\begin{aligned}Z_1=&\big(([OB]\times\{C\})\cup(\{B\}\times[OC])\big)\cup\big(([OC]\times\{B\})\cup(\{C\}\times[OB])\big)\cup\\ &\cup\big(([OA]\times\{C\})\cup([\{A\}\times[OC])\big)\cup\big(([OC]\times\{A\})\cup(\{C\}\times[OA])\big)\cup\\ &\cup\big(([OA]\times\{B\})\cup(\{A\}\times[OB])\big)\cup\big(([OB]\times\{A\})\cup(\{B\}\times[OA])\big)\cup\\ &\cup\left(\left(\left[O\frac{A}{2}\right]\times\{C\}\right)\cup\left(\left\{\frac{A}{2}\right\}\times[OC]\right)\right)\cup\left(\left([OC]\times\left\{\frac{A}{2}\right\}\right)\cup\left(\{C\}\times\left[O\frac{A}{2}\right]\right)\right)\cup\\ &\cup\left(\left(\left[O\frac{A}{2}\right]\times\{B\}\right)\cup\left(\left\{\frac{A}{2}\right\}\times[OB]\right)\right)\cup\left(\left([OB]\times\left\{\frac{A}{2}\right\}\right)\cup\left(\{B\}\times\left[O\frac{A}{2}\right]\right)\right)\cup\\ &\cup\left(\left(\left[O\frac{A}{2}\right]\times\{A\}\right)\cup\left(\{O\}\times\left[\frac{3A}{4}A\right]\right)\right)\cup\left(\left(\left[\frac{3A}{4}A\right]\times\{O\}\right)\cup\left(\{A\}\times\left[O\frac{A}{2}\right]\right)\right).\end{aligned}$$

If (p,q) is in Z_1, then $d(p,q)\geqslant\frac{1}{2}$.

Suppose Z_n is a subcontinuum of $T\times T$ such that (a) $\pi_1(Z_n)=\pi_2(Z_n)=T$, (b) Z_n is the union of twelve continua denoted by $\langle OB,OC\rangle$, $\langle OC,OB\rangle$, $\langle OA,OC\rangle$, $\langle OC,OA\rangle$, $\langle OA,OB\rangle$, $\langle OB,OA\rangle$, $\left\langle O\frac{A}{2},OC\right\rangle$, $\left\langle OC,O\frac{A}{2}\right\rangle$, $\left\langle O\frac{A}{2},OB\right\rangle$, $\left\langle OB,O\frac{A}{2}\right\rangle$, $\left\langle O\frac{A}{2},\frac{3A}{4}A\right\rangle$, and $\left\langle\frac{3A}{4}A,O\frac{A}{2}\right\rangle$ where $\langle t,u\rangle$ denotes a continuum K such that $\pi_1(K)=t$ and $\pi_2(K)=u$, and (c) $\langle t,u\rangle^{-1}=\langle u,t\rangle$. Suppose further, (d) there exist four points x_1, x_2, x_3, and x_4 such that x_1 is in $\left[\frac{3A}{4}A\right]$, x_2 is in $\left[\frac{2B}{3}B\right]$, x_3 is in $\left[\frac{C}{2}C\right]$, and x_4 is in $\left[\frac{A}{4}\frac{A}{2}\right]$ and (1) $\langle OA,OB\rangle$, $\langle OA,OC\rangle$, and $\left\langle\frac{3A}{4}A,O\frac{A}{2}\right\rangle$ contain (x_1,O) and thus $\langle OB,OA\rangle$, $\langle OC,OA\rangle$ and $\left\langle O\frac{A}{2},\frac{3A}{4}A\right\rangle$ contain (O,x_1), (2) $\langle OB,OA\rangle$, $\langle OB,OC\rangle$ and $\left\langle OB,O\frac{A}{2}\right\rangle$ contain (x_2,O) and thus $\langle OA,OB\rangle$, $\langle OC,OB\rangle$, and $\left\langle O\frac{A}{2},OB\right\rangle$ contain (O,x_2), (3) $\langle OC,OA\rangle$, $\langle OC,OB\rangle$, and $\left\langle OC,O\frac{A}{2}\right\rangle$ contain (x_3,O) and thus $\langle OA,OC\rangle$, $\langle OB,OC\rangle$, and $\left\langle O\frac{A}{2},OC\right\rangle$ contain (O,x_3), and (4) $\left\langle O\frac{A}{2},OB\right\rangle$ and $\left\langle O\frac{A}{2},OC\right\rangle$ contain (x_4,O) and thus $\left\langle OB,O\frac{A}{2}\right\rangle$ and $\left\langle OC,O\frac{A}{2}\right\rangle$

contain (O, x_4). Moreover, suppose (e) there are two points z_1 and z_2 such that z_1 is in OC, z_2 is in OA, and (1) (B, z_1) is in $\langle OB, OC\rangle$ and thus (z_1, B) is in $\langle OC, OB\rangle$ and (2) (B, z_2) is in $\langle OB, OA\rangle$ and thus (z_2, B) is in $\langle OA, OB\rangle$. Finally, suppose that if $n > 1$, $f\times f(Z_n) = Z_{n-1}$.

Note that Z_1 satisfies all the conditions of the preceding paragraph. We now construct Z_{n+1}. The following notation will be convenient in this construction. If $\langle t, u\rangle$ is a subcontinuum of Z_n and v and w are arcs in T such that $f|v$ is a homeomorphism throwing v onto t and $f|w$ is a homeomorphism throwing w onto u then $L = (f|v)^{-1}\times(f|w)^{-1}(\langle t, u\rangle)$ is a continuum, called the lifting of $\langle t, u\rangle$ with respect to v and w, such that $\pi_1(L) = v$ and $\pi_2(L) = w$. This continuum will be denoted by $L(\langle t, u\rangle, v, w)$ and in some instances, where no confusion should arise, may be denoted by L.

Let

$$\alpha_1 = L_1^1\Big(\langle OB, OC\rangle, O\frac{B}{3}, \frac{C}{2}C\Big) \cup L_2^1\Big(\Big\langle O\frac{A}{2}, OC\Big\rangle, \frac{B}{3}\frac{B}{2}, \frac{C}{2}C\Big) \cup$$
$$\cup L_3^1\Big(\Big\langle O\frac{A}{2}, OB\Big\rangle, \frac{B}{3}\frac{B}{2}, O\frac{C}{2}\Big) \cup L_4^1\Big(\Big\langle O\frac{A}{2}, OB\Big\rangle, \frac{B}{2}\frac{2B}{3}, O\frac{C}{2}\Big) \cup$$
$$\cup L_5^1\Big(\langle OC, OB\rangle, \frac{2B}{3}B, O\frac{C}{2}\Big).$$

That α_1 is a continuum follows from the facts that $\Big(\frac{B}{3}, \Big(f|\frac{C}{2}C\Big)^{-1}(x_3)\Big)$ is in $L_1^1\cap L_2^1$, $\Big(\Big(f|\frac{B}{3}\frac{B}{2}\Big)^{-1}(x_4), \frac{C}{2}\Big)$ is in $L_2^1\cap L_3^1$, there is a point y_1 of OB such that $\Big(\frac{A}{2}, y_1\Big)$ is in $\Big\langle O\frac{A}{2}, OB\Big\rangle$ so $\Big(\frac{B}{2}, \Big(f|O\frac{C}{2}\Big)^{-1}(y_1)\Big)$ is in $L_3^1\cap L_4^1$, and $\Big(\frac{2B}{3}, \Big(f|O\frac{C}{2}\Big)^{-1}(x_2)\Big)$ is in $L_4^1\cap L_5^1$. Also, $\pi_1(\alpha_1) = OB$, $\pi_2(\alpha_1) = OC$, and $f\times f(\alpha_1)$ is a subset of Z_n. Let $\alpha_2 = \alpha_1^{-1}$ and α_2 is a continuum such that $\pi_1(\alpha_2) = OC$, $\pi_2(\alpha_2) = OB$, and $f\times f(\alpha_2)$ is a subset of Z_n.

Let

$$\alpha_3 = L_1^3\Big(\langle OB, OC\rangle, O\frac{C}{2}, \frac{3A}{4}A\Big) \cup L_2^3\Big(\langle OB, OA\rangle, O\frac{C}{2}, \frac{A}{2}\frac{3A}{4}\Big) \cup$$
$$\cup L_3^3\Big(\langle OB, OA\rangle, O\frac{C}{2}, \frac{A}{4}\frac{A}{2}\Big) \cup L_4^3\Big(\langle OC, OA\rangle, \frac{C}{2}C, \frac{A}{4}\frac{A}{2}\Big) \cup$$
$$\cup L_5^3\Big(\langle OC, OB\rangle, \frac{C}{2}C, O\frac{A}{4}\Big).$$

That α_3 is a continuum follows from the facts that $\Big(\Big(f|O\frac{C}{2}\Big)^{-1}(x_2), \frac{3A}{4}\Big)$

is in $L_1^3\cap L_2^3$, there is a point y_2 of OB such that (y_2, A) is in $\langle OB, OA\rangle$ so $\Big(\Big(f|O\frac{C}{2}\Big)^{-1}(y_2), \frac{A}{2}\Big)$ is in $L_2^3\cap L_3^3$, $\Big(\frac{C}{2}, \Big(f|\frac{A}{4}\frac{A}{2}\Big)^{-1}(x_1)\Big)$ is in $L_3^3\cap L_4^3$, and $\Big(\Big(f|\frac{C}{2}C\Big)^{-1}(x_3), \frac{A}{4}\Big)$ is in $L_4^3\cap L_5^3$. Also, $\pi_1(\alpha_3) = OC$, $\pi_2(\alpha_3) = OA$ and $f\times f(\alpha_3)$ is a subset of Z_n. Let $\alpha_4 = \alpha_3^{-1}$ and α_4 is a continuum such that $\pi_1(\alpha_4) = OA$, $\pi_2(\alpha_4) = OC$, and $f\times f(\alpha_4)$ is a subset of Z_n.

Let

$$\alpha_5 = L_1^5\Big(\langle OB, OC\rangle, O\frac{B}{3}, \frac{3A}{4}A\Big) \cup L_2^5\Big(\langle OB, OA\rangle, O\frac{B}{3}, \frac{A}{2}\frac{3A}{4}\Big) \cup$$
$$\cup L_3^5\Big(\Big\langle O\frac{A}{2}, \frac{3A}{4}A\Big\rangle, \frac{B}{3}\frac{B}{2}, \frac{A}{2}\frac{9A}{16}\Big) \cup L_4^5\Big(\Big\langle O\frac{A}{2}, \frac{3A}{4}A\Big\rangle, \frac{B}{3}\frac{B}{2}, \frac{7A}{16}\frac{A}{2}\Big) \cup$$
$$\cup L_5^5\Big(\Big\langle O\frac{A}{2}, \frac{3A}{4}A\Big\rangle, \frac{B}{2}\frac{2B}{3}, \frac{7A}{16}\frac{A}{2}\Big) \cup L_6^5\Big(\langle OC, OA\rangle, \frac{2B}{3}B, \frac{A}{4}\frac{A}{2}\Big) \cup$$
$$\cup L_7^5\Big(\langle OC, OB\rangle, \frac{2B}{3}B, O\frac{A}{4}\Big).$$

That α_5 is a continuum follows from the facts that $\Big(\Big(f|O\frac{B}{3}\Big)^{-1}(x_2), \frac{3A}{4}\Big)$ is in $L_1^5\cap L_2^5$, $\Big(\frac{B}{3}, \Big(f|\frac{A}{2}\frac{3A}{4}\Big)^{-1}(x_1)\Big)$ is in $L_2^5\cap L_3^5$ since $\Big(f|\frac{A}{2}\frac{3A}{4}\Big)^{-1}(x_1) = \Big(f|\frac{A}{2}\frac{9A}{16}\Big)^{-1}(x_1)$ because x_1 is in $\frac{3A}{4}A$, there is a point y_3 of $O\frac{A}{2}$ such that (y_3, A) is in $\Big\langle O\frac{A}{2}, \frac{3A}{4}A\Big\rangle$ so $\Big(\Big(f|\frac{B}{3}\frac{B}{2}\Big)^{-1}(y_3), \frac{A}{2}\Big)$ is in $L_3^5\cap L_4^5$, there is a point y_4 in $\frac{3A}{4}A$ such that $\Big(\frac{A}{2}, y_4\Big)$ is in $\Big\langle O\frac{A}{2}, \frac{3A}{4}A\Big\rangle$ so $\Big(\frac{B}{2}, \Big(f|\frac{7A}{16}\frac{A}{2}\Big)^{-1}(y_4)\Big)$ is in $L_4^5\cup L_5^5$, $\Big(\frac{2B}{3}, \Big(f|\frac{A}{4}\frac{A}{2}\Big)^{-1}(x_1)\Big)$ is in $L_5^5\cap L_6^5$ since $\Big(f|\frac{A}{4}\frac{A}{2}\Big)^{-1}(x_1) = \Big(f|\frac{7A}{16}\frac{A}{2}\Big)^{-1}(x_1)$ because x_1 is in $\frac{3A}{4}A$, and $\Big(\Big(f|\frac{2B}{3}B\Big)^{-1}(x_3), \frac{A}{4}\Big)$ is in $L_6^5\cap L_7^5$. Also, $\pi_1(\alpha_5) = OB$, $\pi_2(\alpha_5) = OA$ and $f\times f(\alpha_5)$ is a subset of Z_n. Let $\alpha_6 = \alpha_5^{-1}$ and α_6 is a continuum such that $\pi_1(\alpha_6) = OA$, $\pi_2(\alpha_6) = OB$, and $f\times f(\alpha_6)$ is a subset of Z_n.

Let

$$\alpha_7 = L_1^7\Big(\langle OB, OC\rangle, O\frac{A}{4}, \frac{2B}{3}B\Big) \cup L_2^7\Big(\langle OA, OC\rangle, \frac{A}{4}\frac{A}{2}, \frac{2B}{3}B\Big) \cup$$

$$\cup L_3^7\left(\left\langle\frac{3A}{4}A, 0\frac{A}{2}\right\rangle, \frac{7A}{16}\frac{A}{2}, \frac{B}{2}\frac{2B}{3}\right) \cup L_4^7\left(\left\langle\frac{3A}{4}A, 0\frac{A}{2}\right\rangle, \frac{7A}{16}\frac{A}{2}, \frac{B}{3}\frac{B}{2}\right)\cup$$
$$\cup L_5^7\left(\langle OA, OB\rangle, \frac{A}{4}\frac{A}{2}, 0\frac{B}{3}\right).$$

That α_7 is a continuum follows from the facts that $\left(\frac{A}{4}, \left(f|\frac{2B}{3}B\right)^{-1}(x_3)\right)$ is in $L_1^7\cap L_2^7$, $\left(\left(f|\frac{A}{4}\frac{A}{2}\right)^{-1}(x_1), \frac{2B}{3}B\right)$ is in $L_2^7\cap L_3^7$ since $\left(f|\frac{A}{4}\frac{A}{2}\right)^{-1}(x_1) = \left(f|\frac{7A}{16}\frac{A}{2}\right)^{-1}(x_1)$ because x_1 is in $\frac{3A}{4}A$, there is a point y_4 in $\frac{3A}{4}A$ such that $\left(y_4, \frac{A}{2}\right)$ is in $\left\langle\frac{3A}{4}A, 0\frac{A}{2}\right\rangle$, so $\left(\left(f|\frac{7A}{16}\frac{A}{2}\right)^{-1}(y_4), \frac{B}{2}\right)$ is in $L_3^7\cap L_4^7$, and $\left(\left(f|\frac{A}{4}\frac{A}{2}\right)^{-1}(x_1), \frac{B}{3}\right)$ is in $L_4^7\cap L_5^7$. Also, $\pi_1(\alpha_7) = 0\frac{A}{2}$, $\pi_2(\alpha_7) = OB$, and $f\times f(\alpha_7)$ is a subset of Z_n. Let $\alpha_8 = \alpha_7^{-1}$ and α_8 is a continuum such that $\pi_1(\alpha_8) = OB$, $\pi_2(\alpha_8) = 0\frac{A}{2}$, and $f\times f(\alpha_8)$ is a subset of Z_n.

Let
$$\alpha_9 = L_1^9\left(\langle OB, OC\rangle, 0\frac{A}{4}, \frac{C}{2}C\right)\cup L_2^9\left(\langle OA, OC\rangle, \frac{A}{4}\frac{A}{2}, \frac{C}{2}C\right)\cup$$
$$\cup L_3^9\left(\langle OA, OB\rangle, \frac{A}{4}\frac{A}{2}, 0\frac{C}{2}\right).$$

That α_9 is a continuum follows from the facts that $\left(\frac{A}{2}, \left(f|\frac{C}{2}C\right)^{-1}(x_3)\right)$ is in $L_1^9\cap L_2^9$ and $\left(\left(f|\frac{A}{4}\frac{A}{2}\right)^{-1}(x_1), \frac{C}{2}\right)$ is in $L_2^9\cap L_3^9$. Also, $\pi_1(\alpha_9) = 0\frac{A}{2}$ $\pi_2(\alpha_9) = OC$ and $f\times f(\alpha_9)$ is a subset of Z_n. Let $\alpha_{10} = \alpha_9^{-1}$ and α_{10} is a continuum such that $\pi_1(\alpha_{10}) = OC$, $\pi_2(\alpha_{10}) = 0\frac{A}{2}$, and $f\times f(\alpha_{10})$ is a subset of Z_n.

Let
$$\alpha_{11} = L_1^{11}\left(\langle OB, OC\rangle, 0\frac{A}{4}, \frac{3A}{4}A\right)\cup L_2^{11}\left(\langle OA, OC\rangle, \frac{A}{4}\frac{A}{2}, \frac{3A}{4}A\right).$$

Since $\left(\frac{A}{4}, \left(f|\frac{3A}{4}A\right)^{-1}(x_3)\right)$ is in $L_1^{11}\cap L_2^{11}$, α_{11} is a continuum. Also, $\pi_1(\alpha_{11}) = 0\frac{A}{2}$, $\pi_2(\alpha_{11}) = \frac{3A}{4}A$ and $f\times f(\alpha_{11})$ is a subset of Z_n. Let $\alpha_{12} = \alpha_{11}^{-1}$

and α_{12} is a continuum such that $\pi_1(\alpha_{12}) = \frac{3A}{4}A$, $\pi_2(\alpha_{12}) = 0\frac{A}{2}$, and $f\times f(\alpha_{12})$ is a subset of Z_n.

Let $Z_{n+1} = \bigcup_{i=1}^{12}\alpha_i$. Then Z_{n+1} is the union of twelve continua

$$\langle OB, OC\rangle' = \alpha_1,\ \langle OC, OB\rangle' = \alpha_2,\ \langle OA, OC\rangle' = \alpha_4,\ \langle OC, OA\rangle' = \alpha_3,$$
$$\langle OA, OB\rangle' = \alpha_6,\ \langle OB, OA\rangle' = \alpha_5,\ \left\langle 0\frac{A}{2}, OC\right\rangle' = \alpha_9,\ \left\langle OC, 0\frac{A}{2}\right\rangle' = \alpha_{10},$$
$$\left\langle 0\frac{A}{2}, OB\right\rangle' = \alpha_7, \left\langle OB, 0\frac{A}{2}\right\rangle' = \alpha_8, \left\langle 0\frac{A}{2}, \frac{3A}{4}A\right\rangle' = \alpha_{11}, \left\langle \frac{3A}{4}A, 0\frac{A}{2}\right\rangle' = \alpha_{12}$$

such that if $K = \langle t, u\rangle$ then $\pi_1(K) = t$ and $\pi_2(K) = u$. By construction $\langle t, u\rangle^{-1} = \langle u, t\rangle$. Thus (b) and (c) of the inductive hypothesis are satisfied by Z_{n+1}.

Let $x_1' = \left(f|\frac{3A}{4}A\right)^{-1}(z_1)$, $x_2' = \left(f|\frac{2B}{3}B\right)^{-1}(z_1)$, $x_3' = \left(f|\frac{C}{2}C\right)^{-1}(z_1)$, and $x_4' = \left(f|\frac{A}{4}\frac{A}{2}\right)^{-1}(z_2)$. Clearly, x_1' is in $\left[\frac{3A}{4}A\right]$, x_2' is in $\left[\frac{2B}{3}B\right]$, x_3' is in $\left[\frac{C}{2}C\right]$, and x_4' is in $\left[\frac{A}{4}\frac{A}{2}\right]$. Then, (x_1', O) is a point of α_6, α_4, and α_{12} while (O, x_1') is a point of α_5, α_3, and α_{11}. The point (x_2', O) is a point of α_5, α_1, and α_8 while (O, x_2') is a point of α_6, α_2, and α_7. The point (x_3', O) is a point of α_3, α_2, and α_{10} while (O, x_3') is a point of α_4, α_1, and α_9. The point (x_4', O) is a point of α_7 and α_9 while (O, x_4') is a point of α_8 and α_{10}. Therefore, (d) is satisfied.

Moreover, there is a point z_1' in OC such that (B, z_1') is in α_1 and (z_1', B) is in α_2, and there is a point z_2' in OA such that (B, z_2') is in α_5 and (z_2', B) is in α_6. Thus, (e) is satisfied.

From the above, each set in the following finite sequence is a continuum: $(\alpha_4\cup\alpha_6\cup\alpha_{12})$, $(\alpha_1\cup\alpha_4\cup\alpha_9)$, $(\alpha_5\cup\alpha_1\cup\alpha_8)$, $(\alpha_5\cup\alpha_3\cup\alpha_{11})$, $(\alpha_2\cup\alpha_3\cup\alpha_{10})$, $(\alpha_6\cup\alpha_2\cup\alpha_7)$. Moreover, each term of the sequence (except the last) intersects the term that follows it and the sum of all the terms of the sequence is Z_{n+1}, so Z_{n+1} is a continuum. Since, $\pi_i(\alpha_1\cup\alpha_2\cup\alpha_3\cup\alpha_4) = T$ for $i = 1, 2$, $\pi_1(Z_{n+1}) = \pi_2(Z_{n+1}) = T$. So, (a) is satisfied and Z_{n+1} is a subcontinuum of $T\times T$.

Finally, we show $f\times f(Z_{n+1}) = Z_n$. The continua $\langle OB, OC\rangle$, $\langle OA, OC\rangle$, $\langle OA, OB\rangle$, $\left\langle 0\frac{A}{2}, OC\right\rangle$, $\left\langle 0\frac{A}{2}, OB\right\rangle$, $\left\langle 0\frac{A}{2}, \frac{3A}{4}A\right\rangle$ are all subsets of

$f\times f(a_1\cup a_5\cup a_7\cup a_9)$ while $\langle OC, OB\rangle$, $\langle OC, OA\rangle$, $\langle OB, OA\rangle$, $\left\langle OC, O\frac{A}{2}\right\rangle$, $\left\langle OB, O\frac{A}{2}\right\rangle$, $\left\langle\frac{3A}{4}A, O\frac{A}{2}\right\rangle$ are all subsets of $f\times f(a_2\cup a_6\cup a_8\cup a_{10})$. Thus, Z_n is a subset of $f\times f(Z_{n+1})$. Since for each i, $f\times f(a_i)$ is a subset of Z_n, $f\times f(Z_{n+1}) = Z_n$.

THEOREM 3. *The continuum M has positive span, and thus is not chainable.*

Proof. Suppose k is a positive integer and select $Z_1, Z_2, \ldots$ satisfying Theorem 2. For each n let $T_n = T$ and let $g_n = f$ if $n\leqslant k-1$ while g_n is the identity on T if $n\geqslant k$. Then let Y_k denote the inverse limit of the sequence $\{T_i, g_i\}$. Let h: $T_k\to Y_k$ be defined by $h(x) = (f^{k-1}(x), \ldots, f(x), x, x, \ldots)$. Then h is a homeomorphism throwing T_k onto Y_k. The set $W = h\times h(Z_k)$ is a connected subset of $Y_k\times Y_k$ with both projections onto Y_k. If (p, q) is a point of W, $d(p, q)\geqslant \frac{1}{2}d\big(f^{k-1}(p_k), f^{k-1}(q_k)\big)\geqslant\frac{1}{4}$ since $\big(f^{k-1}(p_k), f^{k-1}(q_k)\big)$ is in Z_1. So $\sigma Y_k\geqslant\frac{1}{4}$.

Since M is the sequential limiting set of $Y_1, Y_2, \ldots$, $\sigma M\geqslant\limsup\sigma Y_k\geqslant\frac{1}{4}$ [5]. So $\sigma M > 0$, and M is not chainable since every chainable continuum has span zero ([5], p. 210).

5. Remarks. In [6] Lelek asks: If X is tree-like, is $\sigma X = \operatorname{lub}\{w(Y)|\ Y$ is a subcontinuum of $X\}$ (where $w(Y)$ denotes the width of Y [3])? Since M is atriodic, M has width zero hereditarily [3], Theorem 5, so the continuum M provides a negative answer to this question.

The following theorem could be used to argue from Theorem 2 that M has positive span.

THEOREM 4. *Suppose X is the inverse limit of the inverse limit sequence $\{X_i, f_i\}$ with each X_i compact, $\varepsilon > 0$, and $f_1^n = f_1\circ f_2\circ\ldots\circ f_{n-1}$. If $\sigma f_1^n\geqslant\varepsilon$ for each n, then $\sigma X > 0$.*

Proof. Suppose $(x_1, x_2, \ldots)$ is a point of X. Then h_i: $X_i\to\prod_{j>0} X_j$ defined by $h_i(y) = (f_1^i(y), \ldots, f_{i-1}(y), y, x_{i+1}, x_{i+2}, \ldots)$ is a homeomorphism for each i. Since $\sigma f_1^i\geqslant\varepsilon$, there is a connected subset Z_i of $X_i\times X_i$ such that $\pi_1(Z_i) = \pi_2(Z_i)$ and if (x, y) is in Z_i then $d\big(f_1^i(x), f_1^i(y)\big)\geqslant\frac{1}{2}\varepsilon$. Thus $h_i\times h_i(Z_i)$ is a connected subset of $h_i(X_i)\times h_i(X_i)$ such that its two projections to $h_i(X_i)$ are the same point set and if (a, b) is in $h_i\times h_i(Z_i)$ then $d(a, b)\geqslant\frac{1}{4}\varepsilon$. Thus, $\sigma h_i(X_i)\geqslant\frac{1}{4}\varepsilon$, and, since X is the sequential limiting set of $h_i(X_1), h_i(X_2), \ldots$, $\sigma X\geqslant\frac{1}{4}\varepsilon$.

In conclusion we remark that M is homeomorphic to a plane continuum which can be constructed as the intersection of a defining sequence of tree chains $T_1, T_2, \ldots$ each having only one junction link with T_{n+1} following the pattern suggested by f in T_n.

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