Binding spaces: A unified completion and extension theory

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0. Introduction. Completions of various topological and uniform structures are often obtained by embedding the structure in some filter space. Common examples are those of embedding a Tychonoff space in the space of Z-ultrafilters or real Z-ultrafilters to obtain the Stone–Čech compactification or the Hewitt real compactification, respectively, [3]. Or, of embedding a uniform space in the space whose members are equivalence classes of Cauchy ultrafilters to obtain a uniform completion [1]. Recently Wallman-type compactifications have been obtained by embedding a Tychonoff (or T_1) space in a space of ultrafilters of sets from a normal base [2] or a separating family [6].

Our goal is to provide a common setting for these and other situations. We will extend the concept of ultrafilter by considering more general objects called clusters. Clusters differ from ultrafilters in two ways. First, every finite subcollection of a cluster will be bound (close in a certain sense), and although a finite number of closed sets having a point in common are bound, these need not necessarily be the only bound collections. Secondly, a family of sets maximal with respect to having each finite subcollection bound is a cluster if and only if it contains at least one set from each cover in a specified family of coverings.

The first notion is used in the construction of clusters in a proximity space [5], and the second in defining Cauchy ultrafilters in a uniform space [4]. Thus we obtain greater flexibility, both in the creation of ultrafilter-like objects, by using the notion of bound sets, and in their selection, by requiring them to pass through a collection of covers.

The covers and bound families of a set X determine a structure on X called a binding structure. The bound families also determine a topology on X. The class of all binding spaces, together with the maps between them, forms a category in which subspace, product, and quotient space may be defined.

A binding space X may be embedded in the set X^* of clusters, and the covers and bound families in X can be extended to define a binding structure on X^*, with respect to which X is a dense subspace.

Clusters are fixed or free depending on whether the closures of their sets have a point in common. The space X^* is complete and is called a completion of X. A finite collection of subsets of X is bound if and only if their closures in X^* have a point in common. Any map from X into a complete binding space whose topology is regular has an extension to X^*. The class of all binding structures whose completions have regular topologies is also a category, and the complete spaces in it constitute a reflective subcategory.

In defining a binding structure, it is often convenient to specify something simpler, called a base, which uniquely determines the structure. We will, therefore, define a base for a binding space and work out the properties of the space it generates before setting forth the axioms of the binding space itself.

The spaces to which we apply the theory are most easily seen to be binding spaces when appropriate bases are chosen.

1. Fundamental concepts. Let $\mathcal{F}$ be a base for closed sets for a T_1-topology on a set X. The pair $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ will be called a *closed base for a binding structure on* X provided $\boldsymbol{B}_{\mathcal{F}}$ is a family of finite collections of nonempty subsets of $\mathcal{F}$ satisfying:

B1. if $F_1, F_2, \ldots, F_n \in \mathcal{F}$ and $\bigcap F_i \neq \emptyset$, then $\{F_i\} \in \boldsymbol{B}_{\mathcal{F}}$;

B2. if $\{G_j\} \subset \{F_i\} \in \boldsymbol{B}_{\mathcal{F}}$, then $\{G_j\} \in \boldsymbol{B}_{\mathcal{F}}$;

B3. if $\{F, F_i\} \in \boldsymbol{B}_{\mathcal{F}}$ and $F \subset G \cup H$ where $G, H \in \mathcal{F}$, then $\{G, F_i\} \in \boldsymbol{B}_{\mathcal{F}}$ or $\{H, F_i\} \in \boldsymbol{B}_{\mathcal{F}}$;

B4. if $x \notin F \in \mathcal{F}$, then there are sets $F_1, F_2, \ldots, F_n \in \mathcal{F}$ such that $x \in \bigcap F_i$ and $\{F, F_i\} \notin \boldsymbol{B}_{\mathcal{F}}$;

and $\boldsymbol{C}_{\mathcal{F}}$ is a family of coverings of X consisting of sets in $\mathcal{F}$.

A *cluster base* in $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ is a subfamily $\mathcal{S} \subset \mathcal{F}$ which contains at least one set from each member of $\boldsymbol{C}_{\mathcal{F}}$ and is maximal with respect to having each finite subcollection in $\boldsymbol{B}_{\mathcal{F}}$.

1.1. Example. Let $\mathcal{F}$ be all the closed sets from some T_1-topology on X, $\boldsymbol{B}_{\mathcal{F}}$ all finite collections of closed sets with nonempty intersections, and $\boldsymbol{C}_{\mathcal{F}}$ all finite closed coverings of X. Then $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ is a closed base and the cluster bases are the ultrafilters of closed sets.

The *cluster* $\mathcal{C}$ *generated by a cluster base* $\mathcal{S}$ is the family of all subsets $A \subset X$ having the property that for each $F \in \mathcal{F}$, $A \subset F$ implies $F \in \mathcal{S}$.

1.2. Remark. If $F \in \mathcal{S}$ and $F \subset G \in \mathcal{F}$, then B3 and the maximality of $\mathcal{S}$ imply that $G \in \mathcal{S}$. Thus each cluster base is contained in the cluster it generates. If $\mathcal{S}_1 \neq \mathcal{S}_2$ are cluster bases, then $\mathcal{S}_1$ cannot be contained in the cluster generated by $\mathcal{S}_2$ and thus distinct cluster bases generate distinct clusters. If $\mathcal{S}$ generates $\mathcal{C}$, then $\mathcal{C} \cap \mathcal{F} = \mathcal{S}$.

1.3. Lemma. *If $F, G \in \mathcal{F}$ and $F \cup G \in \mathcal{S}$, then $F \in \mathcal{S}$ or $G \in \mathcal{S}$.*

Proof. If $F \notin \mathcal{S}$, there are sets $F_1, F_2, \ldots, F_n \in \mathcal{S}$ such that $\{F, F_i\} \notin \boldsymbol{B}_{\mathcal{F}}$. Similarly if $G \notin \mathcal{S}$, there are sets $G_1, \ldots, G_m \in \mathcal{S}$ such that $\{G, G_j\} \notin \boldsymbol{B}_{\mathcal{F}}$. From B2 it follows that neither $\{F, F_i, G_j\}$ nor $\{G, F_i, G_j\}$ is in $\boldsymbol{B}_{\mathcal{F}}$ so by B3, $\{F \cup G, F_i, G_j\} \notin \boldsymbol{B}_{\mathcal{F}}$. Thus $F \cup G \notin \mathcal{S}$.

The *binding structure generated by the closed base* $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ is the pair $(\boldsymbol{B}, \boldsymbol{C})$, where $\boldsymbol{B}$ consists of all finite collections of subsets of X occurring in some cluster and $\boldsymbol{C}$ is the family of all coverings of X which are refined by some member of $\boldsymbol{C}_{\mathcal{F}}$.

The cluster $\mathcal{C}$ generated by the cluster base $\mathcal{S}$ in $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ will be called a *cluster in the binding structure* $(\boldsymbol{B}, \boldsymbol{C})$ generated by $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ and may be characterized as follows.

1.4. Theorem. *A family $\mathcal{U}$ of subsets of X is a cluster in the binding structure $(\boldsymbol{B}, \boldsymbol{C})$ if and only if it contains an element from each member of $\boldsymbol{C}$ and is maximal with respect to having each finite subcollection in $\boldsymbol{B}$.*

Proof. Let $\mathcal{U}$ be a cluster. The second property holds by the definition of $\boldsymbol{B}$. If $c \in \boldsymbol{C}$, then c is refined by some $c' \in \boldsymbol{C}_{\mathcal{F}}$ and $\mathcal{U}$ contains a set $F \in c'$. Since $F \subset A \in c$, any set $G \in \mathcal{F}$ containing A contains F, and is thus in the cluster base generating $\mathcal{U}$. Hence $A \in \mathcal{U} \cap c$.

Now, suppose $\mathcal{U}$ satisfies the above conditions. The family $\mathcal{S} = \mathcal{U} \cap \mathcal{F}$ intersects each cover in $\boldsymbol{C}_{\mathcal{F}}$. If $F_1, \ldots, F_n \in \mathcal{F}$ and $\{F_i\} \notin \boldsymbol{B}_{\mathcal{F}}$, then $\{F_i\}$ cannot be contained in any cluster and so $\{F_i\} \notin \boldsymbol{B}$. Thus any finite subcollection of $\mathcal{S}$ is in $\boldsymbol{B}_{\mathcal{F}}$. The maximality of $\mathcal{U}$ implies that $\mathcal{S}$ is a cluster base. If $A \in \mathcal{U}$ and $A \subset F \in \mathcal{F}$, then $F \in \mathcal{S}$. $\mathcal{U}$ is thus contained in the cluster $\mathcal{C}$ generated by $\mathcal{S}$ and the maximality condition on $\mathcal{U}$ implies that $\mathcal{U} = \mathcal{C}$.

In the definition of an abstract binding space, this characterization will be taken as the definition of a cluster.

For each $A \subset X$, let $\bar{A} = \{x \in X: \{x, A\} \in \boldsymbol{B}\}$.

1.5. Theorem. *The binding structure $(\boldsymbol{B}, \boldsymbol{C})$ generated by the closed base $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ has the following properties:*

P1. *If $A_1, \ldots, A_n \subset X$ and $\bigcap A_i \neq \emptyset$, then $\{A_i\} \in \boldsymbol{B}$.*

P2. *If $\{B_j\} \subset \{A_i\} \in \boldsymbol{B}$, then $\{B_j\} \in \boldsymbol{B}$.*

P3. *If $\{A, A_i\} \in \boldsymbol{B}$ and $A \subset B \cup C$, then $\{B, A_i\} \in \boldsymbol{B}$ or $\{C, A_i\} \in \boldsymbol{B}$.*

P4. *$\{A_i\} \in \boldsymbol{B}$ if and only if $\{\bar{A}_i\} \in \boldsymbol{B}$.*

P5. *If $\{x, y\} \in \boldsymbol{B}$, then $x = y$.*

P6. *A covering is in $\boldsymbol{C}$ if and only if it is refined by a covering in $\boldsymbol{C}$ composed of sets of the form $\bar{A}$.*

P7. *If $\{A_i\} \in \boldsymbol{B}$, then $\{A_i\}$ is contained in some cluster.*

Proof. P1. If $x \in \bigcap A_i$ then $\mathcal{S}_x = \{F \in \mathcal{F}: x \in F\}$ is a cluster base and $\{A_i\}$ is contained in the cluster generated by $\mathcal{S}_x$. Thus $\{A_i\} \in \boldsymbol{B}$.

P2 holds because any cluster containing $\{A_i\}$ contains $\{B_j\} \subset \{A_i\}$.

P3. Let $\mathcal{C}$ be a cluster containing $\{A, A_i\}$ and let $\mathcal{S}$ be a cluster base generating $\mathcal{C}$. If $B \notin \mathcal{C}$, then there is an $F \in \mathcal{F}$ such that $B \subset F$ and $F \notin \mathcal{S}$. If $C \notin \mathcal{C}$, then there is a $G \in \mathcal{F}$ such that $C \subset G$ and $G \notin \mathcal{S}$. Since $A \subset F \cup G$, and $A \in \mathcal{C}$, $F \cup G \in \mathcal{S}$. From 1.3, either $F \in \mathcal{S}$ or $G \in \mathcal{S}$. This contradiction shows that either $B \in \mathcal{C}$ or $C \in \mathcal{C}$ and thus either $\{B, A_i\} \in \boldsymbol{B}$ or $\{C, A_i\} \in \boldsymbol{B}$.

P4. If $x \in A$, then $\{x\}$ and A are both in the cluster generated by $\mathcal{S}_x$ (see proof of P1), so $\{x, A\} \in \boldsymbol{B}$. Thus $A \subset \overline{A}$ and any cluster containing A contains $\overline{A}$. Conversely, let $A \subset F \in \mathcal{F}$ and suppose $x \notin F$. By B4 there are sets $F_1, \ldots, F_n \in \mathcal{F}$ such that $x \in \bigcap F_i$ and $\{F, F_i\} \notin \boldsymbol{B}_{\mathcal{F}}$. Thus no cluster base can contain $\{F, F_i\}$ and hence no cluster can contain both A and $\{x\}$. This implies that if $A \subset F \in \mathcal{F}$ then $\overline{A} \subset F$ and so every cluster containing $\overline{A}$ contains A. P4 follows from the definition of $\boldsymbol{B}$.

P5. If $x \neq y$, then there is an $F \in \mathcal{F}$ such that $x \in F$, $y \notin F$ so by the argument above, $y \notin \{\overline{x}\}$ and $\{x, y\} \notin \boldsymbol{B}$.

P6. From B4 it follows that $F = \overline{F}$ for each $F \in \mathcal{F}$ and so each covering in $\boldsymbol{C}_{\mathcal{F}}$ is composed of sets of the form $\overline{A}$. Any covering refined by a member of $\boldsymbol{C}$ is refined by a member of $\boldsymbol{C}_{\mathcal{F}}$, and is thus in $\boldsymbol{C}$.

P7 follows directly from the definition of $\boldsymbol{B}$.

We now define a *binding structure on* X to be a pair $(\boldsymbol{B}, \boldsymbol{C})$, where $\boldsymbol{B}$ is a family of finite collections of nonempty subsets of X and $\boldsymbol{C}$ is a family of coverings of X, satisfying properties P1–P7. The members of $\boldsymbol{B}$ will be called *bound collections*, and the members of $\boldsymbol{C}$ will be called *covers*.

Note. If $\mathcal{A}$ is a family of subsets of X such that $X \subset \bigcup \mathcal{A}$, $\mathcal{A}$ is a covering of X; if, in addition, $\mathcal{A} \in \boldsymbol{C}$, then $\mathcal{A}$ is a cover of X.

1.6. Lemma. *The operator $A \to \overline{A}$ is a closure operator.*

Proof. From P1 it follows that $A \subset \overline{A}$ and thus that $X = \overline{X}$. Since no collection in $\boldsymbol{B}$ contains $\emptyset$, $\emptyset = \overline{\emptyset}$. $\{x, A \cup B\} \in \boldsymbol{B}$ if and only if $\{x, A\} \in \boldsymbol{B}$ or $\{x, B\} \in \boldsymbol{B}$, by P3, so $\overline{A \cup B} = \overline{A} \cup \overline{B}$. From P5 it follows that $\{\overline{x}\} = \{x\}$ for each $x \in X$ and from P4, $\{x, \overline{A}\} = \{\overline{x}, \overline{A}\} \in \boldsymbol{B}$ implies that $\{x, A\} \in \boldsymbol{B}$, so $\overline{\overline{A}} \subset \overline{A}$. Thus $\overline{\overline{A}} = \overline{A}$ and the operator $A \to \overline{A}$ is a closure operator.

If $(\boldsymbol{B}, \boldsymbol{C})$ is a binding structure on X, the closed sets in the *topology of* $(\boldsymbol{B}, \boldsymbol{C})$ are defined to be those sets A such that $A = \overline{A}$. This is a T_1-topology since $\{\overline{x}\} = \{x\}$, and the closure of a set A in this topology is just $\overline{A}$.

If $(\boldsymbol{B}, \boldsymbol{C})$ is generated by the closed base $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$, then the topology of $(\boldsymbol{B}, \boldsymbol{C})$ coincides with the one generated by $\mathcal{F}$. From property B4 it follows that $F = \overline{F}$ for each $F \in \mathcal{F}$ so the sets in $\mathcal{F}$ are closed in the topology of $(\boldsymbol{B}, \boldsymbol{C})$. If $x \notin \overline{A} \subset X$, then A (and hence $\overline{A}$) is not contained in the cluster generated by $\mathcal{S}_x$. Thus there is an $F \in \mathcal{F}$ such that $\overline{A} \subset F$ and $x \notin F$. $\mathcal{F}$ is closed under finite unions and is therefore a base for the closed sets in the topology of $(\boldsymbol{B}, \boldsymbol{C})$.

1.7. Lemma. *Each binding structure $(\boldsymbol{B}, \boldsymbol{C})$ on a set X is generated by a closed base $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$.*

Proof. Let $\mathcal{F}$ denote the family of all closed sets in the topology of $(\boldsymbol{B}, \boldsymbol{C})$; let $\boldsymbol{B}_{\mathcal{F}}$ be all finite subcollections of sets in $\mathcal{F}$ which are in $\boldsymbol{B}$; and let $\boldsymbol{C}_{\mathcal{F}}$ be all covers in $\boldsymbol{C}$ which are composed entirely of sets in $\mathcal{F}$. Clearly $\boldsymbol{B}_{\mathcal{F}}$ satisfies B1–B3 since $\boldsymbol{B}$ satisfies P1–P3. Since $\{x\} \in \mathcal{F}$ for all $x \in X$, $\boldsymbol{B}_{\mathcal{F}}$ satisfies B4 also and $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ is a closed base for some binding structure on X. Denote this binding structure by $(\boldsymbol{B}^1, \boldsymbol{C}^1)$.

A simple consequence of P6 is that $\boldsymbol{C} = \boldsymbol{C}^1$. To see that $\boldsymbol{B} = \boldsymbol{B}^1$, we observe that each cluster base $\mathcal{S}$ in $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ is the intersection of a unique cluster in $(\boldsymbol{B}, \boldsymbol{C})$ with $\mathcal{F}$, and that each cluster in $(\boldsymbol{B}, \boldsymbol{C})$ gives rise to a cluster base in $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$. If $\mathcal{U}^1$ is the cluster in $(\boldsymbol{B}^1, \boldsymbol{C}^1)$ generated by the cluster base $\mathcal{U} \cap \mathcal{F}$, where $\mathcal{U}$ is a cluster in $(\boldsymbol{B}, \boldsymbol{C})$, it is not difficult to verify that $\mathcal{U}^1 = \mathcal{U}$. From P7 and the definition of $\boldsymbol{B}^1$ it follows that $\boldsymbol{B} = \boldsymbol{B}^1$.

1.8. Remark. We should point out at this time that distinct binding structures may have the same topology.

For example, let N denote the positive integers with the discrete topology. Let $\mathcal{F}_1$ be the family of all finite or cofinite sets and let $\mathcal{F}_2$ be the family of all subsets of N. If $F_1, F_2, \ldots, F_n \in \mathcal{F}_i$ then let $\{F_j\} \in \boldsymbol{B}_{\mathcal{F}_i}$ if and only if $\bigcap F_j \neq \emptyset$, for $i = 1, 2$. Let $\boldsymbol{C}_{\mathcal{F}_i}$ be the family of finite coverings of N by sets in $\mathcal{F}_i$. For $i = 1, 2$, $(\boldsymbol{B}_{\mathcal{F}_i}, \boldsymbol{C}_{\mathcal{F}_i})$ is a closed base for a binding structure $(\boldsymbol{B}_i, \boldsymbol{C}_i)$. The topology of each structure is the discrete topology, but $\boldsymbol{B}_1 \subsetneq \boldsymbol{B}_2$ and $\boldsymbol{C}_1 \subsetneq \boldsymbol{C}_2$.

A set X having a binding structure defined on it, explicitly or implicitly, will be called a *binding space*, or sometimes merely a *space*. A binding space will be called a *Hausdorff space*, a *regular space*, etc., whenever the topology of the binding structure is Hausdorff, regular, etc.

In the following, whenever $\mathcal{S}$ is a set function, $\mathcal{R}$ a family of sets and $\boldsymbol{R}$ a collection of families, we will write $\mathcal{S}(\mathcal{R}) = \{\mathcal{S}(R)\colon R \in \mathcal{R}\}$ and $\mathcal{S}(\boldsymbol{R}) = \{\mathcal{S}(\mathcal{R})\colon \mathcal{R} \subset \boldsymbol{R}\}$.

A cluster $\mathcal{C}$ in a binding space is *free* if $\bigcap \overline{\mathcal{C}} = \emptyset$; otherwise it is *fixed*. A binding space is *complete* if each cluster is fixed. If the cluster $\mathcal{C}$ is generated by the cluster base $\mathcal{S}$, then $\mathcal{S} \subset \overline{\mathcal{C}}$ and $\bigcap \mathcal{S} = \emptyset$ implies $\bigcap \overline{\mathcal{C}} = \emptyset$. Conversely, if $x \in \bigcap \mathcal{S}$, then $\{x\} \in \mathcal{C}$; thus $x \in \overline{A}$ for each $A \in \mathcal{C}$ and $x \in \bigcap \overline{\mathcal{C}}$. To determine the completeness of a binding space, it is therefore sufficient to consider only cluster bases.

The cluster bases of the binding structure generated by the closed base of example 1.1 are the ultrafilters of closed sets. Thus this binding space is complete if and only if its topology is compact.

1.9. Lemma. *A cluster is fixed if and only if it contains a singleton.*

Proof. If $\{x\} \in \mathcal{C}$, then $\{x, A\} \in \boldsymbol{B}$ for each $A \in \mathcal{C}$. Thus $x \in \bigcap \{\overline{A}: A \in \mathcal{C}\}$ and $\mathcal{C}$ is fixed. Conversely, if $x \in \bigcap \{\overline{A}: A \in \mathcal{C}\}$, then $\{x, \overline{A}_i\} \in \boldsymbol{B}$ for each finite subcollection $\{A_i\}$ of $\mathcal{C}$. From P4, $\{x, A_i\} \in \boldsymbol{B}$ and so the maximality of $\mathcal{C}$ implies $\{x\} \in \mathcal{C}$. Actually, $x \in \bigcap \overline{\mathcal{C}}$ implies $\{x\} = \bigcap \overline{\mathcal{C}}$ by P5.

If $(\boldsymbol{B}, \boldsymbol{C})$ and $(\boldsymbol{B}^1, \boldsymbol{C}^1)$ are binding structures on X and X^1, respectively, a function f from X to X^1 is a *mapping* if $f^{-1}(\boldsymbol{C}^1) \subset \boldsymbol{C}$ and $f(\boldsymbol{B}) \subset \boldsymbol{B}^1$. Thus a mapping takes bound collections forward to bound collections and covers backward to covers.

1.10 Theorem. *A mapping is a continuous function.*

Proof. Let f be a mapping from X to X^1 and let F be closed in the topology of $(\boldsymbol{B}^1, \boldsymbol{C}^1)$. If $x \notin f^{-1}(F)$, then $f(x) \notin F$ and thus $\{f(x), F\} \notin \boldsymbol{B}^1$. Since $f(\boldsymbol{B}) \subset \boldsymbol{B}^1$, $\{x, f^{-1}(F)\} \notin \boldsymbol{B}$ so $x \notin \overline{f^{-1}(F)}$. Thus $f^{-1}(F)$ is closed in the topology of $(\boldsymbol{B}, \boldsymbol{C})$ and f is continuous.

If $\boldsymbol{C}$ and $\boldsymbol{C}^1$ are uniformities on X and X^1, respectively, then a mapping from X to X^1 is clearly a uniformly continuous function.

If f is a one-to-one mapping of X onto X^1 and f^{-1} is also a mapping, then X and X^1 are *isomorphic* and f is an *isomorphism*. Identities and compositions of mappings are mappings; thus binding spaces and mappings constitute the *category of binding spaces*.

1.11. If $(\boldsymbol{B}, \boldsymbol{C})$ and $(\boldsymbol{B}^1, \boldsymbol{C}^1)$ are binding structures on the same set X and the identity functions on X is a mapping of $(\boldsymbol{B}, \boldsymbol{C})$ to $(\boldsymbol{B}^1, \boldsymbol{C}^1)$, then $(\boldsymbol{B}, \boldsymbol{C})$ is *finer than* $(\boldsymbol{B}^1, \boldsymbol{C}^1)$ and $(\boldsymbol{B}^1, \boldsymbol{C}^1)$ is *coarser than* $(\boldsymbol{B}, \boldsymbol{C})$.

The finest binding structure on a set X is the pair $(\boldsymbol{B}_1, \boldsymbol{C}_1)$ where $\{A_i\} \in \boldsymbol{B}_1$ if and only if $\bigcap A_i \neq \emptyset$ and $\boldsymbol{C}_1$ consists of all coverings of X. The topology of $(\boldsymbol{B}_1, \boldsymbol{C}_1)$ is the discrete topology since $A = \overline{A}$ for each $A \subset X$. The covering composed entirely of singleton sets is in $\boldsymbol{C}_1$, so each cluster must contain a singleton and must therefore be fixed, 1.9. Thus $(\boldsymbol{B}_1, \boldsymbol{C}_1)$ is complete.

The coarset binding structure on a set X is the pair $(\boldsymbol{B}_0, \boldsymbol{C}_0)$ where $\{A_i\} \in \boldsymbol{B}_0$ if and only if either $\bigcap A_i \neq \emptyset$ or some A_i is infinite, and $\boldsymbol{C}_0$ consists of all coverings refined by the covering $C = \{X\}$. If A is infinite, $\overline{A} = X$ and if A is finite, $\overline{A} = A$. Thus the topology of $(\boldsymbol{B}_0, \boldsymbol{C}_0)$ is the cofinite topology. From the definition of $\boldsymbol{B}_0$ it follows that each cluster must contain a singleton, and is therefore fixed, 1.9; hence $(\boldsymbol{B}_0, \boldsymbol{C}_0)$ is complete.

Actually, the family of all binding structures on a set X forms a complete lattice, but the proof of this will be delayed until the next section, where it follows from a theorem on products.

2. Fundamental constructions. If X and Y are binding spaces and Y is a subset of X, Y will be called a *subspace* of X if the binding structure on Y is the coarsest one which makes the injection $i\colon Y \to X$ a mapping.

In the following, we will construct this coarsest binding on Y. Let $(\boldsymbol{B}, \boldsymbol{C})$ denote the binding structure on X and $\mathcal{G} = \{\overline{A} \cap Y: A \subset X\}$. Let $\boldsymbol{C}_{\mathcal{G}}$ be the trace on Y of covers in $\boldsymbol{C}$ and if $G_1, G_2, \ldots, G_n \in \mathcal{G}$, let $\{G_i\} \in \boldsymbol{B}_{\mathcal{G}}$ if and only if $\{G_i\} \in \boldsymbol{B}$. It is not difficult to verify that $(\boldsymbol{B}_{\mathcal{G}}, \boldsymbol{C}_{\mathcal{G}})$ is a closed base for a binding structure. Let $\big(\boldsymbol{B}(Y)\boldsymbol{C}(Y)\big)$ denote the binding structure on Y generated by $(\boldsymbol{B}_{\mathcal{G}}, \boldsymbol{C}_{\mathcal{G}})$.

It is easy to observe that $C \in \boldsymbol{C}(Y)$ if and only if C is the trace on Y of a cover in $\boldsymbol{C}$ $(\boldsymbol{C}(Y) = \boldsymbol{C} \cap Y)$, and that $\boldsymbol{B}(Y) \subseteq \boldsymbol{B}$. However, there may be sets $A_1, \ldots, A_n \subset Y$ such that $\{A_1\} \in \boldsymbol{B}$, but $\{A_i\} \notin \boldsymbol{B}(Y)$.

2.1. Lemma. *The injection $i\colon Y \to X$ is a mapping.*

2.2. Lemma. *If $(\boldsymbol{B}^1, \boldsymbol{C}^1)$ is any binding structure on Y making the injection a mapping, then $(\boldsymbol{B}^1, \boldsymbol{C}^1)$ is finer than $\big(\boldsymbol{B}(Y), \boldsymbol{C}(Y)\big)$.*

Proof. First $\boldsymbol{C}(Y) = i^{-1}(C) \subset \boldsymbol{C}^1$. Second, if $\{A_1, \ldots, A_n\} \in \boldsymbol{B}^1$, then $\{i(A_j)\} \in \boldsymbol{B}$ and so $\{\overline{A}_j \cap Y\} \in \boldsymbol{B}_{\mathcal{G}}$. Let $\mathcal{C}^1$ be a cluster in $(\boldsymbol{B}^1, \boldsymbol{C}^1)$ containing $\{A_i\}$, and let $\mathcal{E} = \{\overline{A} \cap Y: A \in \mathcal{C}^1\}$. Thus finite collections of $\mathcal{E}$ are in $\boldsymbol{B}_{\mathcal{G}}$ and $\mathcal{E}$ contains a member of each cover in $\boldsymbol{C}_{\mathcal{G}}$. $\mathcal{E}$ is contained in a cluster $\mathcal{C}$ in $\big(\boldsymbol{B}(Y), \boldsymbol{C}(Y)\big)$, $\{A_i\} \subseteq \mathcal{C}$ and thus $\{A_i\} \in \boldsymbol{B}(Y)$.

Lemmas 2.1 and 2.2 show that $\big(\boldsymbol{B}(Y), \boldsymbol{C}(Y)\big)$ is the subspace binding structure on Y. It follows that if f is a mapping from a space W into a space X, then f is a mapping from W onto the subspace $f[W]$ of X.

A subspace Y of X is called a *dense subspace* provided $\overline{\big(\boldsymbol{B}(Y), \boldsymbol{C}(Y)\big)}$ is a closed base for the binding structure of X, where closures are taken in X.

If Y is a dense subspace of X then the following hold:

(a) The closures (in X) of sets in Y are a base for the closed sets of X.
(b) Y is topologically dense in X, i.e. $\overline{Y} = X$.
(c) The closures of covers in $\boldsymbol{C}(Y)$ are in $\boldsymbol{C}$.
(d) If $A_1, \ldots, A_n \subset Y$, then $\{A_i\} \in \boldsymbol{B}(Y)$ if and only if $\{\overline{A}_i\} \in \boldsymbol{B}$.
(e) For each $x \in X - Y$, there is a cluster $\mathcal{C}$ in Y such that $\bigcap \overline{\mathcal{C}} = \{x\}$.

The following example shows that a topologically dense subspace need not be dense as a binding space. Let $X = [0, 1]$, $\boldsymbol{C}$ be all coverings of X, and let $\{A_i\} \in \boldsymbol{B}$ if and only if $\bigcap \mathrm{cl} A_i \neq \emptyset$, where $\mathrm{cl} A$ is the closure of A in the usual topology on $[0, 1]$. It follows that $x \in \overline{A}$ if and only if $x \in \mathrm{cl} A$, and $\boldsymbol{C}$ contains the covering consisting of singletons. Let $Y = [0, 1)$ be a subspace of X. Then the covering c of Y which consists of singletons in Y is a cover in $\boldsymbol{C}(Y)$. Clearly, $\overline{c} = \{\{\overline{y}\}: y \in Y\} = c$ is not a covering of X, so $\overline{c} \notin \boldsymbol{C}$. By property (c) above, Y is not a dense subspace of X.

The *product* of a family $\{X_a\}$ of binding spaces is the coarsest binding structure on the cartesian product of the sets X_a making each projection a mapping.

2.3. THEOREM. *The product of any family of binding spaces exists.*

Proof. Let $(\boldsymbol{B}_a, \boldsymbol{C}_a)$ denote the binding structure on X_a and let $\mathcal{F}_a$ be the family of closed sets in X_a with respect to the topology of $(\boldsymbol{B}_a, \boldsymbol{C}_a)$. Let π be the cartesian product set of the factors X_a and let p_a denote the coordinate projection of π onto X_a.

The family $\{p_a^{-1}(F)\colon F \in \mathcal{F}_a\}_a$ is a subbase for the product topology on π which is a T_1-topology. Let $\mathcal{F}$ be the family of finite unions of these subbase elements and let $\boldsymbol{C}_{\mathcal{F}}$ be the family of covers of π of the form $p_a^{-1}(c)$ where c is a closed cover in $\boldsymbol{C}_a$. If $F_1, \ldots, F_n \in \mathcal{F}$, let $\{F_i\} \in \boldsymbol{B}_{\mathcal{F}}$ if and only if for each i a subbase element $A_i \subset F_i$ can be chosen so that $\{p_a(A_i)\} \in \boldsymbol{B}_a$ for each a.

We need to verify that $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ satisfies conditions B1–B4. First, if $x \in \bigcap_{i=1}^{n} F_i$ for $F_1, F_2, \ldots, F_n \in \mathcal{F}$, then there is a subbase element $G_i \subset F_i$ such that $x \in \bigcap_{i=1}^{n} G_i$. Thus $p_a(x) \in \bigcap_{i=1}^{n} p_a(G_i)$ for each a, so $\{p_a(G_i)\} \in \boldsymbol{B}_a$ and $\{F_i\} \in \boldsymbol{B}_{\mathcal{F}}$. Thus B1 holds. Clearly B2 is valid.

To see that B4 holds, suppose $x \notin p_a^{-1}(F_a)$ for $F_a \in \mathcal{F}_a$. Then $p_a(x) \notin F_a$ so $\{p_a^{-1}p_a(x), p_a^{-1}(F_a)\} \notin \boldsymbol{B}_{\mathcal{F}}$. (Since $\mathcal{F}_a$ is the family of all closed sets in the topology of $(\boldsymbol{B}_a, \boldsymbol{C}_a)$ which is T_1, $p_a(x) \in \mathcal{F}_a$ so $p_a^{-1}p_a(x) \in \mathcal{F}$.) Now, if $x \notin F = \bigcup_{i=1}^{m} p_{a_i}^{-1}(F_{a_i})$, then, as above, $\{p_{a_1}^{-1}p_{a_1}(x), \ldots, p_{a_m}^{-1}p_{a_m}(x), F\} \notin \boldsymbol{B}_{\mathcal{F}}$ and $x \in \bigcap_{i=1}^{m} p_{a_i}^{-1}p_{a_i}(x)$.

To show that B3 holds, we will use the fact that if $p_\beta^{-1}(A_\beta) \subset \bigcup_{i=1}^{m}\{\bigcup_{j=1}^{n_i} p_{a_i}^{-1}(A_{j,i})\}$ where $A_\beta \in \mathcal{F}_\beta$ and $A_{j,i} \in \mathcal{F}_{a_i}$, then $p_\beta^{-1}(A_\beta) \subset \bigcup_{j=1}^{n_i} p_{a_i}^{-1}(A_{j,i})$ for some i, $1 \leqslant i \leqslant m$.

Now assume $\{F_i, G\} \in \boldsymbol{B}_{\mathcal{F}}$ and $G \subset H \in \mathcal{F}$. There are subbase elements $A_i \subset F_i$ and $B \subset G$ such that $\{p_a(A_i), p_a(B)\} \in \boldsymbol{B}_a$ for each a. $B \subset G \subset H = \bigcup_{j=1}^{m}\{\bigcup_{k=1}^{n_j} p_{a_j}^{-1}(D_{k,j})\}$ so $B \subset \bigcup_{k=1}^{n_j} p_{a_j}^{-1}(D_{k,j})$ for some $j, 1 \leqslant j \leqslant m$. $p_{a_j}(B) \subset \bigcup p_{a_j}p_{a_j}^{-1}(D_{k,j}) = \bigcup_{k=1}^{n_j} D_{k,j}$ so by P3, $\{p_{a_j}(A_i), D_{k,j}\} \in \boldsymbol{B}_{a_j}$ for some k. If $a \neq a_j$, then $p_a p_{a_j}^{-1}(D_{k,j}) = X_a$ and thus $\{p_a(A_i), p_a p_{a_j}^{-1}(D_{k,j})\} \in \boldsymbol{B}_a$. Since $p_{a_j}^{-1}(D_{k,j}) \subset H$, $\{F_i, H\} \in \boldsymbol{B}_{\mathcal{F}}$. Thus $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ is a closed base for a binding structure, say $(\boldsymbol{B}, \boldsymbol{C})$, on π.

To see that the projections are mappings, let $C_a \in \boldsymbol{C}_a$. By P6, there is a closed cover $d_a \in \boldsymbol{C}_a$ such that d_a refines C_a. From the definition of $\boldsymbol{C}_{\mathcal{F}}$, $p_a^{-1}(d_a) \in \boldsymbol{C}_{\mathcal{F}}$ and $p_a^{-1}(d_a)$ refines $p_a^{-1}(C_a)$, so $p_a^{-1}(C_a) \in \boldsymbol{C}$. Thus $p_a^{-1}(\boldsymbol{C}_a) \subset \boldsymbol{C}$ for each a.

The topology of $(\boldsymbol{B}, \boldsymbol{C})$ is just the product topology of π so the projections are continuous. This implies that p_a takes bound families into bound families, for if $A_1, \ldots, A_n \subset \pi$ and $\{p_a(\overline{A_i})\} \notin \boldsymbol{B}_a$, then $\{\overline{p_a(A_i)}\} \notin \boldsymbol{B}_a$. From the definition of $\boldsymbol{B}_{\mathcal{F}}$, $\{p_a^{-1}\overline{p_a(A_i)}\} \notin \boldsymbol{B}_{\mathcal{F}}$ and thus $\{p_a^{-1}\overline{p_a(A_i)}\} \notin \boldsymbol{B}$. Since $A_1 \subset p_a^{-1}\overline{p_a(A_i)}$, $\{A_i\} \notin \boldsymbol{B}$.

Finally, suppose $(\boldsymbol{B}^1, \boldsymbol{C}^1)$ is another binding structure on π making each projection a mapping. Each cover of π which is refined by a cover of the form $p_a^{-1}(c_a)$ for $c_a \in \boldsymbol{C}_a$, is in $\boldsymbol{C}^1$ and thus $\boldsymbol{C} \subseteq \boldsymbol{C}^1$. If $F_1, \ldots, F_n \notin \mathcal{F}$ and $\{F_i\} \in \boldsymbol{B}^1$, then by P3, there are sets $A_i \in \bigcup_a p_a^{-1}(\mathcal{F}_a)$ such that $A_i \subset F_i$ and $\{A_i\} \in \boldsymbol{B}^1$. Thus $\{p_a(A_i)\} \in \boldsymbol{B}_a$ for each a and by definition, $\{F_i\} \in \boldsymbol{B}_{\mathcal{F}}$. Now, suppose $\{G_i\} \in \boldsymbol{B}^1$. There is a cluster $\mathcal{U}$ in $(\boldsymbol{B}^1, \boldsymbol{C}^1)$ containing $G_1, \ldots, G_m$. Let $\mathcal{E}^1 = \{F\colon F \in \mathcal{U} \cap \mathcal{F}\}$. Since $\boldsymbol{C}_{\mathcal{F}} \subseteq \boldsymbol{C}^1$, $\mathcal{U}$ (and hence $\mathcal{E}^1$) contains a set from every cover in $\boldsymbol{C}_{\mathcal{F}}$. Each finite subcollection from $\mathcal{E}^1$ is in $\boldsymbol{B}_{\mathcal{F}}$ so $\mathcal{E}^1$ is contained in a cluster base $\mathcal{E}$ in $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$, and by 1.2, $\mathcal{E} \subset \mathcal{V}$, a cluster in $(\boldsymbol{B}, \boldsymbol{C})$. If $G_i \subset F \in \mathcal{F}$, then $F \in \mathcal{U}$. Thus $F \in \mathcal{U} \cap \mathcal{F} = \mathcal{E}^1 \subset \mathcal{E}$ and $G_i \in \mathcal{V}$. From the definition of $\boldsymbol{B}$, $\{G_i\} \in \boldsymbol{B}$. Thus $(\boldsymbol{B}^1, \boldsymbol{C}^1)$ is finer than $(\boldsymbol{B}, \boldsymbol{C})$ and the theorem is proved.

In any category, an object Z is a product of a family $\{X_a\}$ if there exist mappings $p_a\colon Z \to X_a$ such that (1) if $f\colon W \to Z$ and $g\colon W \to Z$ are two different mappings, then for some a, $p_a f \neq p_a g$, and (2) for every family of mappings $f_a\colon W \to X_a$, there exists a mapping $f\colon W \to Z$ such that $p_a f = f_a$ for each a. Clearly π with the projection mappings satisfies (1), and (2) also holds when f is defined by $f(w)_a = f_a(w)$. Thus π is a product in the categorical sense and we have:

2.4. THEOREM. *A function f from a space W into the product π of spaces X_a is a mapping if and only if $p_a f$ is a mapping for each a.*

Theorem 2.3 can now be used to prove that the family of all binding structures on a given set X is a complete lattice with respect to the partial order defined in 1.11.

Let $\{(\boldsymbol{B}_a, \boldsymbol{C}_a)\}$ be a family of binding structures on X and let $(\boldsymbol{B}, \boldsymbol{C})$ denote the product binding structure on π, the product of these binding spaces. The injection $e\colon X \to \pi$ defined by $p_a e(x) = x$ for all $x \in X$ is one-to-one and eX is a subset of π. If $\big(\boldsymbol{B}(eX), \boldsymbol{C}(eX)\big)$ denotes the subspace binding structure on eX, then $\big(e^{-1}\boldsymbol{B}(eX), e^{-1}\boldsymbol{C}(eX)\big)$ is a binding structure on X. Let $\boldsymbol{B}^1 = e^{-1}\boldsymbol{B}(eX)$ and $\boldsymbol{C}^1 = e^{-1}\boldsymbol{C}(eX)$.

2.5. LEMMA. $(\boldsymbol{B}^1, \boldsymbol{C}^1)$ is finer than $(\boldsymbol{B}_a, \boldsymbol{C}_a)$ for each a.

Proof. From the definition of subspace, e is a mapping from X into π. Thus $p_a e$ is a mapping of $(\boldsymbol{B}^1, \boldsymbol{C}^1)$ onto $(\boldsymbol{B}_a, \boldsymbol{C}_a)$.

2.6. THEOREM. *$(\boldsymbol{B}^1, \boldsymbol{C}^1)$ is the coarsest binding structure on X which is finer than $(\boldsymbol{B}_a, \boldsymbol{C}_a)$ for each a.*

Proof. Let $(\boldsymbol{B}_1, \boldsymbol{C}_1)$ be a binding structure on X which is finer than $(\boldsymbol{B}_a, \boldsymbol{C}_a)$ for each a. Then the function $p_a e\colon (\boldsymbol{B}_1, \boldsymbol{C}_1) \to (\boldsymbol{B}_a, \boldsymbol{C}_a)$ is a mapping (since $p_\alpha e$ is just the identity). From 2.4 it follows that $e\colon (\boldsymbol{B}_1, \boldsymbol{C}_1) \to \pi$ is a mapping. The definition of the subspace binding implies that $(\boldsymbol{B}_1, \boldsymbol{C}_1)$ is finer than $(\boldsymbol{B}^1, \boldsymbol{C}^1)$. This, together with 2.5, completes the proof.

The binding structure $(\boldsymbol{B}^1, \boldsymbol{C}^1)$ may thus be denoted as $\bigvee_\alpha (\boldsymbol{B}_\alpha, \boldsymbol{C}_\alpha)$. It can be shown that $\boldsymbol{C}^1 = \bigcup_\alpha \boldsymbol{C}_\alpha$ and that $\{x, A\} \in \boldsymbol{B}^1$ if and only if $\{x, A\} \in \boldsymbol{B}_\alpha$ for each α.

There is a largest and a smallest binding structure on X (1.11). Since, in any partial order, the existence of suprema and a smallest element implies the existence of infima, it follows from 2.6 that the family of all binding structures on X forms a complete lattice.

A *quotient mapping* $q\colon X \to Q$ is an onto mapping such that whenever q is a composition of two mappings $q = gf$, where $g\colon Q^1 \to Q$ is one-to-one and onto, then g is an isomorphism.

2.7. THEOREM. *Every mapping $f\colon X \to Y$ has the form $f^1 q$, where $q\colon X \to Q$ is a quotient mapping and $f^1\colon Q \to Y$ is one-to-one.*

Proof. Consider the family of all binding structures on the set $f[X]$ which make f a mapping. The binding structure on $f[X]$ induced by that on Y is such a structure. That f is a mapping with respect to the supremum of this family of structures on $f[X]$ follows from 2.4 and the definition of the supremum. Let Q be the set $f[X]$ with this finest structure making f a mapping. Let $q\colon X \to Q$ be coincident with f and let $f^1\colon Q \to Y$ be the identity mapping.

If $q\colon X \to Q$ factors as $q = g_1 g_2$ where $g_2\colon X \to Q^1$ and $g_1\colon Q^1 \to Q$ is one-to-one and onto, then the binding structure on Q^1 is finer than that on Q and makes f a mapping. From the definition of Q, g_1 must be an isomorphism. Thus q is a quotient mapping.

2.8. The topology of the product, or subspace binding structure is the product, or subspace topology, respectively. This is obvious since the product and subspace binding structures were generated by bases for closed sets of the product and subspace topology, respectively. The following example shows that this is not the case for quotients.

Let $X = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, $\mathcal{F}$ the family of closed (in the usual topology) subsets of X, and $\boldsymbol{C}_{\mathcal{F}}$ the family of finite covers from $\mathcal{F}$. A finite subcollection $F_1, F_2, \ldots, F_n \in \mathcal{F}$ will be in $\boldsymbol{B}_{\mathcal{F}}$ if and only if $\bigcap \overline{F}_i \neq \emptyset$ where $\overline{F}_i$ is the closure of F_i in the usual topology on $[0, 1]$. It is easy to verify that $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ is a closed base for a binding structure $(\boldsymbol{B}, \boldsymbol{C})$ on X. Moreover, $\{[0, \frac{1}{2}), (\frac{1}{2}, 1]\} \in \boldsymbol{B}$. Let $Y = [0, \frac{1}{2}]$ have the binding structure generated by finite covers of closed sets (in usual topology) and finite subcollections of closed sets with nonempty intersection. The function $f\colon X \to Y$ defined by $f(x) = x$, $x \in [0, \frac{1}{2})$ and $f(x) = \frac{1}{2}$, $x \in (\frac{1}{2}, 1]$ is a mapping. Since $\{[0, \frac{1}{2}), (\frac{1}{2}, 1]\} \in \boldsymbol{B}$, $\{[0, \frac{1}{2}), \{\frac{1}{2}\}\}$ is bound in the quotient structure on Y, that is, $\frac{1}{2}$ is in the closure of $[0, \frac{1}{2})$ with respect to the topology of the

quotient binding structure. However, since f is the identity on $[0, \frac{1}{2})$ and $[0, \frac{1}{2})$ is closed in the topology of $(\boldsymbol{B}, \boldsymbol{C})$ on X, $[0, \frac{1}{2})$ is closed in the quotient topology on Y.

In general, since every mapping is continuous (1.10), the quotient topology is at least as fine as the topology of the quotient binding structure, and in the above example, it is finer.

3. Complete spaces and completions. In section 1, a binding structure $(\boldsymbol{B}, \boldsymbol{C})$ on a set X is defined to be complete provided each cluster (or equivalently, each cluster base) is fixed. This definition is analogous to that of a complete uniform space in which each Cauchy ultrafilter converges, or to a compact topological space in which each ultrafilter of closed sets has a nonempty intersection. As might be expected, some of the theorems regarding complete uniform spaces and compact topological spaces carry over to complete binding spaces.

3.1. THEOREM. *A closed subspace of a complete binding space is complete.*

Proof. Let Y be a closed subspace of X and let $\mathcal{C}$ be a cluster in $(\boldsymbol{B}_Y, \boldsymbol{C}_Y)$, the binding structure on Y induced by $(\boldsymbol{B}, \boldsymbol{C})$ on X. By 2.2, $\boldsymbol{B}_Y \subset \boldsymbol{B}$. Let $\mathcal{U} = \{E \subset X\colon A \subset E$ for some $A \in \mathcal{C}\}$. Property P3 and $\boldsymbol{B}_Y \subset \boldsymbol{B}$ imply that every finite subfamily of $\mathcal{U}$ is in $\boldsymbol{B}$. If $C \in \boldsymbol{C}$, then $C \cap Y \in \boldsymbol{C}_Y$, so there is an $A \in C \cap Y$ such that $A \in \mathcal{C}$. But $A = B \cap Y$ for some $B \in C$ and B is in $\mathcal{U}$. Thus $\mathcal{U}$ is a cluster in $(\boldsymbol{B}, \boldsymbol{C})$ and $\mathcal{C} \subset \mathcal{U}$. Since X is complete, $\bigcap \overline{\mathcal{C}} \supset \bigcap \overline{\mathcal{U}} \neq \emptyset$, and since Y is closed, $\bigcap \overline{\mathcal{C}} \subset Y$.

The converse of 3.1 is not true, i.e. a complete subspace of a complete binding space need not be closed. To see this, it suffices to notice that if $\boldsymbol{C}$ contains the cover composed of singletons, then any subspace of X will be complete, but not necessarily closed. The binding structure $(\boldsymbol{B}_1, \boldsymbol{C}_1)$ defined in 1.11 is such an example. It can be said, however, that a complete subspace of a complete binding space is not a dense subspace.

3.2. THEOREM. *The product of complete spaces is complete.*

Proof. Let π denote the product of spaces X_α having complete binding structures $(\boldsymbol{B}_\alpha, \boldsymbol{C}_\alpha)$, and let $(\boldsymbol{B}, \boldsymbol{C})$ denote the product binding structure on π. By 2.3, the projections $p_\alpha\colon \pi \to X_\alpha$ are mappings, so if $\mathcal{C}$ is a cluster in $(\boldsymbol{B}, \boldsymbol{C})$, $p_\alpha(\mathcal{C})$ is contained in a cluster $\mathcal{U}_\alpha$ in $(\boldsymbol{B}_\alpha, \boldsymbol{C}_\alpha)$. Let $x_\alpha = \bigcap \overline{\mathcal{U}}_\alpha$ and let $x = (x_\alpha) \in \pi$.

If $A \subset \pi$ and $x \notin \overline{A}$, then there are closed sets $F_{\alpha_i} \subset X_{\alpha_i}$ such that $x \notin \bigcup \{p_{\alpha_i}^{-1}(F_{\alpha_i})\colon 1 \leqslant i \leqslant n\} \supset A$. Thus $x_{\alpha_i} = p_{\alpha_i}(x) \notin p_{\alpha_i}[p_{\alpha_i}^{-1}(F_{\alpha_i})] = F_{\alpha_i}$, i.e. $F_{\alpha_i} \notin \mathcal{U}_{\alpha_i}$ and consequently $p_{\alpha_i}^{-1}(F_{\alpha_i}) \notin \mathcal{C}$. By P3, $\bigcup \{p_{\alpha_i}^{-1}(F_{\alpha_i})\colon 1 \leqslant i \leqslant n\}$ is not in $\mathcal{C}$ and thus $\overline{A} \notin \mathcal{C}$. This shows $x \in \bigcap \overline{\mathcal{C}}$ and π is complete.

A space Y is said to be a *completion* of X if Y is complete and X is isomorphic to a dense subspace of Y. When no confusion will arise, we will simply assume that X is a dense subspace of Y.

3.3. THEOREM. *Each binding space has a completion.*

Proof. Let $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ be any closed base for the binding structure $(\boldsymbol{B}, \boldsymbol{C})$ on X and let X^* be the set of all cluster bases in $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$. For each $F \in \mathcal{F}$, let $F^* = \{\mathcal{S} \in X^*\colon F \in \mathcal{S}\}$ and let $\mathcal{F}^* = \{F^*\colon F \in \mathcal{F}\}$. Define $\boldsymbol{B}_{\mathcal{F}^*} = \big\{\{F^*\}\colon \{F\} \in \boldsymbol{B}_{\mathcal{F}}\big\}$ and $\boldsymbol{C}_{\mathcal{F}^*} = \big\{C^* = \{F^*\}\colon C = \{F\} \in \boldsymbol{C}_{\mathcal{F}}\big\}$.

First, $\mathcal{F}^*$ is a base for the closed sets of some T_1-topology on X^*. If $\mathcal{S} \in X^*$ and $F_1^*, F_2^* \in \mathcal{F}^*$ with $\mathcal{S} \notin F_1^* \cup F_2^*$, then $F_1 \notin \mathcal{S}$ and $F_2 \notin \mathcal{S}$. If $\mathcal{U}$ is the cluster in $(\boldsymbol{B}, \boldsymbol{C})$ generated by $\mathcal{S}$, then $F_1 \cup F_2 \notin \mathcal{U}$. Thus there is an $F_3 \in \mathcal{F}$ such that $F_1 \cup F_2 \subset F_3$ and $F_3 \notin \mathcal{S}$. This shows $\mathcal{S} \notin F_3^*$. It is easy to verify that $F_1^* \cup F_2^* \subset F_3^*$. Clearly, $\mathcal{S} \in \bigcap \{F^*\colon F \in \mathcal{S}\}$. If $\mathcal{S}^1 \neq \mathcal{S}$, then there is a $F \in \mathcal{F}$ such that $F \in \mathcal{S}$ and $F \notin \mathcal{S}^1$. Thus $\mathcal{S}^1 \notin \bigcap \{F^*\colon F \in \mathcal{S}\}$ and $\{\mathcal{S}\} = \bigcap \{F^*\colon F \in \mathcal{S}\}$.

The verification that $\boldsymbol{B}_{\mathcal{F}^*}$ satisfies B1–B4 is straightforward. Since $\mathcal{S} \in X^*$ contains an element from each $C \in \boldsymbol{C}_{\mathcal{F}}$, $\boldsymbol{C}_{\mathcal{F}^*}$ is a family of coverings of X^*. Thus $(\boldsymbol{B}_{\mathcal{F}^*}, \boldsymbol{C}_{\mathcal{F}^*})$ is a closed base for a binding structure $(\boldsymbol{B}^*, \boldsymbol{C}^*)$ on X^*, which is the desired completion.

To see that $(\boldsymbol{B}^*, \boldsymbol{C}^*)$ is complete, it is sufficient to observe that for each $\mathcal{S} \in X^*$, the family $\mathcal{S}^* = \{F^*\colon F \in \mathcal{S}\}$ is a cluster base in $(\boldsymbol{B}_{\mathcal{F}^*}, \boldsymbol{C}_{\mathcal{F}^*})$, and that $\mathcal{S} = \bigcap \mathcal{S}^*$.

Define $e\colon X \to X^*$ by $e(x) = \mathcal{S}_x = \{F \in \mathcal{F}\colon x \in F\}$. Clearly, e is one-to-one. To see that e is a mapping, let $d^* \in \boldsymbol{C}^*$. There is a cover $C^* \in \boldsymbol{C}_{\mathcal{F}^*}$ which refines d^*. For each $F^* \in \mathcal{F}^*$, $e^{-1}(F^*) = F$ and thus $e^{-1}(C^*) = C \in \boldsymbol{C}_{\mathcal{F}}$. Since C refines $e^{-1}(d^*)$, $e^{-1}(d^*) \in \boldsymbol{C}$.

To show that $e(\boldsymbol{B}) \subset \boldsymbol{B}^*$, it is sufficient to show that if $\mathcal{U}$ is a cluster in $(\boldsymbol{B}, \boldsymbol{C})$ and $\mathcal{U}$ is generated by the cluster base $\mathcal{S}$, then $e[\mathcal{U}]$ is contained in the cluster in $(\boldsymbol{B}^*, \boldsymbol{C}^*)$ generated by $\mathcal{S}^*$. Let $A \in \mathcal{U}$. If $e(A) \subset F^* \in \mathcal{F}^*$, then $e(A) \subset F^* \cap e(X) = e(F)$. Thus $A \subset F \in \mathcal{F}$ so by the definition of $\mathcal{U}$, $F \in \mathcal{S}$. This says that $F^* \in \mathcal{S}^*$, and that $e(A)$ is in the cluster in $(\boldsymbol{B}^*, \boldsymbol{C}^*)$ generated by $\mathcal{S}^*$.

If $(\boldsymbol{B}_{eX}, \boldsymbol{C}_{eX})$ is the subspace binding structure on $e(X)$, and if $d \in \boldsymbol{C}$, then d is refined by some $C \in \boldsymbol{C}_{\mathcal{F}}$, $e(d)$ is refined by $e(C) = C^* \cap e(X)$ $\in \boldsymbol{C}_{eX}$, and thus $e(d) \in \boldsymbol{C}_{eX}$. To see that $e^{-1}(\boldsymbol{B}_{e\overline{X}}) \subset \boldsymbol{B}$, first observe that if $A \subset e(X)$ and $\mathcal{S} \in \overline{A}$, then $e^{-1}(A)$ is in the cluster in $(\boldsymbol{B}, \boldsymbol{C})$ generated by $\mathcal{S}$. Now, if $\{A_i\} \in \boldsymbol{B}_{eX} \subset \boldsymbol{B}^*$, then $\{\overline{A}_i\} \in \boldsymbol{B}_j^*$; and since X^* is complete, there is an $\mathcal{S} \in X^*$ such that $\mathcal{S} \in \bigcap \overline{A}_i$. Hence $\{e^{-1}(A_1)\}$ is in a cluster in $(\boldsymbol{B}, \boldsymbol{C})$ and $\{e^{-1}(A_i)\} \in \boldsymbol{B}$. This completes the proof the e is an isomorphism of X onto $e(X)$, a subspace of X^*.

It remains only to show that $e(X)$ is dense in X^*, and this will be accomplished by showing that $\overline{eF} = F^*$ for all $F \in \mathcal{F}$. Clearly $\overline{eF} \subset F^*$. If $\mathcal{S} \in X^*$ and $\mathcal{S} \notin \overline{eF}$, then since $\mathcal{F}^*$ is a base for closed sets of X^* (1.7), there is an $F_1^* \in \mathcal{F}$, such that $\overline{eF} \subset F_1^*$ and $\mathcal{S} \notin F_1^*$. But $eF \subset F_1^*$ implies that $F \subset F_1$, and hence that $F^* \subset F_1^*$. Thus if $\mathcal{S} \notin \overline{eF}$, then $\mathcal{S} \notin F^*$, so $\overline{eF} = F^*$. It is now easy to verify that $\boldsymbol{B}_{\mathcal{F}^*} \subset \overline{\boldsymbol{B}_{eX}}$ and that $\boldsymbol{C}_{\mathcal{F}^*} \subset \overline{\boldsymbol{C}_{eX}}$. Now, since $(\boldsymbol{B}_{\mathcal{F}^*}, \boldsymbol{C}_{\mathcal{F}^*})$ is a closed base for $(\boldsymbol{B}^*, \boldsymbol{C}^*)$, every cover in $\boldsymbol{C}^*$ can be refined by one in $\boldsymbol{C}_{\mathcal{F}^*}$, hence in $\overline{\boldsymbol{C}_{eX}}$; since $\boldsymbol{B}_{\mathcal{F}^*} \subseteq \overline{\boldsymbol{B}_{eX}} \subseteq \boldsymbol{B}^*$, $(\overline{\boldsymbol{B}_{eX}}, \overline{\boldsymbol{C}_{eX}})$ must generate $(\boldsymbol{B}^*, \boldsymbol{C}^*)$.

3.4. LEMMA. *If F is a function from a complete space Z into a complete space Y such that $F(\overline{A}) \subset \overline{F(A)}$ for all $A \subset Z$, and the restriction of F to a dense subspace X of Z is a mapping, then F is a mapping.*

Proof. A finite collection of subsets of a complete space is bound if and only if the intersection of the closures of the sets is nonempty. Thus if $\{A_i\}$ is bound in Z, then $\bigcap \overline{A}_i \neq \emptyset$. This implies $\emptyset \neq \bigcap F(\overline{A}_i) \subset \bigcap \overline{F(A_i)}$, hence $\{F(A_i)\}$ is bound in Y.

If d is a cover of Y, then d is refined by a closed cover C of Y. $\overline{(F|X)^{-1}(C)}$ is a cover of Z. But F is continuous, so $\overline{(F|X^{-1}(C)}$ refines $F^{-1}(C)$ which in turn refines $F^{-1}(d)$. Thus $F^{-1}(d)$ is a cover of Z, and F is a mapping.

3.5. THEOREM. *The completion of a binding space is unique up to an isomorphism.*

Proof. Suppose Y and Z are complete binding spaces and that X is isomorphic to $e(X)$ and $g(X)$, dense subspaces of Y and Z, respectively. Let $(\boldsymbol{B}_{eX}, \boldsymbol{C}_{eX})$ and $(\boldsymbol{B}_{gX}, \boldsymbol{C}_{gX})$ denote the binding structures on $e(X)$ and $g(X)$, respectively. A family $\mathcal{U}$ of subsets of $e(X)$ is a cluster in $(\boldsymbol{B}_{eX}, \boldsymbol{C}_{eX})$ if and only if $\overline{\mathcal{U}}$ is a cluster base in the closed base $\overline{(\boldsymbol{B}_{eX}, \boldsymbol{C}_{eX})}$.

If $f\colon e(X) \to g(X)$ is defined by $f(e(x)) = g(x)$ for each $x \in X$, then f is an isomorphism of $e(X)$ onto $g(X)$. If $y \in Y$, then $y = \bigcap \overline{\mathcal{U}}$ for some cluster $\mathcal{U}$ in $(\boldsymbol{B}_{eX}, \boldsymbol{C}_{eX})$.

Define $F(y) = \bigcap \overline{f(\mathcal{U})}$. Then F is a one-to-one function from Y onto Z and $F\big(e(x)\big) = f\big(e(x)\big)$ for all $x \in X$. If $E \subset e(X)$ and $y \in \overline{E}$, then $\overline{E}$ is in the cluster base whose intersection is $\{y\}$. Consequently, $\overline{F(E)} = \overline{f(E)}$ is in the cluster base (in Z) whose intersection is $F(y)$. Thus $F(\overline{E}) \subset \overline{F(E)}$. The inclusion $\overline{F(E)} \subset F(\overline{E})$ follows in a similar manner. Since $\{\overline{E}\colon E \subset e(X)\}$ and $\{\overline{f(E)}\colon E \subset e(X)\}$ are bases for the closed sets in Y and Z, respectively, F is a homeomorphism. Applying 3.4 to F and F^{-1}, it follows that F is an isomorphism of Y onto Z which satisfies $Fe = g$.

Because of the one-to-one correspondence between cluster bases and clusters, the points in X^* may be thought of as clusters in X. Or, the points in $X^* - X$ may be thought of as "ideal points" to which the free clusters in X now converge.

3.6. THEOREM. *Every mapping on a binding space X into a complete regular binding space Y has a unique extension to the completion X^* of X.*

Proof. Let $f\colon X \to Y$ be a mapping where Y is complete and regular.

For each $p \in X^*$, let $\mathcal{U}_p = \{A \subset X: p \in \bar{A}\}$. If $f^{\#}(\mathcal{U}_p) = \{B \subset Y: f^{-1}(B) \in \mathcal{U}_p\}$, then $f^{\#}(\mathcal{U}_p)$ is contained in a cluster in Y. Let $y \in \bigcap f^{\#}(\mathcal{U}_p)$. If $z \neq y$, there are sets $B_1, B_2 \subset Y$ such that $\bar{B}_1 \cup \bar{B}_2 = Y$, $y \notin \bar{B}_1$ and $z \notin \bar{B}_2$. Since $f^{-1}(\bar{B}_1) \cup f^{-1}(\bar{B}_2) = X$, either $f^{-1}(\bar{B}_1)$ or $f^{-1}(\bar{B}_2)$ is in $\mathcal{U}_p$. This says either $\bar{B}_1$ or $\bar{B}_2$ is in $f^{\#}(\mathcal{U}_p)$. The condition $y \notin \bar{B}_1$ implies $\bar{B}_2 \in f^{\#}(\mathcal{U}_p)$, i.e. $z \notin \bigcap f^{\#}(\mathcal{U}_p)$.

Define $F: X^* \to Y$ by $F(p) = \bigcap f^{\#}(\mathcal{U}_p)$. If $x \in X$ and $B \in f^{\#}(\mathcal{U}_x)$, then $x \in \overline{f^{-1}(B)}$. Since f is a mapping, $f(x) \in \bar{B}$ and thus $F(x) = f(x)$ for all $x \in X$.

In light of 3.4, to show that F is a mapping it is sufficient to prove that $F(\bar{A}) \subset \overline{F(A)}$ for all $A \subset X^*$. Let $p \in \bar{A}$ and suppose $F(p) \notin \overline{F(A)}$. Since Y is regular, there are sets $B_1, B_2 \subset Y$ such that $\bar{B}_1 \cup \bar{B}_2 = Y$, $\overline{F(A)} \subset \bar{B}_2$, $F(p) \notin \bar{B}_2$ and $\bar{B}_1 \cap \overline{F(A)} = \emptyset$. The first condition implies that for each $q \in A$, either $f^{-1}(\bar{B}_1)$ or $f^{-1}(\bar{B}_2)$ is in $\mathcal{U}_q$. But if $f^{-1}(\bar{B}_1) \in \mathcal{U}_q$, then $F(q) \in \bar{B}_1$, which contradicts $\bar{B}_1 \cap \overline{F(A)} = \emptyset$. Thus $f^{-1}(\bar{B}_2) \in \mathcal{U}_q$ for each $q \in A$, i.e. $q \in \overline{f^{-1}(\bar{B}_2)}$ for all $q \in A$. But $A \subset \overline{f^{-1}(\bar{B}_2)}$ implies that $p \in \bar{A} \subset \overline{f^{-1}(\bar{B}_2)}$, which contradicts $F(p) \notin \bar{B}_2$. Thus $F(\bar{A}) \subset \overline{F(A)}$.

Since Y is Hausdorff and $\bar{X} = X^*$, F is the unique extension of f.

In order to obtain a categorical result, we need to restrict our attention to those binding spaces whose completion are regular spaces.

A binding space will be called an *R-binding space* if it satisfies (R): if $\mathcal{C}$ is a cluster and $F = \bar{F} \notin \mathcal{C}$, then there are sets A_1 and A_2 in X such that

(i) $X = A_1 \cup A_2$,

(ii) $A_1 \notin \mathcal{C}$,

(iii) $\{F, A_2\} \notin \boldsymbol{B}$.

3.7. Lemma. *The topology of an R-binding space is regular. A complete regular space is an R-binding space.*

Proof. If F is a closed subset of X and $x \notin F$, then $F \notin \mathcal{C}_x = \{E \subset X: x \in \bar{E}\}$. If A_1 and A_2 satisfy (i)–(iii), then $X - \bar{A}_1$ and $X - \bar{A}_2$ are disjoint open sets containing x and F, respectively.

If X is complete, each cluster contains a singleton; so if $F = \bar{F} \notin \mathcal{C}$, then there is an $\{x\} \in \mathcal{C}$ such that $x \notin F$. If U and V are disjoint open sets containing x and F, respectively, then $X - U$ and $X - V$ satisfy (i)–(iii).

3.8. Theorem. *A space is an R-binding space if and only if its completion is regular.*

Proof. The elements of X^* are just the cluster of X and the closures in X^* of sets in X form a base for the closed sets of X^*. From this, the proof follows as in 3.7.

The family $\mathfrak{R}$ of R-binding spaces and maps constitutes a category, and if $\mathcal{K}$ is the subfamily of $\mathfrak{R}$ consisting of complete spaces, then $\mathcal{K}$ is a subcategory of $\mathfrak{R}$.

Theorem 3.6 shows that the embedding $e: X \to X^*$ is a $\mathcal{K}$-reflection of X (whenever $X^* \in \mathcal{K}$); and theorem 3.3 and lemma 3.8 guarantee the existence of $\mathcal{K}$-reflections for each $X \in \mathfrak{R}$.

3.9 Theorem. *$\mathcal{K}$ is a reflective subcategory of $\mathfrak{R}$.*

4. Applications. For any set X there is a finest and a coarsest binding structure on X, both of which are complete, 1.11. If (X, τ) is a topological space, a binding structure $(\boldsymbol{B}, \boldsymbol{C})$ on X will be said to be *compactible with* τ if the topology of $(\boldsymbol{B}, \boldsymbol{C})$ is τ.

4.1. Lemma. *For any T_1 topological space (X, τ) there is a finest binding structure on X which is compatible with τ.*

Proof. Let $\mathcal{F}$ be the family of all closed sets in (X, τ) and $\boldsymbol{C}_{\mathcal{F}}$ be the family of all closed coverings of X. If $F_1, F_2, \ldots, F_n \in \mathcal{F}$, then let $\{F_i\} \in \boldsymbol{B}_{\mathcal{F}}$ if and only if $\bigcap F_i \neq \emptyset$. It is easily seen that $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ is a closed base for a binding structure $(\boldsymbol{B}, \boldsymbol{C})$ on X and that the topology of $(\boldsymbol{B}, \boldsymbol{C})$ is just τ (1.7). If $(\boldsymbol{B}^1, \boldsymbol{C}^1)$ is any other binding structure on X compatible with τ, then P1 and P4 imply that $\boldsymbol{B} \subseteq \boldsymbol{B}^1$ and P6 implies that $\boldsymbol{C}^1 \subseteq \boldsymbol{C}$ Thus the identity function of $(\boldsymbol{B}, \boldsymbol{C})$ onto $(\boldsymbol{B}^1, \boldsymbol{C}^1)$ is a mapping.

4.2. For a given T_1-topological space (X, τ) there are several binding structures on X which are compatible with τ. Some of the more common ones are given below.

(a) Let $\mathcal{F}$ be a family of closed subsets of X which separates points and closed sets, i.e. if H is closed and $x \notin H$, then there are sets $F_1, F_2 \in \mathcal{F}$ such that $x \in F_1$, $H \subset F_2$ and $F_1 \cap F_2 = \emptyset$. If $F_1, \ldots, F_n \in \mathcal{F}$, let $\{F_i\} \in \boldsymbol{B}_{\mathcal{F}}$ if and only if $\bigcap F_i \neq \emptyset$. Let $\boldsymbol{C}_{\mathcal{F}}$ consist of all finite coverings of X from $\mathcal{F}$.

The cluster bases in $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ are subfamilies of $\mathcal{F}$, maximal with respect to having the finite intersection property. The completion X^* of the binding structure generated by $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ has a compact topology. From 2.3 and 2.17, it follows that as a topological space the completion is a compactification of (X, τ), [6]. If $\mathcal{F}$ is the family of all closed sets of (X, τ), then as a topological space, X^* is the Wallman compactification of (X, τ).

(b) If $\mathcal{F}$ and $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ are as in (a), but now (X, τ) is a Tychonoff space and $\mathcal{F}$ is also a normal family, i.e. disjoint sets in $\mathcal{F}$ can be separated by disjoint complements of sets in $\mathcal{F}$, then the topology of the completion is also Hausdorff and the completion is, topologically, usually referred to as a Wallman-type compactification, [2], [6].

(c) If (X, τ) is a Tychonoff space, $\mathcal{F}$ is the family of all zero-sets in X, finite subcollections of $\mathcal{F}$ are in $\boldsymbol{B}_{\mathcal{F}}$ if and only if they have non-empty intersection, and $\boldsymbol{C}_{\mathcal{F}}$ is all finite coverings of X from $\mathcal{F}$, then the cluster bases in $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ are just Z-ultrafilters (ultrafilters of zero-sets, see [3], chapter 6). From the definition of the topology on the completion, it is, topologically, the Stone–Čech compactification βX of (X, τ).

The Z-ultrafilters are all fixed if and only if τ is compact, so a Tychonoff space is compact if and only if it is complete with respect to the binding structure just defined. This structure will be called the *β-binding* on X compatible with τ. It is easy to see that the β-binding on a Tychonoff space is an R-binding.

4.3. LEMMA. *A continuous function from a Tychonoff space (X, τ) into a compact Hausdorff space (Y, σ) is a mapping from X into Y when X and Y have the β-binding compatible with τ and σ, respectively.*

Proof. The inverse, under a continuous function, of a zero-set is a zero-set, and since any cover of Y can be refined by a finite cover of zero-sets, its inverse under f is a cover of X.

If $A_1, \ldots, A_n \subset X$ and $\{f(A_i)\}$ is not bound in Y, then since Y is complete, $\bigcap \overline{f(A_i)} = \emptyset$. There are zero-sets $Z_1, \ldots, Z_n$ in (Y, σ) such that $\overline{f(A_i)} \subset Z_i$ and $\bigcap Z_i = \emptyset$. Thus $A_i \subset f^{-1}(Z_i)$ and $\bigcap f^{-1}(Z_i) = \emptyset$. Since $\{f^{-1}(Z_i)\}$ is not bound in X, neither is $\{A_i\}$. f must then take bound collections of X into bound collections of Y and hence f is a mapping.

Lemma 4.3, together with theorem 3.6, prove that every continuous function from a Tychonoff space (X, τ) into a compact Hausdorff space can be extended (to a mapping, and hence continuously, 1.10) to βX.

(d) By modifying example (c) to let $\boldsymbol{C}_{\mathcal{F}}$ consist of all countable coverings of zero-sets, the cluster bases become Z-ultrafilters with the countable intersection property, and the completion of the structure generated by $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ is, topologically, the Hewitt realcompactification, νX of (X, τ). The binding structure on a Tychonoff space (X, τ) just defined, will be called the *ν-binding on X compatible with τ.*

A Tychonoff space (Y, σ) is realcompact if and only if Y is complete with respect to the ν-binding compatible with σ. Exactly as in 4.3, a continuous function from a Tychonoff space (X, τ) into a realcompact space (Y, σ) is a mapping from X into Y when X and Y have the ν-binding compatible with τ and σ, respectively. Thus, again from 3.6 and the fact that the ν-binding is an R-binding, any continuous function from a Tychonoff space (X, τ) into a realcompact space can be continuously extended to νX.

4.4. It might be worthwile to mention that if (Y, σ) is any Hausdorff compactification of a topological space (X, τ), then there is a binding structure on X, compatible with τ, whose completion, topologically, is (Y, σ). It is not too difficult to check that the closed base $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ generates such a structure, where $\mathcal{F}$ is all closed sets of (X, τ), $\boldsymbol{C}_{\mathcal{F}}$ is all finite coverings of X by members of $\mathcal{F}$, and if $F_1, \ldots, F_n \in \mathcal{F}$, then $\{F_i\} \in \boldsymbol{B}_{\mathcal{F}}$ if and only if $\bigcap \mathrm{cl}_{(Y,\sigma)} F_i \neq \emptyset$.

4.5. Now suppose (X, μ) is a Hausdorff uniform space. Let $\mathcal{F}$ be the family of all closed subsets of X with respect to the uniform topology and let $\boldsymbol{C}_{\mathcal{F}}$ be all closed uniform coverings in μ. If $F_1, \ldots, F_n \in \mathcal{F}$, let $\{F_i\} \in \boldsymbol{B}_{\mathcal{F}}$ if and only if for each cover $C \in \boldsymbol{C}_{\mathcal{F}}$ there is an $F \in C$ such that $F \cap F_i \neq \emptyset$, $1 \leqslant i \leqslant n$.

It is easy to see that $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ satisfy B1 and B2. Each uniform covering can be refined by a uniform covering of closed sets. Thus if $\{G, F_i\}$ and $\{H, F_i\}$ are not in $\boldsymbol{B}_{\mathcal{F}}$, there are covers $C_1, C_2 \in \boldsymbol{C}_{\mathcal{F}}$ such that no set in C_1 intersects G and each F_i, and no set in C_2 intersects H and each F_i. There is a $C_3 \in \boldsymbol{C}_{\mathcal{F}}$ which refines both C_1 and C_2 and no set in C_3 can intersect $G \cup H$ and each F_i. Thus $\{G \cup H, F_i\} \notin \boldsymbol{B}_{\mathcal{F}}$ and B3 follows. For B4, if $x \notin F \in \mathcal{F}$ then there is a $C \in \boldsymbol{C}_{\mathcal{F}}$ such that $\mathrm{st}(x, C) \cap F = \emptyset$. Thus $\{x, F\} \notin \boldsymbol{B}_{\mathcal{F}}$.

The binding structure on X generated by $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ will be called the *μ-binding*. This binding structure is compatible with the topology of μ and the covers are precisely the uniform coverings.

4.6. THEOREM. *A uniform space is complete if and only if the μ-binding is a complete binding structure on X.*

Proof. Each Cauchy ultrafilter in (X, μ) can be extended to a cluster in the μ-binding, so completeness of the binding structure implies completeness of the uniform structure.

To show that converse, suppose (X, μ) is complete and let $\mathcal{C}$ be a cluster in the μ-binding. Define a family $\mathcal{G} = \{A \in \mathcal{C}\colon \mathrm{st}(F, C) \subset A$ for some $C \in \boldsymbol{C}$, and $F \in \mathcal{F} \cap \mathcal{C}\}$. Since each cover in $\boldsymbol{C}$ is star-refined by another cover in $\boldsymbol{C}$, $\mathcal{G}$ contains a member of each uniform covering in $\mu = \boldsymbol{C}$. For $A_1, A_2, \ldots, A_n \in \mathcal{G}$, there are covers $C_1, C_2, \ldots, C_n$ and sets $F_1, F_2, \ldots, F_n$ in $\mathcal{F} \cap \mathcal{C}$ such that $\mathrm{st}(F_i, C_i) \subset A_i$. There is a closed cover $C \in \boldsymbol{C}$ which refines each C_i. Since $\{F_i\}$ is bound, there is a set $H \in C$ such that $H \cap F_i \neq \emptyset$ for each i. Thus $H \subset \bigcap \mathrm{st}(F_i, C) \subset \bigcap \mathrm{st}(F_i, C_i) \subset \bigcap A_i$, and $\mathcal{G}$ has the finite intersection property. Since (X, μ) is complete, $\bigcap \overline{\mathcal{G}} \neq \emptyset$.

If $x \in \bigcap \overline{\mathcal{G}}$, we will show $x \in \bigcap \overline{\mathcal{C}}$. Suppose on the contrary that $x \notin \bigcap \overline{\mathcal{C}}$. Then there is a set $A \in \mathcal{C}$ such that $x \notin \overline{A}$. There is a cover $C \in \boldsymbol{C}$ such that $\mathrm{st}(x, C) \cap \overline{A} = \emptyset$. Let C^1 be a closed cover, which star-refines C and let $G \in \mathcal{G} \cap C^1$. Since $\mathrm{st}(G, C^1) \subset \mathrm{st}(x, C)$, there can be no set in C^1 which intersects both G and $\overline{A}$. Thus $\{G, \overline{A}\}$ are not bound which contradicts $G, \overline{A} \in \mathcal{C}$. This completes the proof that the μ-binding on X is complete.

In order to show that the completion X^* of the μ-binding on X and the uniform completion of the uniform space (X, μ) coincide, we will need the following two lemmas.

4.7. LEMMA. *The μ-binding is an R-binding.*

Proof. Let $\mathcal{C}$ be a cluster and H a closed set of X not in $\mathcal{C}$. There are then closed sets $F_1, F_2, \ldots, F_n \in \mathcal{C}$ such that $\{H, F_1, \ldots, F_n\} \notin \boldsymbol{B}_{\mathcal{F}}$. From the definition of $\boldsymbol{B}_{\mathcal{F}}$ (4.5), there is a cover $C_0 \in \boldsymbol{C}_{\mathcal{F}}$ such that no set in C_0 intersects H and each F_i. Let $C_1, C_2, C_3 \in \boldsymbol{C}$ be such that C_i star-refines C_{i-1} for $i = 1, 2, 3$. Let $F \in \mathcal{C} \cap C_3$. There are sets $K_i \in C_i$, $i = 0, 1, 2$ such that $\operatorname{st}(F, C_3) \subset K_2$, $\operatorname{st}(K_2, C_2) \subset K_1$, and $\operatorname{st}(K_1, C_1) \subset K_0$. Since $\{F, F_1, \ldots, F_n\} \in \boldsymbol{B}_{\mathcal{F}}$, $\operatorname{st}(F, C_3) \cap F_i \neq \emptyset$ for each i and thus $K_0 \cap F_i \neq \emptyset$ for each i. From the choice of C_0, $K_0 \cap H = \emptyset$ and $\{K_1, H\} \notin \boldsymbol{B}_{\mathcal{F}}$. If $E = \operatorname{cl}_x(X - K_1)$, then $K_1 \cup E = X$. It remains to show $E \notin \mathcal{C}$. For $x \in K_2$, $\operatorname{st}(x, C_2) \subset \operatorname{st}(K_2, C_2) \subset K_1$ so $x \notin E$. Thus $\operatorname{st}(F, C_3) \cap E = \emptyset$ and $\{E, F\} \notin \boldsymbol{B}_{\mathcal{F}}$, and $E \notin \mathcal{C}$.

4.8. Lemma. *If X^* is the completion of the μ-binding on X, then $\boldsymbol{C}^*$ is a uniformity for X^*.*

Proof. It suffices to show that $\boldsymbol{C}_{\mathcal{F}^*}$ is a base for a uniformity since each cover in $\boldsymbol{C}^*$ is refined by one in $\boldsymbol{C}_{\mathcal{F}^*}$. If $C_3 = C_1 \wedge C_2$ for $C_1, C_2, C_3 \in \boldsymbol{C}_{\mathcal{F}}$, then $\overline{C}_3 = \overline{C}_1 \wedge \overline{C}_2$. If C_1 star-refines C_2 and C_2 star-refines C_3, then $\overline{C}_1$ star-refines $\overline{C}_3$. $\boldsymbol{C}_{\mathcal{F}}$ is a base for μ, so $\boldsymbol{C}_{\mathcal{F}^*}$ is a base for a uniformity, namely $\boldsymbol{C}^*$.

4.9. Theorem. *The completion of the μ-binding on X is, as a uniform space, the uniform completion of (X, μ).*

Proof. To show $(X^*, \boldsymbol{C}^*)$ is a completion of (X, μ), we must show that the topology of the binding structure $(\boldsymbol{B}^*, \boldsymbol{C}^*)$ agrees with the uniform topology induced by $\boldsymbol{C}^*$ and that $(X^*, \boldsymbol{C}^*)$ is complete.

If A is closed in $(\boldsymbol{B}^*, \boldsymbol{C}^*)$ and $x \notin A$, then there is an $F \in \mathcal{F}$ such that $A \subset F^*$, $x \notin F^*$. By 4.7, there are sets $F_1, F_2 \in \mathcal{F}$ such that $F_1 \cup F_2 = X$, $\{F, F_1\} \notin \boldsymbol{B}_{\mathcal{F}}$ and $x \notin F_2^*$. From the definition of $\boldsymbol{B}_{\mathcal{F}}$, there is a $C \in \boldsymbol{C}_{\mathcal{F}}$ such that for each $H \in C$, if $x \in H^*$ then $H^* \cap F^* = \emptyset$. By 4.8, there is a $C_1 \in \boldsymbol{C}_{\mathcal{F}}$ such that $\overline{C}_1$ star-refines $\overline{C}$. Thus if $x \in K^* \in \overline{C}_1$ then $\operatorname{st}(K^*, \overline{C}_1) \cap F^* = \emptyset$. Since $x \in \operatorname{st}(K^*, \overline{C}_1)$ and $\operatorname{st}(K^*, \overline{C}_1) \cap A = \emptyset$, A is closed in the uniform topology on $(X^*, \boldsymbol{C}^*)$.

Conversely, suppose A is closed in $(X^*, \boldsymbol{C}^*)$ and $x \notin A$. There is a $\overline{C} \in \boldsymbol{C}_{\mathcal{F}^*}$ such that $\operatorname{st}(x, \overline{C}) \cap A = \emptyset$. Let $\overline{C}_1 \in \boldsymbol{C}_{\mathcal{F}^*}$ star-refine $\overline{C}$. For each $a \in A$, $\operatorname{st}(a, \overline{C}_1) \cap \operatorname{st}(x, \overline{C}_1) = \emptyset$. Let $K = \bigcup \{\operatorname{st}(a, \overline{C}_1) \colon a \in A\}$ and $\operatorname{cl}K$ be the closure of K in $(X^*, \boldsymbol{C}^*)$. Since $\operatorname{st}(x, \overline{C}_1) \cap K = \emptyset$, $x \notin \operatorname{cl}K$.

If $E = \operatorname{cl}_{(X,\mu)} \bigcup \{F \in C_1 \colon A \cap F^* \neq \emptyset\}$, then $A \subset E^*$. It remains to show that $x \notin E^*$. First, $E \subset \operatorname{cl}K$ and $x \notin \operatorname{cl}K$. Thus there are covers $\overline{b}_1, \overline{b}_2 \in \boldsymbol{C}_{\mathcal{F}^*}$ such that $\overline{b}_2$ star-refines $\overline{b}_1$ and $\operatorname{st}(x, \overline{b}_1) \cap \operatorname{cl}K = \emptyset$. Let $x \in F^* \in \overline{b}_2$. If $G \in b_2$ and $G \cap F \neq \emptyset$, then $G^* \subset \operatorname{st}(F^*, \overline{b}_2) \subset H^* \in \overline{b}_1$. This implies that $H^* \subset \operatorname{st}(x, \overline{b}_1)$, and thus that $H^* \cap \operatorname{cl}K = \emptyset$. It follows that $G \cap E = \emptyset$ and $\{E, F\} \notin \boldsymbol{B}_{\mathcal{F}}$. But $x \in F^*$ and $F^* \cap E^* = \emptyset$ imply that $x \notin E^*$. This completes the proof that the two topologies are identical.

If $\mathcal{E}$ is a Cauchy ultrafilter on $(X^*, \boldsymbol{C}^*)$ composed of closed sets, then $\mathcal{E}$ is contained in a cluster $\mathcal{C}$ in $(\boldsymbol{B}^*, \boldsymbol{C}^*)$. Since $\bigcap \overline{\mathcal{C}} \neq \emptyset$, $\mathcal{E}$ converges.

4.10. If f is a uniformly continuous function from a uniform space (X, μ) into a uniform space (Y, μ^1), then f is a mapping from X into Y where X and Y have the μ-binding and μ^1-binding, respectively. Clearly the inverses of covers are covers and if $\{f(U_i)\}$ is not bound, then there is a closed cover C of Y, no member of which intersects each $f(\overline{U}_i)$. Clearly, no member of $f^{-1}(C)$ intersects each $\overline{U}_i$ so $\{U_i\}$ is not bound in X.

Since (Y, μ^1) is complete if and only if the μ^1-binding is complete (4.6), and the μ^1-binding is R-binding (4.7), theorems 3.6 and 4.9 imply that every uniformly continuous function on (X, μ) into a complete uniform space can be extended uniformly to X^*, considered as a uniform completion of (X, μ).

4.11. Let (X, δ) be a proximity space and μ the uniformity on X generated by δ [4]. The clusters in the μ-binding are precisely the δ-clusters [5] and the completion of X with the μ-binding is the Smirnov compactification of (X, δ). It follows that any δ-continuous function from one proximity space into another, may be extended to a δ-continuous function from the Smirnov compactification of one to the Smirnov compactification of the other.

4.12. Each paracompactification of a topological space may be obtained as the completion of a binding structure in a natural way. Suppose $(X, \tau/X)$ is a topologically dense subspace of (Y, τ), where (Y, τ) is paracompact and T_3. If $\mathcal{F}$ is the family of all closed subsets of Y, let $\boldsymbol{C}_{\mathcal{F}}$ be all locally-finite closed coverings of Y, and let $\boldsymbol{B}_{\mathcal{F}}$ consist of all finite subcollections of $\mathcal{F}$ whose members have a point in common. Then $(\boldsymbol{B}_{\mathcal{F}}, \boldsymbol{C}_{\mathcal{F}})$ is a closed base for a complete binding structure on Y which is compatible with τ. X, with the subspace binding, is dense in Y and thus the completion of X is isomorphic to Y.

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