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Rings in which ideals are annihilators

by

Michał Jaegermann and **Jan Krempa** (Warszawa)

Introduction. The aim of this paper is to present some structure theorems about associative rings in which every left ideal is a left annihilator (LA-rings). We develop here the ideas given in paper [8] by C. R. Yohe. In particular, we obtain answers to the problems mentioned there.

In § 1 we prove a structure theorem for semiprime rings in which every two-sided ideal is an annihilator (ILA-rings). This theorem enables us in § 2 to give a complete characterization of semiprime LA-rings. In § 3 we shall prove that every semiprime LA-ring R which satisfies some simple finiteness conditions is a finite direct sum of matrix rings.

In § 4 we consider L(1)A-R(1)A-rings (i.e. rings in which every one-sided ideal is an annihilator of one element). These are rings with unity and Theorem 4.2 characterizes such rings as finite direct sums of matrix rings over rings of some special kind.

§ 5 contains some remarks on commutative LA-rings.

In § 6 we give some examples.

In this paper by an ideal of a ring we shall mean a two-sided ideal. R is a simple ring if $R^2 \neq 0$ and if R has no ideals different from 0 and R. A semiprime ring is a ring without non-zero nilpotent ideals. All one-sided definitions and results are generally stated in their left versions. The right versions are used without any further mention.

§ 1. Semiprime ILA-rings. If S is a non-empty subset of a ring R the $l_R(S) = \{x \in R|\ xS = 0\}$ will be called a *left annihilator* and $r_R(S) = \{x \in R|\ Sx = 0\}$ a *right annihilator* of the set S. We shall use $l(S)$ instead of $l_R(S)$ and $r(S)$ instead of $r_R(S)$ if there is no confusion; $l(S)$ and $r(S)$ defined in this way are a left and a right ideal of R, respectively.

A left ideal I of R is said to be a *left annihilator in* R if $I = l(S)$ for some $S \subseteq R$.

The proofs of the following two propositions are straightforward and we omit them.

Proposition 1.1. (i) *Let S and T be non-empty subsets of a ring R. Then $l(S) = l(r(l(S)))$, $l(S \cup T) = l(S) \cap l(T)$ and if $S \subseteq T$, then $l(S) \supseteq l(T)$.*

(ii) *Let $I = l(S)$ and $J = l(T)$. If $r(I) = r(J)$, then $I = J$.* ■

Proposition 1.2. *Let A be a direct summand of a ring R. If $I \subseteq A$ is a left annihilator in R, then I is a left annihilator in A.* ■

Proposition 1.3. *For any ideal of a semiprime ring R we have $l(I) = r(I)$.*

Proof. We have $(I \cdot l(I))^2 = 0$. But $I \cdot l(I)$ is a left ideal of a semiprime ring R. Hence $I \cdot l(I) = 0$. Therefore $l(I) \subseteq r(I)$. Considering the right ideal $r(I)I$, we obtain the converse inclusion. ■

In view of Proposition 1.3 we agree to write, for any ideal of a semiprime ring R, $a(I)$ instead of $l(I) = r(I)$. Of course $a(I)$ is an ideal of R.

Definition. A ring R is called an ILA-*ring* if each of its ideals is a left annihilator in R.

Using Propositions 1.1 and 1.3 we can easily see that for any ideal I of a semiprime ILA-ring $a(a(I)) = I$.

Proposition 1.4. *If I is an ideal of a semiprime* ILA-*ring R, then $R = I \oplus a(I)$.*

Proof. We have $(I \cap a(I))^2 \subseteq I \cdot a(I) = 0$. Hence $I \cap a(I) = 0$ since R is semiprime. Let us take $S = I \oplus a(I)$. We have $S \supseteq I \cup a(I)$; thus by Proposition 1.1 $a(S) \subseteq a(I) \cap a(a(I)) = a(I) \cap I = 0$. Since R is semiprime, $a(R) = 0$. Then $a(S) = a(R)$. Applying Proposition 1.1, we get $R = S$. ■

Corollary 1.5. *Let R be an* ILA-*ring. Then R is prime if and only if R is simple.* ■

Corollary 1.6. *Let R be a semiprime* ILA-*ring and let P be a prime ideal of R. Then the annihilator $a(P)$ is a simple ring and P is a maximal ideal of R.* ■

Theorem 1.1. *A ring R is a semiprime* ILA-*ring if and only if R is a direct sum of simple rings.*

Proof. Let R be a semiprime ILA-ring and let $\mathfrak{P}$ be the set of all prime ideals of R. Let A be the sum (as the sum of ideals) of all annihilators $a(P)$, where $P \in \mathfrak{P}$. For any $P \in \mathfrak{P}$ we have $A \supseteq a(P)$. Then $a(A) \subseteq \bigcap a(a(P)) = \bigcap P = 0$ since R is semiprime. Therefore $R = A \oplus a(A) = A$; i.e. R is a sum of ideals $a(P), P \in \mathfrak{P}$, which by Corollary 1.6 are simple as rings. Now it is not hard to verify that R is a direct sum of simple rings.

The converse implication is obvious. ■

§ 2. Semiprime LA-rings.

Definition. A ring R is said to be a *left annihilator ring* (an LA-*ring*, for short) if each of its left ideals is a left annihilator in R.

In an analogous way one can define a *right annihilator ring* (an RA-*ring*). Each LA-ring is of course an ILA-ring.

Proposition 2.1. *Let R be an* LA-*ring and $r(R) = 0$. Then for any subset $S \subseteq R$ we have $S \subseteq RS$, where RS is the set of all finite sums of the form $\sum a_i s_i$, $a_i \in R$, $s_i \in S$.*

Proof. Since $R \cdot Sr(RS) = 0$, we have $Sr(RS) \subseteq r(R) = 0$. Therefore $S \subseteq l(r(RS)) = RS$. The last equality follows from Proposition 1.1 (i) because R is an LA-ring.

Proposition 2.2. A *prime* LA-*ring is simple and contains a minimal one-sided ideal.*

Proof. Let R be a prime LA-ring. Then by Corollary 1.5 R is simple. We will show that R contains a minimal one-sided ideal. Let $x \in R$, $x \neq 0$. Then by Proposition 2.1 $x \in Rx$. Thus $x = yx$ for some $y \in R$. It is easy to check that y cannot be quasi regular. Therefore the ring R cannot be radical in Jacobson's sense. It is well known ([5], Theorem I.6.1) that R contains a maximal left ideal I. Since R is an LA-ring, we have $I = l(S)$ for some $S \subseteq R$. By Proposition 1.1 $I = l(K)$ for the right ideal $K = r(l(S))$. Since $I \neq R$, we have $K \neq 0$. Hence $K^2 \neq 0$ because R is prime. Therefore $pK \neq 0$ for some $p \in K$. Hence $l(pK) \neq R$, since otherwise $(pK)^2 = 0$, which is impossible. Since $IpK = 0$, we have $I \subseteq l(pK)$. But I is maximal, and therefore $I = l(pK)$.

Now we shall prove that $Rp \cap I = 0$. Let us take any $s \in Rp \cap I$. Then $s = s_1 p$ for some $s_1 \in R$. But, on the other hand, $s \in I = l(K)$. Hence $s_1 pK = 0$, i.e. $s_1 \in l(pK) = I$. Then $s = s_1 p \in Ip \subseteq IK = 0$. Therefore $Rp \cap I = 0$.

Since $Rp \neq 0$ and I is a maximal left ideal of R, a left R-module R is a direct sum of R-modules Rp and I. Therefore Rp is a minimal left ideal of R.

It is well known ([5], Theorem IV.16.3) that a ring R is isomorphic to the complete ring of linear transformations of finite rank of a vector space over a division ring if and only if R is a simple LA-ring containing minimal one-sided ideals. Therefore we have

Theorem 2.1. *A ring R is a prime* LA-*ring if and only if R is isomorphic to the complete ring of linear transformations of finite rank of a vector space over a division ring.*

It is also known ([5], Theorem IV. 16.4) that a simple ring with minimal one-sided ideals is right and left Artinian if and only if R is an LA-RA-ring. Therefore, applying the Wedderburn theorem, we get

THEOREM 2.2. *A ring R is a prime* LA-RA-*ring if and only if R is isomorphic to the matrix ring over a division ring.*

PROPOSITION 2.3. *Let R be a direct sum of rings R_t, $t \in T$, and let $r_{R_t}(R_t) = 0$ for every $t \in T$. Then R is an* LA-*ring if and only if the rings R_t, $t \in T$, are* LA-*rings.*

Proof. If R is an LA-ring, then by Proposition 1.2 R_t, $t \in T$, is also an LA-ring.

Conversely, let R_t, $t \in T$, be an LA-ring and let I be a left ideal of R. By K we denote the sum of all $R_t \cap I$, $t \in T$. Then obviously $K \subseteq I$. Now let $a \in I$. Then $a = a_1 + a_2 + \ldots + a_k$, where $a_i \in R_{t_i}$. By Proposition 2.1 $a_i = y_i a_i$ for some $y_i \in R_{t_i}$. Therefore $a_i = y_i a_i = y_i a \in I$, i.e. $a \in K$. Hence we have proved that every left ideal I of R is a direct sum of left ideals $R_t \cap I$.

Since the rings R_t, $t \in T$, are LA-rings, we have $R_t \cap I = l_{R_t}(S_t)$ for some $S_t \in R_t$. It is not hard to check that $I = l(S)$, where S is a set-theoretical union of S_t, $t \in T$. ■

Applying Theorem 1.1, Proposition 2.3, Theorem 2.1 and respectively Theorem 2.2 we obtain

THEOREM 2.3. *A ring R is a semiprime* LA-*ring if and only if R is a direct sum of complete rings of linear transformations of finite rank of vector spaces over division rings.* ■

THEOREM 2.4. *A ring R is a semiprime* LA-RA-*ring if and only if R is a direct sum of matrix rings over division rings.* ■

§ 3. Semiprime LA-rings with finiteness conditions.

DEFINITION. A ring R is an LFA-*ring* if each of its left ideals can be expressed as an annihilator of a finite subset of R. A ring R is an L(n)A-*ring* if each of its left ideals can be expressed as an annihilator of a set with at most n elements from R.

In an analogous way one can define RFA- and R(n)A-*rings*.

PROPOSITION 3.1. *If an* LFA-*ring (resp.* L(n)A-*ring) is a direct sum of rings ($\neq 0$), then that sum is finite and every direct summand is an* LFA-*ring (resp.* L(n)A-*ring).*

Proof. Let an LFA-ring R be a direct sum of rings R_t, $t \in T$. By assumption, $0 = l(S)$ for some

$$S = \left\{\sum s_{1t}, \sum s_{2t}, \ldots, \sum s_{kt}\right\} \subseteq R\,, \quad \text{where} \quad t \in T,\ s_{it} \in R_{t_i},$$

and all but finite s_{it} equal 0. T_1 is a finite subset of T which contains such $t \in T$ that there exist $s_{it} \neq 0$ for some $i = 1, 2, \ldots, k$. Since $t_0 \in T \setminus T_1$ implies $T_{t_0} \subseteq l(S) = 0$, we have $T = T_1$ and T is finite.

The proof of the second part is straightforward. ■

PROPOSITION 3.2. *Let $R = \bigoplus_{i=1}^{m} R_i$ and $r_{R_i}(R_i) = 0$ for $i = 1, 2, \ldots, m$. Then* (i) *if every R_i is an* L(p_i)A-*ring, then R is an* L(q)A-*ring, where $q = \max_i p_i$;* (ii) *if every R_i is an* LFA-*ring, then R is also an* LFA-*ring.*

Proof of (i). Let I be a left ideal of R. As in Proposition 2.3 $I = \bigoplus_{i=1}^{m} (I \cap R_i)$. There exist s_{ik}, $i = 1, 2, \ldots, m$, $k = 1, 2, \ldots, q$, such that $s_{ik} \in R_i$, $s_{ik} = 0$ if $k > p_i$ and $I \cap R_i = l_{R_i}(s_{i1}, s_{i2}, \ldots, s_{iq})$. It is not hard to check that $I = l_R(\sum_i s_{i1}, \sum_i s_{i2}, \ldots, \sum_i s_{iq})$.

Analogically one can prove (ii). ■

THEOREM 3.1. *For a semiprime* LA-*ring R the following conditions are equivalent:*

(i) *R is an* L(1)A-R(1)A-*ring.*
(ii) *R is an* LFA-*ring.*
(iii) *R is finitely generated as an R-module (left or right).*
(iv) *R is a ring with a unity element.*
(v) *R satisfies* ACC *or* DCC *on left ideals.*
(vi) *R satisfies* ACC *or* DCC *on right annihilators or right ideals.*
(vii) *R is an* RA-*ring and does not contain infinite direct sums of two-sided ideals.*
(viii) *R is a finite direct sum of matrix rings over division rings.*

Proof. Every left ideal of a matrix ring M over a division ring has the form Me, where e is an idempotent. Hence $Me = l(1-e)$ and M is an L(1)A-ring. Analogously M is an R(1)A-ring. Now, as a simple consequence of Theorem 2.4 and Propositions 3.1 and 3.2 (with their right versions), we obtain the equivalence of conditions (i), (vii) and (viii).

We have obvious implications (i) ⇒ (ii), (viii) ⇒ (iv) ⇒ (iii), (viii) ⇒ ⇒ (vi) ⇒ (v). Hence it is enough to prove (ii) ⇒ (viii), (iii) ⇒ (viii), (v) ⇒ (viii).

By $\mathcal{F}(\mathfrak{M})$ we denote the ring of all linear transformations of finite rank of a vector space $\mathfrak{M}$ over a division ring D. If the dimension of $\mathfrak{M}$ is finite and equals n, we shall identify, in the same way as in [5], $\mathcal{F}(\mathfrak{M})$ and the ring of all $n \times n$ matrices over D.

If $R = \mathcal{F}(\mathfrak{M})$ for some vector space $\mathfrak{M}$, then every left ideal of R has the form

$$I = I(\mathfrak{N}) = \{F \in R \mid \mathfrak{M}F \subseteq \mathfrak{N}\}$$

for some subspace $\mathfrak{N}$ of $\mathfrak{M}$ (cf. [5], Theorem IV. 16.1). Let us observe that a vector x belongs to $\mathfrak{N}$ if and only if there exists an $F \in I(\mathfrak{N})$ such that $\mathfrak{M}F = Dx$. From this it follows that $I(\mathfrak{N}) \supseteq I(\mathfrak{N}')$ if and only if $\mathfrak{N} \supseteq \mathfrak{N}'$, where $\mathfrak{N}, \mathfrak{N}'$ are subspaces of $\mathfrak{M}$.

(A). Proof (ii) ⇒ (viii). From Theorem 2.3 and Proposition 3.1 it follows that we have only to prove that if an LFA-ring R is of the form $\mathcal{F}(\mathfrak{M})$ for some vector space $\mathfrak{M}$, then $\mathfrak{M}$ is finite-dimensional.

Let us consider a zero-dimensional subspace $\mathfrak{N}$ of $\mathfrak{M}$. $I(\mathfrak{N})$ is a non-zero left ideal of $\mathcal{F}(\mathfrak{M})$. Thus there exist linear transformations $F_1, F_2, \ldots, F_k$ such that

$$I(\mathfrak{N}) = l(F_1, F_2, \ldots, F_k) = \{F \in \mathcal{F}(\mathfrak{M}) \mid \mathfrak{M}F \subseteq \bigcap_{i=1}^{k} \ker F_i\} = I(\bigcap_{i=1}^{k} \ker F_1).$$

This implies $\mathfrak{N} = \bigcap_{i=1}^{k} \ker F_i$.

Let us consider the linear transformation

$$G\colon \mathfrak{M} \to \mathfrak{M}F_1 \times \mathfrak{M}F_2 \times \ldots \times \mathfrak{M}F_k,$$

$$x \to (xF_1, xF_2, \ldots, xF_k), \qquad x \in \mathfrak{M}.$$

Then $\ker G = \bigcap \ker F_i = \mathfrak{N}$. Hence $\mathfrak{M}$ has the same dimension as a $\mathfrak{M}G$ which is a subspace of the finite-dimensional space $\mathfrak{M}F_1 \times \mathfrak{M}F_2 \times \ldots \times \mathfrak{M}F_k$.

(B). Proof (iii) ⇒ (viii). From Theorem 2.3 it follows immediately that R is a finite direct sum of rings of type $\mathcal{F}(\mathfrak{M})$ which are finitely generated as $\mathcal{F}(\mathfrak{M})$-modules.

If $R = \mathcal{F}(\mathfrak{M})$ is a finitely generated left R-module with generators $F_1, F_2, \ldots, F_k$, then for every $F \in R$ there exist transformations $G_1, G_2, \ldots, G_k$ from R such that

$$F = G_1F_1 + G_2F_2 + \ldots + G_kF_k.$$

By $\varrho(H)$ we denote the rank of transformation H. Now from the properties of the rank of linear transformation we obtain for every $F \in R$

$$\varrho(F) \leqslant \varrho(F_1) + \varrho(F_2) + \ldots + \varrho(F_k) = m.$$

Therefore $\dim \mathfrak{M} \leqslant m$.

If R is a right R-module, the proof is analogous.

(C). Proof (v) ⇒ (viii). As in (B) we may assume that $R = \mathcal{F}(\mathfrak{M})$. From the remark on the form of left ideals in $\mathcal{F}(\mathfrak{M})$ it follows that $\mathfrak{M}$ satisfies ACC or DCC on its subspaces, whence $\mathfrak{M}$ is finite-dimensional. ■

§ 4. L(1)A-R(1)A-rings. The following useful proposition results directly from Proposition 1.1.

PROPOSITION 4.1. *Let R be an LA-RA-ring. Then the following maps $I \to r(I)$, $K \to l(K)$, where I is a left and K is a right ideal of R, are mutually inverse antiisomorphisms of lattices of left and right ideals of R. In particular, $l(R) = r(R) = 0$.* ■

THEOREM 4.1. *For any LA-RA-ring R the following conditions are equivalent:*

(i) *R is an RFA-ring.*

(ii) *R satisfies ACC on left ideals (equivalently DCC on right ideals).*

(iii) *R is a left and right Artinian ring.*

(iv) *R is an $L(n)A$-$R(n)A$-ring with unity for some n.*

We give a cyclic proof.

(A). Proof (i) ⇒ (ii). Let I be a left ideal of a ring R. By Proposition 4.1 $I = l(r(I))$. Since R is an RFA-ring, there exists a finite set $S = \{s_1, s_2, \ldots, s_n\} \subseteq R$ such that the right ideal $r(I)$ can be expressed in the form $r(I) = r(S) = r(S^*)$, where S^* is a left ideal generated by S. Thus $I = S^*$. Hence every left ideal of R is finitely generated. This implies that R satisfies ACC on left ideals. By Proposition 4.1 this is equivalent to DCC on right ideals.

(B). Proof (ii) ⇒ (iii). We start with the proof that R has a unity element. Since R satisfies DCC on right ideals, the Jacobson radical $J = J(R)$ of R is a nilpotent ideal. Hence $l(J) \neq 0$, and thus $r(l(J)) = J \neq R = r(0)$.

Let q be the index of nilpotency of J. We shall prove that for every k, $1 \leqslant k \leqslant q$, a ring R/J^k has a unity. We proceed by induction. The ring R/J is semisimple and satisfies DCC on right ideals, and so it must have a unity element. Let us suppose that for every $m < k$ R/J^m is a ring with a unity. By Proposition 2.1, $Rx \subseteq J^k$ implies $x \in J^k$. Hence $r(R/J^k) = 0$. Analogically $l(R/J^k) = 0$. Let us consider ideals J/J^k and J^{k-1}/J^k of the ring R/J^k. Rings

$$R/J^k \big/ J/J^k = R/J \quad \text{and} \quad R/J^k \big/ J^{k-1}/J^k = R/J^{k-1}$$

have unity elements by the induction assumption. Moreover, $J/J^k \cdot J^{k-1}/J^k = 0$. Lemma 2 from [3] implies that R/J^k is a ring with unity.

Now it remains to prove that R satisfies DCC on left ideals or equivalently ACC on right ideals. But this follows directly from the fact that R satisfies DCC on right ideals and from the Hopkins theorem ([4], Theorem IV. 29).

(C). Proof (iii) ⇒ (iv). Because R satisfies ACC and DCC on left ideals, there exists a composition series of left ideals of a length n. By Proposition 4.1 we infer that correspondent right annihilators form a composition series of right ideals of the length n. In particular, every left and right ideal has a set of generators with at most n elements in it. Hence every left ideal of R can be expressed in the form $I = l(s_1, s_2, \ldots, s_k)$, where $k \leqslant n$ and $s_1, s_2, \ldots, s_k$ are generators of a right ideal $r(I)$. That means that R is an $L(n)A$-ring. Analogously R is an $R(n)A$-ring.

(D). Proof (iv) $\Rightarrow$ (i). Obvious. ■

DEFINITION. An LA-RA-ring which satisfies one of the conditions of Theorem 4.1 is called a *quasi-Frobenius ring* (a QF-*ring*).

DEFINITION. A ring R is called a *pli ring* if R has a unity element and each left ideal of R is principal, i.e. has the form Rx for some $x \in R$.

Analogously we define a *pri ring*.

COROLLARY 4.2. *If R is an* LA-R(1)A-*ring, then R is a right and left Artinian pli ring.*

Proof. We have only to prove that R is a pli ring. But this fact is contained in the proof of Theorem 4.1, part (A). ■

Our aim in this section is to give a complete characterization of LA-R(1)A-rings. Since every such ring has a unity element, in the sequel we shall only deal with rings with unity.

DEFINITION. A ring R is a *left self-injective ring* if each homomorphism (of left R-modules) from a left ideal of R into R is a right multiplication by a suitable element of R.

It is known (cf. [2]) that R is a QF-ring if and only if R is a left self-injective and left Noetherian ring.

PROPOSITION 4.3. *If R is a pli ring and every principal right ideal xR is a right annihilator in R, then R is a* QF-*ring.*

Proof. It is enough to prove that R is a left self-injective ring, because a pli ring is obviously a left Noetherian ring.

Let I be a left ideal of R and let $\varphi\colon I \to R$ be a homomorphism. There exists an element $x \in R$ such that $I = Rx$. If for some $y \in R$ we have $yx = 0$, then $\varphi(yx) = y\varphi(x) = 0$. Hence $l(xR) = l(x) \subseteq l(\varphi(x)) = l(\varphi(x)R)$. But xR and $\varphi(x)R$ are right annihilators and hence by Proposition 1.1 (ii) $xR \supseteq \varphi(x)R$. Therefore there exists such a $t \in R$ that $\varphi(x) = xt$. Since every $a \in I$ has a form $a = bx$ for some $b \in R$, we obtain $\varphi(a) = \varphi(bx) = b\varphi(x) = bxt = at$, which means that R is self-injective. ■

As usual, by R_n we mean the ring of all $n \times n$-matrices over R.

PROPOSITION 4.4. *If R_n is an* RA-*ring, then R is also an* RA-*ring.*

The proof can easily be obtained by standard arguments. ■

DEFINITION. A ring R is a *primary ring* if for every ideal A, B of R, $AB = 0$ implies either $A = 0$ or $B^n = 0$ for some n. A ring is called a *completely primary ring* if $R/J(R)$ is a division ring, where $J(R)$ is the Jacobson radical of R.

For one-sided Artinian rings this definition coincides with that in [6].

The next proposition is a part of Theorem II.4.12 from [6] and we state it only for the sake of completeness.

PROPOSITION 4.5. *Let R be a primary left Artinian ring in which the Jacobson radical $J(R)$ is a principal left ideal of R. Then R is a pli ring and there exists a completely primary left Artinian pli ring B such that $R = B_n$ for some natural n. Moreover, every left ideal of B is two-sided and is of the form $(Bw)^k = Bw^k$, where $Bw = J(B)$ and $0 \leqslant k \leqslant q =$ the index of nilpotency of w.* ■

PROPOSITION 4.6. *Let R be as above. If every principal right ideal of R is a right annihilator, then there exists a completely primary left and right Artinian pli pri ring B such that $R = B_n$ for some natural n.*

Proof. By Proposition 4.5, $R = B_n$ for some completely primary left Artinian pli ring B and R is a pli ring. Then by Proposition 4.3 R is a QF-ring, and in particular an RA-ring. Proposition 4.4 implies that B is also an RA-ring. That means that every right ideal of B is an annihilator of some left ideal of the ring B. Hence there exist at most $q+1$ right ideals of B (q is the index of nilpotency of $J(B)$). But the left ideals $B, Bw, \ldots, Bw^{q-1}, Bw^q = 0$ are two-sided, and therefore the only right ideals of B are Bw^k, $k = 0, 1, \ldots, q$. Therefore B is right Artinian.

Since $1 \notin wB$, $w \in wB$ and Bw is the only right ideal of B with these properties, we have $wB = Bw$. This implies $w^kB = Bw^k$ for $k = 0, 1, \ldots, q$, which means that B is a pri ring. ■

PROPOSITION 4.7. *If B is a left and right Artinian completely primary pli pri ring, then for every natural n the ring B_n is an* L(1)A-R(1)A-*ring.*

Proof. From Proposition 4.5 it follows directly that all left ideals of B are of the form Bw^k, where $0 \leqslant k \leqslant q =$ the index of nilpotency of w. Now it is easy to note that $Bw^k = l(w^{q-k})$, which means that B is an LA-ring. Analogically B is an RA-ring. Since B is left and right Artinian, B is a *QF*-ring ([1], Theorem 59.7). Moreover, B_n is a pli pri ring ([4], Theorem IV.40). This implies that B_n is an L(1)A-R(1)A-ring.

DEFINITION. A ring R is called an *ipli ring* if every two-sided ideal of R can be expressed in the form Rx for some $x \in R$.

In an analogous way we define an *ipri ring*. Obviously, every pli ring is an ipli ring.

THEOREM 4.2. *For every ring R the following conditions are equivalent:*

(i) *R is an* L(1)A-R(1)A-*ring.*

(ii) *R is an* LA-R(1)A-*ring.*

(iii) *R is a pli ring and every principal right ideal xR is a right annihilator.*

(iv) *R is a left Artinian ipli ring and every principal right ideal xR is a right annihilator.*

(v) *R is a left and right Artinian ipli ipri ring.*

(vi) *R is a left Artinian ipli ring and the Jacobson radical $J(R)$ is a principal right ideal.*

(vii) *R is a finite direct sum of ideals R_i, $i = 1, 2, \ldots, m$, and every R_i is the ring of all $n_i \times n_i$-matrices over a completely primary left and right Artinian pli pri ring B_i.*

Moreover, m, R_i, B_i, and n_i in condition (vii) *are uniquely determined up to an isomorphism.*

Remark. Contrary to the case of semiprime LA-rings it is not possible to drop the assumptions on right ideals of R.

Proof. (A). The implication (i) $\Rightarrow$ (ii) is obvious. The implications (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) follow directly from Corollary 4.2 and (iii) $\Rightarrow$ (iv) from Proposition 4.3.

(B). Proof (iv) $\Rightarrow$ (vii). Let R' be a direct summand of a ring R which satisfies condition (iv). Then, for any $x \in R'$, $xR' = xR$, and hence xR' is a right annihilator in R; therefore, by Proposition 1.2, xR' is a right annihilator in R'. Moreover, R' is obviously a left Artinian ipli ring. It is easy to see that every left Artinian ipli ring satisfies the assumptions of Johnson's theorems ([6], Theorems II.4.1 and II.4.2). Now, applying the first theorem of Johnson we could represent R as a finite direct sum of primary rings R_i, $i = 1, 2, \ldots, m$. Every ring R_i satisfies the assumptions of Proposition 4.6. Then for every R_i there exists a completely primary left and right Artinian pli pri ring B_i such that R_i is the ring of all $n_i \times n_i$-matrices over B_i.

(C). The implication (i) $\Rightarrow$ (v) follows directly from Corollary 4.2. The implication (v) $\Rightarrow$ (vi) is obvious.

(D). Proof (vi) $\Rightarrow$ (vii). As in (B) we could represent R as a direct sum of rings R_i, $i = 1, 2, \ldots, m$, where R_i are primary left Artinian ipli rings. The Second Theorem of Johnson [6] forced that every $R_i/J(R_i)$ is a prime ring. Hence $R_i/J(R_i)$, as a left Artinian ring, is simple. Of course $J(R_i)$ is a principal right ideal of R_i. Now, applying Theorems IV.38 and IV.39 from [4], we directly infer that R_i are of the required form.

(E). Proof (vii) $\Rightarrow$ (i). This is a simple corollary of Propositions 4.7 and 3.2.

(F). Proof of the uniqueness of the representation. Let $\bigoplus_{i=1}^{m} R_i$ be a representation as was described in (vii). By Proposition 4.5 all proper ideals of B_i are contained in $J(B_i)$ and hence are nilpotent. This implies that every proper ideal of R_i is nilpotent. Hence R_i, as a ring with a unity element, could not be the sum of its proper ideals. Applying standard arguments, we infer that m and the ideals R_i are determined uniquely up to the ordering of summands. The fact that n_i and B_i are uniquely determined by R_i up to an isomorphism is mentioned in [4], Theorem IV.10.

Corollary 4.8. *Every homomorphic image and a finite direct sum of* L(1)A-R(1)A*-rings is also an* L(1)A-R(1)A*-ring.*

Proof. This follows directly from the equivalence of conditions (i) and (v) of Theorem 4.2. ■

§ 5. Remarks on commutative A-rings. In the commutative case we will use the terms A-ring, FA-ring and (n)A-ring for the terms LA-ring, LFA-ring and L(n)A-ring, respectively.

By Theorem 4.1 and well-known facts about commutative rings ([9] Theorem IV.3, [7] Theorem 1.1) one could obtain

Theorem 5.1. *For a commutative ring R the following conditions are equivalent:*

(i) *R is an* A*-ring and R satisfies* DCC *(or equivalently* ACC*) on ideals.*

(ii) *R is an* FA*-ring.*

(iii) *There exists such an n that R is* (n)A*-ring.*

(iv) *R is a ring with a unity element and R is a finite direct sum of completely primary, i.e. local, Artinian rings B_i, $i = 1, 2, \ldots, m$, such that every B_i has exactly one minimal ideal. Moreover, there exists such an n that every ideal of B_i has at most n generators.*

The representation described above is unique up to an isomorphism. ■

Remark 1. From Proposition 2.3 it follows that every direct sum of local Artinian A-rings is an A-ring. But Example C will provide an A-ring which is not of this form.

Remark 2. From Proposition 4.7 it follows that every local Artinian ring in which every ideal is principal is an A-ring. This means that R has exactly one minimal ideal. (This fact was proved directly in [8].) Similiar theorem is not generally true if ideals of R have (at most) n generators, $n > 1$. For example $R = Z_2[X, Y]/(X^2, Y^2, XY)$ is a local Artinian ring and all ideals of R are $R = (1)$, (X, Y), 0, (X), (Y), $(X+Y)$. The last three ideals are minimal, whence R could not be an A-ring.

There also exists a local ring with exactly one minimal ideal which is not an A-ring. For this see Example C.

§ 6. Examples.

A. An example of an L(n)A-ring which is not an L$(n-1)$A-ring.

From Theorem 3.1 it follows that every semiprime LFA-ring is an L(m)A-ring for every natural m. It is not true in the general case. For example, let $R = K[X_1, X_2, \ldots, X_n]/N$, where $K[X_1, X_2, \ldots, X_n]$ is a polynomial ring over a (commutative) field K of characteristic at least n and let N be the ideal of $K[X_1, X_2, \ldots, X_n]$ generated by the set

$$\{X_1^2, X_2^2, \ldots, X_n^2\} \cup \cup \{(X_1X_2 - X_iX_j) \mid (1 \leqslant i \leqslant n) \text{ and } (1 \leqslant j \leqslant n) \text{ and } (i \neq j)\}.$$

One could verify that R is a local Artinian ring in which $(X_1 X_2)$ is a single minimal ideal, that every ideal has at most n generators and that $(X_1, X_2, \ldots, X_n)$ has exactly n generators. By Theorem 5.1 R is an A-ring. Now it is easy to see that R is an (n)A-ring and is not an $(n-1)$A-ring.

B. An example of a left and right Artinian completely primary pli L(1)A-ring with unity which is not an RA-ring.

Let $K(X)$ be the field of rational functions over a field K and let φ be the endomorphism of $K(X)$ which maps $p(X)$ onto $p(X^2)$.

Let R be the ring of all formal sums p_1+p_2Y, $p_1, p_2 \in K(X)$, where addition and multiplication is defined as follows

$$(p_1+p_2Y)+(q_1+q_2Y) = (p_1+q_1)+(p_2+q_2)Y,$$

$$(p_1+p_2Y)(q_1+q_2Y) = p_1q_1+\big(p_1q_2+p_2\cdot\varphi(q_1)\big)Y.$$

Then every element of a proper one-sided ideal of R has the form pY, $p \in K(X)$. Thus all left ideals of R are R, RY and 0. Now one could immediately infer that R is a left Artinian completely primary pli L(1)A-ring with unity.

The ring R is a 4-dimensional left $K(X^2)$-vector space and every right ideal is a $K(X^2)$-subspace of R. This implies that R is right Artinian. The right ideals R, $YR = K(X^2)Y$, $XYR = K(X^2)XY$ and 0 are different. By Proposition 4.1 we infer that R could not be an RA-ring.

C. Examples of a local A-ring which is not an Artinian ring and a local ring with exactly one minimal ideal which is not an A-ring.

Let Z be the set of such reals from $[0,1]$ that $0 \in Z$ and the sum of any two numbers from Z either belongs to Z or is greater than 1. Let T be the set of symbols t_α, where $\alpha \in Z$. And let R be a vector space spanned over some fixed field K by the set T. We define multiplication on the set T in the following way:

$$t_\alpha \cdot t_\beta = \begin{cases} t_{\alpha+\beta} & \text{if} \quad \alpha+\beta \leqslant 1, \\ 0 & \text{if} \quad \alpha+\beta > 1 \end{cases}$$

and we extend this definition to R by distributivity. $1t_0$ is the unity element of R. All ideals of R are of the form: either $I = I(\alpha) = Rt_\alpha$, where $\alpha \in Z$, or $I = I(\bar{\alpha}) = RS$, where $S = \{t_\beta \in T \mid \alpha < \beta \leqslant 1 \text{ and } \alpha \in [0,1]\}$.

If $Z = [0,1]$, then $I(\alpha) = l\big(I(\overline{1-\alpha})\big)$ and $I(\bar{\alpha}) = l\big(I(1-\alpha)\big)$, i.e. R is an A-ring. But R is not an Artinian ring.

If Z consists of 0 and all reals α, $\frac{1}{2} < \alpha \leqslant 1$, then R has a single minimal ideal $I(1)$. Annihilators of all proper ideals equal $I(\bar{\tfrac{1}{2}})$, which means that R is not an A-ring.

D. A remark about LFA-rings.

We have proved in Theorem 4.1 that every LFA-RA-ring is an L(n)A-R(n)A-ring for some n. We have also proved that a semiprime LFA-ring is an L(n)A-ring for every n.

The authors do not know whether there exists an LFA-ring which is not an L(n)A-ring.

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INSTITUTE OF MATHEMATICS OF THE WARSAW UNIVERSITY
INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

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