Characterizations of uniformity-dependent dimension functions

by

Leonard E. Soniat (Tampa, Fla.)

1. Introduction. Let (E, ϱ) be a metric space and let $\dim(E)$ denote the covering dimension of E. M. Katetov gave the first definition of a metric-dependent dimension function which he designated $d_0(E, \varrho)$ [3]. $d_0(E, \varrho)$ was not a true topological dimension function but was an integer valued function on metric spaces which depended on the particular metric defined for a given space. His definition was modeled on a characterization of the covering dimension of a space. J. H. Roberts and K. Nagami expanded the idea of metric-dependent dimension by constructing several new metric-dependent dimension functions [6] which they designated $d_2(E, \varrho)$, $d_3(E, \varrho)$, and $d_4(E, \varrho)$. Their definitions were analogously modeled on certain defining properties of the covering dimension of a space. In this paper the author is interested in pursuing a generalization to uniform spaces for $d_0(E, \varrho)$ of Katetov and for $d_2(E, \varrho)$ and $d_3(E, \varrho)$ of Nagami and Roberts.

The denotation for a uniform space, its definition, and the definition of uniformly continuous function, along with the notations and conventions employed below, are found in N. Bourbaki (*General Topology*). The following definitions and results concerning uniform spaces will find particular use in the remainder of this paper.

Definition. Let $(E, \mathcal{U})$ be a uniform space and let C and C' be subsets of E. If there exists $U \in \mathcal{U}$ such that U is symmetric and $(C \times C') \cap U = \emptyset$, then C and C' are said to be *uniformly separated* in $(E, \mathcal{U})$ or $\mathcal{U}$-*separated* in E.

Theorem 1.1. *Let C and C' be $\mathcal{U}$-separated sets in a uniform space $(E, \mathcal{U})$. Then there exists a uniformly continuous function $f\colon (E, \mathcal{U}) \to I$ such that $f(C) = -1$ and $f(C') = 1$* [4, Theorem 3, p. 90].

Definition. A covering $\mathcal{L}$ of a uniform space $(E, \mathcal{U})$ is called *Lebesgue* if there exists $U \in \mathcal{U}$ such that $\{U(x) | x \in E\}$ refines $\mathcal{L}$. An *open Lebesgue cover* is a Lebesgue cover each of whose elements is open.

2. Definition and characterizations of d_0 for uniform spaces.

DEFINITION 2.1. Let $(E, \mathcal{U})$ be a uniform space. $d_0(E, \mathcal{U}) \leqslant n$ if for every $U \in \mathcal{U}$ there exists $\mathcal{G}$, an open refinement of $\{U(x) | x \in E\}$ with order of $\mathcal{G}$ (denoted $\operatorname{ord}\mathcal{G}$) $\leqslant n+1$. If $d_0(E, \mathcal{U}) \leqslant n$ is true, and $d_0(E, \mathcal{U}) \leqslant n-1$ is false, then we say $d_0(E, \mathcal{U}) = n$.

DEFINITION 2.2. Let $(E, \mathcal{U})$ be a uniform space and let $f: E \to Y$ be a continuous function. Let $U \in \mathcal{U}$. f is called a *U-mapping* if for every $p \in Y$ there exists G_p, an open neighborhood of p such that $\{f^{\leftarrow}(G_p) | p \in Y\}$ is a refinement of $\{U(x) | x \in E\}$.

Remark 2.3. Let P be a polyhedron (i.e., the geometric realization of a simplicial complex). If v is a vertex of P then $\operatorname{St}(v)$ denotes the union of all the interiors of simplices which have v as a vertex. Observe that $\{\operatorname{St}(v) | v$ a vertex of $P\}$ is an open covering of P. If E is a normal space and $\mathcal{G}$ is a locally finite open cover of E such that $\operatorname{ord}\mathcal{G} \leqslant n+1$ then there exists a metric polyhedron P with $\dim(P) \leqslant n$ and a continuous mapping $f: E \to P$ such that $\{f^{\leftarrow}(\operatorname{St}(v)) | v$ a vertex of $P\}$ refines $\mathcal{G}$ [1, Theorem 5.4, p. 172].

THEOREM 2.4. *The following are equivalent for a paracompact uniform space* $(E, \mathcal{U})$.

1) $d_0(E, \mathcal{U}) \leqslant n$.

2) *For every $U \in \mathcal{U}$ there exists a U-mapping $f: E \to P$ where P is a metric polyhedron with dimension less than or equal to n.*

3) *Every open Lebesgue cover has an open refinement of order less than or equal to $n+1$.*

4) *Every Lebesgue cover has an open refinement of order less than or equal to $n+1$.*

5) *Every Lebesgue cover has a one-one open refinement of order less than or equal to $n+1$.*

6) *Every Lebesgue cover has a locally finite one-one open refinement of order less than or equal to $n+1$.*

Proof. 1) implies 2): Let $U \in \mathcal{U}$ and let $\mathcal{V} = \{V_a: a \in A\}$ be an open refinement of $\{U(x): x \in E\}$ such that $\operatorname{ord}\mathcal{V} \leqslant n+1$. Let $\mathcal{G} = \{G_a: a \in A\}$ be a locally finite open refinement of $\mathcal{V}$ (and hence of $\{U(x): x \in E\}$) with $\operatorname{ord}\mathcal{G} \leqslant n+1$. By Remark 2.3 there exists a metric polyhedron P with dimension less than or equal to n, and a continuous mapping $f: E \to P$ such that $\{f^{\leftarrow}(\operatorname{St}(v)): v$ a vertex of $P\}$ refines $\mathcal{G}$. Thus f is a U-mapping since $\mathcal{G}$ refines $\{U(x): x \in E\}$.

2) implies 3): Let $\mathcal{L}$ be an open Lebesgue cover of E and let $U \in \mathcal{U}$ such that $\{U(x): x \in E\}$ refines $\mathcal{L}$. Let $f: E \to P$ be a U-mapping where P is a metric polyhedron with dimension less than or equal to n. For every point $p \in P$, let W_p be an open neighborhood of p such that $f^{\leftarrow}(W_p) \subseteq V$ for some $V \in \{U(x): x \in E\}$. Let $\mathcal{G}$ be a locally finite open refinement of the cover $\{W_p: p \in P\}$. Since $\dim(P) \leqslant n$, let $\mathcal{V}$ be an open refinement of $\mathcal{G}$ of order less than or equal to $n+1$ [5, Theorem 2.1, p. 18]. Then $\{f^{\leftarrow}(W): W \in \mathcal{V}\}$ is an open refinement of $\{U(x): x \in E\}$ (and thus of $\mathcal{L}$) of order less than or equal to $n+1$.

3) implies 4): Let $\mathcal{L}$ be a Lebesgue cover and let $U \in \mathcal{U}$ such that U is open and $\{U(x): x \in E\}$ is a refinement of $\mathcal{L}$. Then $\{U(x): x \in E\}$ is an open Lebesgue cover for E which has a refinement or order less than or equal to $n+1$.

4) implies 5): Let $\mathcal{L} = \{L_a: a \in A\}$ be a Lebesgue cover of E. Let $\mathcal{V} = \{V_\beta: \beta \in B\}$ be an open refinement of $\mathcal{L}$ of order less than or equal to $n+1$. Let A, the index set for $\mathcal{L}$, be well ordered. Define $B_a = \{\beta \in B: V_\beta \subseteq L_a\} - \bigcup_{\gamma < a} B_\gamma$. Define $W_a = \bigcup_{\beta \in B_a} V_\beta$ for every $a \in A$. Then $\mathcal{W} = \{W_a: a \in A\}$ is an open refinement of $\mathcal{L}$ and $\operatorname{ord}\mathcal{W} \leqslant n+1$, also $\mathcal{W}$ refines $\mathcal{L}$ in a one-one fashion since $W_a \subseteq L_a$ for every $a \in A$.

5) implies 6): Let $\mathcal{L}$ be a Lebesgue cover of E and let $\mathcal{V}$ be an open one-one refinement of $\mathcal{L}$ of order less than or equal to $n+1$. Let $\mathcal{G}$ be a locally finite refinement of $\mathcal{V}$. Then $\mathcal{G}$ can be amalgamated inside $\mathcal{V}$ as in 4) implies 5), to produce a locally finite one-one refinement of order less than or equal to $n+1$.

6) implies 1): Let $U \in \mathcal{U}$; then $\{U(x): x \in E\}$ is a Lebesgue cover of E and thus has a refinement of order less than or equal to $n+1$.

COROLLARY 2.5. *Statements* (1), (3), (4), *and* (5) *are equivalent in any uniform space.*

3. Definition and characterizations of d_2 for uniform spaces. Consider a uniform space $(E, \mathcal{U})$ and a subset F of E. Then $(F, \mathcal{U}_F)$ will denote a uniform space where $\mathcal{U}_F$ is the natural restriction of $\mathcal{U}$ to F.

DEFINITION 3.1. Let $(E, \mathcal{U})$ be a uniform space. $d_2(E, \mathcal{U}) < n$, if for every closed set F of E and every uniformly continuous function $f: (F, \mathcal{U}_F) \to S^n$, there exists $f': E \to S^n$ a continuous function such that $f'|F = f$. If $d_2(E, \mathcal{U}) \leqslant n$ and $d_2(E, \mathcal{U}) \leqslant n-1$ is false, then $d_2(E, \mathcal{U}) = n$.

THEOREM 3.2. *For every normal uniform space $(E, \mathcal{U})$, the following properties are equivalent to the property $d_2(E, \mathcal{U}) \leqslant n$.*

$P_1(n)$: *For every collection $\{C_1, C_1', \ldots, C_{n+1}, C_{n+1}'\}$ such that C_i and C_i' are closed $\mathcal{U}$-separated subsets of E, there exists a collection $\{B_i: i = 1, \ldots, n+1\}$ of closed subsets of E such that B_i separates C_i and C_i' and* $\bigcap_{i=1}^{n+1} B_i = \emptyset$.

$P_2(n)$: *This property is the same as $P_1(n)$ except that C_i and C_i' are not necessarily closed.*

$P_3(n)$: *For every uniformly continuous function $f: (E, \mathcal{U}) \to I^{n+1}$, the point $p = (0, 0, \ldots, 0)$ is an unstable value of f.*

$P_4(n)$: *For every uniformly continuous function* $f\colon (E, \mathfrak{U}) \to I^{n+1}$, *if* $p \in I^{n+1} - S^n$ *then* p *is an unstable value of* f.

$P_5(n)$: *This property is the same as* $P_4(n)$ *except that* p *may be any point of* I^{n+1}.

$P_6(n)$: *For every subset* C *of* E *and every uniformly continuous function* $f(C, \mathfrak{U}_C) \to S^n$, *there exists a continuous function* $f'\colon E \to S^n$ *such that* f' *extends* f.

$P_7(n)$: *Every uniformly continuous function* $f\colon (E, \mathfrak{U}) \to I^{n+1}$ *is inessential.*

Proof. $d_2(E, \mathfrak{U}) \leqslant n$ implies $P_1(n)$: Let $\{C_1, C_1', \ldots, C_{n+1}, C_{n+1}'\}$ be any $n+1$ pairs of closed $\mathfrak{U}$-separated subsets of E. Let $F = \bigcup_{i=1}^{n+1} (C_i \cup C_i')$; then since C_i and C_i' are $\mathfrak{U}_F$-separated in the uniform space $(F, \mathfrak{U}_F)$, by Theorem 1.1 there exists $f_i\colon (F, \mathfrak{U}_F) \to I$ such that $f_i(C_i) = -1$ and $f_i(C_i') = 1$ and f_i is uniformly continuous. The function $f\colon (F, \mathfrak{U}_F) \to I^{n+1}$ defined by $f(x) = (f_1(x), \ldots, f_{n+1}(x))$ is uniformly continuous. Also, the range of f is a subset of $S^n = \operatorname{Bdry} I^{n+1}$ since $x \in F$ implies there exists i such that $x \in C_i$ or C_i'; hence, $f_i(x) = \pm 1$. Thus since $d_2(E, \mathfrak{U}) \leqslant n$, there exists a continuous function $g\colon E \to S^n$ such that $g|F = f$. Let $B_i = g_i^{\leftarrow}(0)$. B_i is a closed subset of E which separates C_i and C_i' and $\bigcap_{i=1}^{n+1} B_i = \emptyset$ since the range of g is a subset of S^n.

$P_1(n)$ implies $P_2(n)$: Let $\{C_1, C_1', \ldots, C_{n+1}, C_{n+1}'\}$ be any $n+1$ pairs of $\mathfrak{U}$-separated sets in E. Then, it is easily seen that $\{\bar{C}_1, \bar{C}_1', \ldots, \bar{C}_{n+1}, \bar{C}_{n+1}'\}$ are $n+1$ pairs of $\mathfrak{U}$-separated closed sets in E and by $P_1(n)$ there exists $\{B_1, \ldots, B_{n+1}\}$ such that B_i separates $\bar{C}_i$ and $\bar{C}_i'$ and $\bigcap_{i=1}^{n+1} B_i = \emptyset$, and B_i is closed for every i. However, a fortiori, B_i also separates C_i and C_i'; therefore, $P_2(n)$ is true.

$P_2(n)$ implies $P_3(n)$: Let $f(x) = (f_1(x), \ldots, f_{n+1}(x))$ be a uniformly continuous function from $(E, \mathfrak{U})$ into I^{n+1} and let $\varepsilon > 0$ be given. Then the coordinate function f_i is uniformly continuous for every i. Let

$$C_i = \left\{x \in E\colon f_i(x) \geqslant \frac{\varepsilon}{2\sqrt{n+1}}\right\} \quad \text{and} \quad C_i' = \left\{x \in E\colon f_i(x) \leqslant -\frac{\varepsilon}{2\sqrt{n+1}}\right\}.$$

Since f_i is uniformly continuous, C_i and C_i' are $\mathfrak{U}$-separated. Let $\{B_1, \ldots, B_{n+1}\}$ be closed sets having empty intersection such that B_i separates C_i and C_i'. Let $\{V_1, \ldots, V_{n+1}\}$ be an open collection such that $B_i \subseteq V_i \subseteq E - (C_i \cup C_i')$ for every i and $\bigcap_{i=1}^{n+1} V_i = \emptyset$. Let $h_i\colon E \to [0, 1]$ be a continuous mapping such that $h_i(B_i) = 0$ and $h_i(E - V_i) = 1$. $E - B_i = U_i \cup U_i'$ where U_i and U_i' are disjoint open subsets of E such that $C_i \subseteq U_i$ and $C_i' \subseteq U_i'$.

Define $f_i'\colon E \to I$ as follows:

$$f_i'(x) = \begin{cases} f_i(x), & x \in (C_i \cup C_i'), \\ \left(\dfrac{\varepsilon}{2\sqrt{n+1}}\right) h_i(x), & x \in (U_i \cup B_i - \operatorname{int} C_i), \\ -\left(\dfrac{\varepsilon}{2\sqrt{n+1}}\right) h_i(x), & x \in (U_i' \cup B_i - \operatorname{int} C_i'). \end{cases}$$

Then f_i' is continuous and $f_i'(x) = 0$ implies that $x \in V_i$.

Define $f'(x) = (f_1'(x) \ldots f_{n+1}'(x))$; then

$$|f(x) - f'(x)| \leqslant \left(\sum_{i=1}^{n+1} |f_i(x) - f_i'(x)|^2\right)^{1/2} \leqslant \left(\sum_{i=1}^{n+1} \left|2\left(\frac{\varepsilon}{2\sqrt{n+1}}\right)\right|^2\right)^{1/2} = \varepsilon$$

and

$$f'^{\leftarrow}(0) \subseteq \bigcap_{i=1}^{n+1} V_i = \emptyset.$$

$P_3(n)$ implies $P_4(n)$: Let $p \in (I^{n+1} - S^n)$. Let $f\colon E \to I^{n+1}$ be uniformly continuous and let $h\colon I^{n+1} \to I^{n+1}$ be a homeomorphism with $h(p) = 0$. $h^{\leftarrow}$ is uniformly continuous and if $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|h^{\leftarrow}(x) - h^{\leftarrow}(y)| < \varepsilon$. $h \circ f$ is uniformly continuous thus there exists $f'\colon E \to I^{n+1}$ such that $f'^{\leftarrow}(0) = \emptyset$ and $|h \circ f(x) - f'(x)| < \delta$ for every $x \in E$. Observe that $(h^{\leftarrow} \circ f')^{\leftarrow}(p) = \emptyset$ and also that

$$\begin{aligned} |f(x) - h^{\leftarrow} \circ f'(x)| &= |(h^{\leftarrow} \circ h) \circ f(x) - h^{\leftarrow} \circ f'(x)| \\ &= |h^{\leftarrow}(h \circ f(x)) - h^{\leftarrow}(f'(x))| < \varepsilon \end{aligned}$$

since $|h \circ f(x) - f'(x)| < \delta$, thus $h^{\leftarrow} \circ f'$ is the desired function.

$P_4(n)$ implies $P_5(n)$: We need only show that the condition holds for every point p in S^n. Let $p \in S^n$ and let S be the surface of the ε-ball about p. We can obtain a function $g\colon I^{n+1} \to I^{n+1}$ by retracting the ε-ball about p onto S and letting g be the identity elsewhere in I^{n+1}. Then p is not in the range of g. Now, letting $f' = g \circ f$ we obtain the desired function.

$P_5(n)$ implies $P_6(n)$: Let $C \subseteq E$ and let $f\colon (C, \mathfrak{U}_C) \to S^n$ be uniformly continuous. By the application of theorem 3 of [4] to each of the coordinate functions of f, there exists $g\colon (E, \mathfrak{U}) \to I^{n+1}$ such that g extends f. Then by $P_5(n)$ and by [2, VI, I, B] there is a continuous function $g'\colon E \to I^{n+1}$ such that 0 is not in the range of g and $g' = g$ on S^n. Let $\alpha\colon (I^{n+1} - \{0\}) \to S^n$

be the radial projection mapping, then $f' = a \circ g'$ is the desired extension of f.

$P_6(n)$ implies $P_7(n)$: Let $f\colon (E, \mathcal{U}) \to I^{n+1}$ be uniformly continuous. Then $f^{\leftarrow}(S^n) \subseteq E$ and the restriction of f to $f^{\leftarrow}(S^n)$ is uniformly continuous and thus extendable to all of E.

$P_7(n)$ implies $P_1(n)$: Let $\{C_1, C'_1, \dots, C_{n+1}, C'_{n+1}\}$ be $n+1$ pairs of subsets of E such that C_i and C'_i are $\mathcal{U}$-separated. By Theorem 1.1 there exists $f_i\colon (E, \mathcal{U}) \to I$ uniformly continuous such that $f_i(C_i) = -1$ and $f_i(C'_i) = 1$. Let $f = (f_1, \dots, f_{n+1})\colon (E, \mathcal{U}) \to I^{n+1}$. Then f is uniformly continuous and $\bigcup_{i=1}^{n+1} (C_i \cup C'_i) \subseteq f^{\leftarrow}(S^n)$. Let $f'\colon E \to S^n$ be a continuous function which extends $f|f^{\leftarrow}(S^n)$ and define $B_i = f_i'^{\leftarrow}(0)$, $1 \leqslant i \leqslant n+1$, where f'_i is the ith coordinate function for f'. Then B_i separates C_i and C'_i and $\bigcap_{i=1}^{n+1} B_i = \emptyset$.

$P_6(n)$ implies $d_2(E, \mathcal{U}) \leqslant n$: Trivial.

Remark 3.3. For a Lebesgue covering characterization of d_2, see Smith [7].

4. Definition and a characterization of d_3 for uniform spaces.

DEFINITION 4.1. Let $(E, \mathcal{U})$ be a uniform space. $d_3(E, \mathcal{U}) \leqslant n$ if for every finite Lebesgue cover $\mathcal{L}$ of E, there exists $\mathcal{G}$, an open refinement of $\mathcal{L}$ such that ord $\mathcal{G} \leqslant n+1$. If $d_3(E, \mathcal{U}) \leqslant n$ is true and $d_3(E, \mathcal{U}) \leqslant n-1$ is false, then we say $d_3(E, \mathcal{U}) = n$.

THEOREM 4.2. *Let $(E, \mathcal{U})$ be a normal uniform space. Then $d_3(E, \mathcal{U}) \leqslant n$ if and only if given $\{C_1, C'_1, \dots, C_m, C'_m\}$ m pairs of closed $\mathcal{U}$-separated subsets of E, there exists $\{B_1, \dots, B_m\}$ a collection of closed subsets of E such that B_i separates C_i and C'_i, $1 \leqslant i \leqslant m$, and* ord$\{B_1, \dots, B_m\} \leqslant n$.

Proof. Sufficiency: As was shown in the proof of Theorem 2.4, we need only prove that every finite open Lebesgue cover has a refinement of order less than or equal to $n+1$. Let $\mathcal{L} = \{L_1, \dots, L_m\}$ be a finite open Lebesgue cover. Let $U \in \mathcal{U}$ such that $\{U(x)\colon x \in E\}$ refines $\mathcal{L}$. Let $W \in \mathcal{U}$ such that W is symmetric and $W \circ W \subseteq U$. $\{W(x)\colon x \in E\}$ is an open refinement of $\mathcal{L}$. Define $H_i = L_i - \bigcup W(x)$, $x \in E - L_i$, $1 \leqslant i \leqslant m$. Let $p \in E$. Then there exists j such that $U(p) \subseteq L_j$. Let $y \in E - L_j$ and assume $p \in W(y)$. If $x \in W(y)$, we would then have $(x, y) \in W$ and also $(y, p) \in W$ since $p \in W(y)$ and W was chosen symmetric. Thus $(x, p) \in W \circ W \subseteq U$ and $x \in U(p)$ implying that $W(y) \subseteq U(p)$, a contradiction because of the choice of y. Thus $p \notin W(y)$ for every $y \in E - L_j$, hence $p \in H_j$, and we have shown that $\mathcal{H} = \{H_1, \dots, H_m\}$ is a cover of E. Since H_i and $E - L_i$ are $\mathcal{U}$-separated, $1 \leqslant i \leqslant m$, there exists $\{B_1, \dots, B_m\}$, a collection of closed subsets of E, such that B_i separates H_i and $E - L_i$ and ord $\{B_1, \dots, B_m\} \leqslant n$. Since E is a normal space, there exists an open collection $\{V_1, \dots, V_m\}$ such that $B_i \subseteq V_i \subseteq L_i$ and ord $\{V_1, \dots, V_m\} \leqslant n$. For $1 \leqslant i \leqslant m$, $E - B_i = U_i \cup U'_i$, where $H_i \subseteq U_i$ and $E - L_i \subseteq U'_i$ and U_i and U'_i are disjoint open sets.

Define $W_1 = U_1$ and $W_1 = U_i \cap (\bigcap_{j=1}^{i-1} U'_j)$ for $2 \leqslant i \leqslant m$. Observe that ord $\{W_i\colon i = 1, \dots, m\} = 1$. Then $\{W_i\colon i = 1, \dots, m\} \cup \{V_i\colon i = 1, \dots, m\}$ is an open refinement of $\mathcal{L}$ of order less than or equal to $n+1$.

Necessity: Let $\{C_1, C'_1, \dots, C_m, C'_m\}$ be a collection of m pairs of closed sets such that C_i and C'_i are $\mathcal{U}$-separated for every $1 \leqslant i \leqslant m$. Define $\mathcal{L}$ as the collection of all sets of the form $\bigcap_{i=1}^{m} Y_i$ where $Y_i \in \{E - C_i, E - C'_i\}$ for every $1 \leqslant i \leqslant m$. Then clearly $\mathcal{L}$ is a finite Lebesgue cover of E. Let $\mathcal{V}$ be an open refinement of $\mathcal{L}$ of order less than or equal to $n+1$. We may assume $\mathcal{V}$ is a one-one refinement of $\mathcal{L}$ and thus finite, $\mathcal{V} = \{V_1, \dots, V_{2^m}\}$. Let $\mathcal{F} = \{F_1, \dots, F_{2^m}\}$ be a closed one-one refinement of $\mathcal{V}$. Let $\mathcal{G}^i = \{G_1^i, \dots, G_{2^m}^i\}$, for every $1 \leqslant i \leqslant m$, be a sequence of open covers of E such that

$$F_j \subseteq G_j^1 \subseteq \bar{G}_j^1 \subseteq \dots \subseteq G_j^i \subseteq \bar{G}_j^i \subseteq G_j^{i+1} \subseteq \dots \subseteq V_j \tag{4.1}$$

for every $1 \leqslant j \leqslant 2^m$ and for every $1 \leqslant i \leqslant m$. Let $K_i = \bigcup \{G \in \mathcal{G}^i\colon G \cap C_i \neq \emptyset\}$ for every $1 \leqslant i \leqslant m$. Then $C_i \subseteq K_i \subseteq E - C'_i$ since $\mathcal{G}^i$ covers E and refines $\mathcal{V}$. Let B_i = boundary of K_i. Then B_i separates C_i and C'_i for every $1 \leqslant i \leqslant m$. To show that ord$\{B_1, \dots, B_m\} \leqslant n$ assume the contrary, i.e., $\bigcap_{i=1}^{n+1} B_{j_i} \neq \emptyset$ where $j_i \neq j_k$ if $i \neq k$. Then since $\mathcal{G}^i$ is finite for every $1 \leqslant i \leqslant m$, there exists $G_{k_i}^{j_i} \in \mathcal{G}^{j_i}$ for every $1 \leqslant i \leqslant n+1$ such that

$$\bigcap_{i=1}^{n+1} (\bar{G}_{k_i}^{j_i} - G_{k_i}^{j_i}) \neq \emptyset. \tag{4.2}$$

If $k_t = k_s$ when $t \neq s$ then by (4.1) we obtain $(\bar{G}_{k_t}^{j_t} - G_{k_t}^{j_t}) \cap (\bar{G}_{k_s}^{j_s} - G_{k_s}^{j_s}) = \emptyset$, a contradiction to (4.2); hence $k_s \neq k_t$ whenever $s \neq t$ and thus the k_i are distinct indices. Let $p \in \bigcap_{i=1}^{n+1} (\bar{G}_{k_i}^{j_i} - G_{k_i}^{j_i})$. From (4.1) it follows that $p \notin \bigcup_{i=1}^{n+1} F_{k_i}$, however, $p \in V_{k_i}$ for every $1 \leqslant i \leqslant n+1$. Since $\mathcal{F}$ covers E, there exists q such that $p \in F_q$ where $q \neq k_i$ for every $1 \leqslant i \leqslant n+1$. Consequently, $p \in V_q \cap (\bigcap_{i=1}^{n+1} V_{k_i})$ which contradicts ord $\mathcal{V} \leqslant n+1$. Thus ord $\{B_1, \dots, B_m\} \leqslant n$.

COROLLARY 4.3. *In a normal uniform space $(E, \mathcal{U})$ we have $d_2(E, \mathcal{U}) \leqslant d_3(E, \mathcal{U})$.*

Proof. Use $P_1(n)$ of Theorem 3.2 and Theorem 4.2.

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UNIVERSITY OF SOUTH FLORIDA
Tampa, Florida

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Closed mappings and the Freudenthal compactification

by

Krzysztof Nowiński (Warszawa)

The main purpose of this paper is to give a characterization of closed mappings of locally compact weakly paracompact spaces into compact spaces and to apply this characterization in a study of the problem of extending closed mappings over the Freudenthal compactification. In the first section we state Theorem 1, giving a necessary condition for the closedness of a mapping $f\colon X\to Y$ from a weakly paracompact space X into a compact space Y, and give some applications of this theorem. The above-mentioned characterization of closed mappings is given in Theorem 2. The second section contains results about extensions of closed mappings over some compactifications. The main theorem of this part is Theorem 5, an essential generalization of a result of Morita ([7], Theorem 5). Lastly, the third section contains some facts on the Freudenthal compactification. In particular, Theorem 7 gives a characterization of the Freudenthal compactification of some subsets of manifolds.

All notions and notations are taken from [1] with a small modification: if rX is a compactification of X then we regard X as lying in rX and we write shortly $rX\setminus X$ instead of $rX\setminus r(X)$. All spaces are assumed to be $T_{3\frac{1}{2}}$ and all mappings are assumed to be continuous. The weakly paracompact (metacompact) spaces are called shortly WPC spaces.

We define, moreover, some useful notation: if $\mathcal{A}$ is a collection of disjoint subsets of the space X, then $X/\mathcal{A}$ denotes the quotient space X/R_A, where the equivalence relation R_A is defined as follows:

$$xR_Ay \quad \text{iff} \quad x=y \quad \text{or} \quad x,y\in A \quad \text{for some } A\in\mathcal{A}.$$

1. Closed mappings.

Definition 1. A mapping $f\colon X\to Y$ is *closed* iff for every closed subset A of X its image $f(A)$ is closed in Y.

Let us notice the following obvious

Proposition 1. *If there exists a compact subset Z of X such that $f(X\setminus Z)$ is finite, then the mapping $f\colon X\to Y$ is closed.*