On semi-closed sets and semi-open sets and their applications (*)

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1. Introduction. The notion of a semi-closed set was originally introduced in 1936 by G. T. Whyburn in [19]. The results in § 2 of this paper were reproduced in [20]. However, the notion lay dormant for twenty years until in 1956 J. Nash defined the concept of a connectivity function in [15]. Nash asked whether Brouwer's fixed point theorem held for connectivity functions. In 1957 O. H. Hamilton answered this question affirmatively in [6]. J. Stallings noticed, however, that Hamilton's proof contained a gap, and in [17] he showed that this gap could be filled by observing that the inverse of a closed set under a connectivity function was a semi-closed set. Since the appearance of Stalling's paper in 1959, several papers have appeared developing the properties of connectivity functions, notable among them being [5], [1], [21], [22], [2] and [3], and in each of them the notion of a semi-closed set has been used prominently, even when it has not been named as such, as in [1] and [2].

We shall first briefly indicate the extent to which semi-closed sets play a part in the above-mentioned papers. In [1], [22] and [3] only simple results involving semi-closed sets are established, and these are used to prove theorems about certain non-continuous functions (mainly connectivity functions). In the proof of theorem 4 of [17], lemma (2.1) of the present paper is proved. This is a fundamental result on semi-closed sets, as will be apparent from the sequel, though it is not isolated as a result on semi-closed sets in [17]. Lemma 1 of [2] is a significant result on totally disconnected sets (i.e., on a subclass of semi-closed sets), and it is the key to proving the theorem of [2]. In [5] the monotone-light factorization for certain non-continuous functions is established. In the course of doing this several theorems are proved concerning the usc decompositions which certain semi-closed collections of sets induce on

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a space. Several topics are covered in [21], new among them being the results concerning the quasi-components of loosely closed and quasi-closed sets (both of which are special cases of semi-closed sets). In [19] various theorems are proved concerning the connectedness and local connectedness of certain semi-open sets. These are similar to the corollaries of lemma (2.1) given below.

In this paper we make a study of semi-closed sets and semi-open sets *per se*. This is done in § 2. While a number of properties of these sets are given—most of them similar to those for closed sets and open sets—the objective of the paper is to prove theorems (2.1) and (2.2), the first of which concerns semi-closed sets and the second of which concerns semi-open sets.

The formulation of these two theorems arises from drawing an analogy between semi-closed sets and semi-open sets on the one hand, and quasi-closed sets and quasi-open sets on the other. This analogy we shall now explain. In [6] Hamilton introduced the notion of a peripherally continuous function, and the study of these functions has been closely related to that of connectivity functions. However, unlike the connectivity function, the peripherally continuous function has a particularly useful characterization: a function is peripherally continuous if and only if the inverse of every closed set under the function is quasi-closed (see [22]). In [22] Whyburn proved two particularly useful theorems about quasi-closed sets and quasi-open sets, namely theorems (2.1) and (2.2) of that paper. Theorem (2.1) of [22] was given its most satisfactory form in [4], where it reads as follows.

THEOREM I. *Let A and B be closed sets in a locally cohesive regular T_1-space X. Any quasi-closed set L which weakly separates A and B in X contains a closed set K which separates $A-K$ and $B-K$ in X.*

Theorem (2.2) of [22] is the following.

THEOREM II. *If X is locally cohesive (and regular and T_1), any connected set lying in the union of two disjoint quasi-open sets lies entirely in one of them.*

Quasi-closed sets and quasi-open sets are special cases of semi-closed sets and semi-open sets, as we have remarked [1], and theorems (2.1) and (2.2) of the present paper are the analogues of theorems I and II for semi-closed sets and semi-open sets, respectively.

Whyburn used theorems (2.1) and (2.2) of [22] to simplify the proofs of existing theorems on peripherally continuous functions. Similarly,

[1] Every totally disconnected set is semi-closed. Thus, for example, the totally disconnected set which becomes connected upon the adjunction of a single point, described in [13], is semi-closed but not quasi-closed.

using theorems (2.1) and (2.2) of this paper, we are able to simplify the proofs of existing theorems on connectivity functions and pseudo-continuous functions. Using these two theorems, we are also able to prove new results for semi-closed sets and semi-open sets in particular spaces (viz., cubes and spheres) which are more general than the corresponding results for quasi-closed sets and quasi-open sets. These applications are dealt with in § 3.

However, in spite of the analogy between the statements and applications of theorems (2.1) and (2.2) of this paper and theorems I and II, the proofs of the former theorems are quite different from those of the latter. This is because each point of a quasi-open set has a base of neighbourhoods whose frontiers lie in the quasi-open set, and this property is used continually in the proofs of theorems I and II (and in the proofs of other theorems on quasi-closed sets and quasi-open sets—see [21], [22]). For semi-open sets there is no such simple and useful property. Thus in the proofs of both theorems (2.1) and (2.2) we have relied on the more complicated result of lemma (2.3).

Finally, the results in § 2 require that the spaces be at least Peano spaces. We shall not, however, use locally cohesive Peano spaces (cf., theorems I and II). We shall simply use cyclic, unicoherent Peano spaces. The reason for this is twofold. On the one hand, it will save us from complications which clutter the proofs. On the other hand, locally cohesive Peano spaces are not the most general spaces in which the results can be proved (see [9]). In fact, it is probable that all the results hold in certain infinitely multicoherent spaces (e.g., a plane from which the interiors of a null sequence of disjoint disks have been removed).

2. The main theorems on semi-closed sets and semi-open sets.

Throughout this section X will denote a *cyclic, unicoherent Peano space* unless otherwise stated.

We shall use several characterizations of unicoherence in a Peano space, all of which can be found in or are immediately derivable from the results on p. 429 of [18] or pp. 47–51 of [23].

We shall denote the frontier of an open set G by $\operatorname{Fr} G$; i.e., $\operatorname{Fr} G = \bar{G}-G$. We shall use the terms "cyclic", "non-degenerate", "convergence" for a sequence of sets, "separated sets" and "to separate", and the definitions of these terms can be found in [20] or [23]. For convenience we restate the following two definitions. Let L be a subset of X and C, D two non-empty subsets (or one or both of them may be points) of $X-L$. We say that *L separates C in X* if $X-L$ is the union of two non-empty separated sets both of which meet C. We say that *L separates C and D in X* if $X-L$ is the union of two non-empty separated sets one of which contains C and the other of which contains D.

A subset L of X is said to be *semi-closed in* X if for each convergent sequence $F_1, F_2, \ldots$ of components of L, $\lim F_n$ is a single point or a subset of L. This definition is given in [19] and [20]. We shall say that a subset G of X is *semi-open in* X if its complement is semi-closed in X.

Notice that the components of a semi-closed subset of X are closed in X.

Let L be a semi-closed subset of X, and C a non-empty subset of $X-L$ which L does not separate in X. We shall denote by $[L]$ the union of L and the set of all points in $X-L$ that L does not separate from C in X. We shall call $[L]$ the *enclosure of* L *with respect to* C.

Using the notation of the previous paragraph, let Q be the quasi-component of $X-L$ in which C lies. Notice, then, that $Q = X-[L]$ and that $[L]$ is also the enclosure of L with respect to $Q = X-[L]$.

It will often be convenient to refer to an enclosure of a semi-closed set L without indicating the set with respect to which the enclosure is taken. To facilitate this we shall say that an arbitrary set $[L]$ is an *enclosure of* L if it is the complement of some (unspecified) quasi-component of $X-L$.

Lemma (2.1). *Let L be a semi-closed set in X and $[L]$ an enclosure of L such that $X-[L]$ is non-degenerate. Then each component of $[L]$ is the enclosure of a component of L with respect to $X-[L]$.*

Although this lemma is not isolated as such in [17], its proof is given in the course of—and occupies the major part of—the proof of theorem 4 of [17]. The space X in this theorem is an lpc polyhedron, the operative properties of which are those of being a cyclic Peano space with a base of unicoherent regions. Under the hypotheses of lemma (2.1), it follows from lemma (2.1) that *$[L]$ is a semi-closed set and $X-[L]$ is connected.* From this we obtain the following simple corollaries.

Corollary 1. *Let H be a non-degenerate component of a semi-open subset G of X. Then H is itself a semi-open subset of X.*

Corollary 2. *Let G be a semi-open subset of X. Then the quasi-components of G are connected.*

Corollaries 1 and 2 have been partially proved, or proved for different classes of spaces, in several places. Corollary 2 can easily be deduced from lemma 1 of [2] for the case in which the space is a finitely multicoherent Peano continuum. Theorems (3.2), (4.1) and (4.2) of [19] have much in common with corollaries 1 and 2, and also suggest that the set H of corollary 1 is locally connected. Although this is so, we shall not digress to prove it.

In order to prove lemma (2.3) we need the following result, the proof of which is straightforward and is omitted. (The proof is in any case given in [11].)

Lemma (2.2). *Let L be a semi-closed set in X, and V an open set in X with a compact closure. Denote by M the union of $X-V$ and all the components of L that meet $X-V$. Then M is a closed set.*

In order to prove lemma (2.3) we also need to know the easily proved fact that *a cyclic Peano space has a base of open sets whose complements are connected.*

Lemma (2.3). *Let L be a semi-closed subset of X and H a component of $X-L$. Then there is an arbitrarily small neighbourhood U of H such that* $\mathrm{Fr}\, U \subset L$.

In fact, if H is non-degenerate, then $\mathrm{Fr}\, U \subset \overline{H}-H$.

Proof. In the first case we suppose that H is a degenerate component of $X-L$, and put $H = \{p\}$. We let V be an arbitrary neighbourhood of p such that $\overline{V}$ is compact and $X-V$ is connected. If there is a component F of L lying in V and separating $p, X-V$ in X, then the component U of $X-F$ to which p belongs satisfies the requirements of the lemma. So we suppose that this does not occur, and we let M be the union of $X-V$ and all the components of L which meet $X-V$. Then, by lemma (2.2), M is a closed set. We let U be the component of $X-M$ to which p belongs.

To see that $\mathrm{Fr}\, U \subset L$, suppose on the contrary that there is a point $q \in (\mathrm{Fr}\, U)-L$. Then $q \in (X-V)-L$ and, by corollary 2 to lemma (2.1), L separates p, q in X. Thus in fact a component F of L separates p, q in X, because X is unicoherent. Now F does not lie in V, for otherwise F would separate p and the connected set $X-V$ in X, contrary to supposition. So $F \subset M$. Consequently F separates q and the connected set U in X; i.e., $q \notin \overline{U}$, which is a contradiction. This shows that $\mathrm{Fr}\, U \subset L$.

Secondly, we suppose that H is a non-degenerate component of $X-L$. Then, by corollary 1 to lemma (2.1), H is a semi-open set; i.e., $X-H$ is a semi-closed set. Let V be an arbitrary neighbourhood of H and, for each $p \in H$, let V_p be a neighbourhood of p such that $\overline{V}_p$ is compact and $V_p \subset V$. Let M_p be the union of $X-V_p$ and all the components of the semi-closed set $X-H$ that meet $X-V_p$. By lemma (2.2), M_p is a closed set. Let $U_p = X-M_p$ and $U = \bigcup\{U_p\colon p \in H\}$. Then $\mathrm{Fr}\, U \subset (\overline{H}-H) \cap L$.

In order to prove this, we first show that $\mathrm{Fr}\, U_p \subset \overline{H}$ for each $p \in H$. To see this, let $x \in \mathrm{Fr}\, U_p$ and let yz be an arc in an arbitrarily small neighbourhood of x such that $yz-\{z\} \subset U_p$ and $z \in \mathrm{Fr}\, U_p$. Since the components of $X-H$ which meet U_p are closed sets lying in U_p, it follows that $yz-\{z\} \not\subset X-H$; i.e., $yz-\{z\}$ meets H. Thus $x \in \overline{H}$, and this proves that $\mathrm{Fr}\, U_p \subset \overline{H}$. It follows from the local connectedness of X that $\mathrm{Fr}\, U$ is contained in the closure of $\bigcup\{\mathrm{Fr}\, U_p\colon p \in H\}$. This now gives us $\mathrm{Fr}\, U \subset \overline{H}$. But $U \supset H$, and so $\mathrm{Fr}\, U \subset \overline{H}-H$. Since $\overline{H}-H \subset L$, it follows that $\mathrm{Fr}\, U \subset (\overline{H}-H) \cap L$. This completes the proof.

Let A, B be two subsets or two points of X. We shall say that a subset L of X *disconnects* A, B *in* X if no component of $X-L$ meets both A and B. This definition is given on p. 439 of [18]. In [22], [4] and [9] the same definition is given, but in these papers the phrase "weakly separates" is used instead of "disconnects". Notice that the definition permits $A \cap B \cap L \neq \emptyset$.

THEOREM (2.1). *Let L be a semi-closed subset of X and A, B two closed subsets of X which are disconnected in X by L. Then L contains a closed subset K of X which separates $A-K$ and $B-K$ in X.*

Proof. In order to prove this theorem we introduce the following notation. Let $\{H_\alpha\}_\alpha$ be the collection of all components of $X-L$ which meet A, and let $M = \bigcup\limits_\alpha H_\alpha$. For each α, let U_α be a neighbourhood of H_α such that $U_\alpha \cap B \neq \emptyset$ and $\mathrm{Fr}\, U_\alpha \subset L$ (that there is such a neighbourhood follows from lemma (2.3)). Let $\{H_\beta\}_\beta$ be the collection of all components of $X-L$ which do not meet A, and let $N = \bigcup\limits_\alpha H_\beta$. For each β, let U_β be a neighbourhood of H_β such that $U_\beta \cap A = \emptyset$ and $\mathrm{Fr}\, U_\beta \subset L$.

We first show that $M \cap \overline{U}_\beta = \emptyset$ for each β. For suppose on the contrary that $M \cap \overline{U}_\beta \neq \emptyset$ for some β. Then we have $M \cap U_\beta \neq \emptyset$, because $\mathrm{Fr}\, U_\beta \subset L$. This implies that $H_\alpha \cap U_\beta \neq \emptyset$ for some α. However, $H_\alpha \cap (X-U_\beta) \neq \emptyset$, because H_α meets A and U_β does not. But this is a contradiction, because $H_\alpha \cap \mathrm{Fr}\, U_\beta = \emptyset$. This establishes that $M \cap \overline{U}_\beta = \emptyset$ for each β.

Now let $U_{\alpha_1}, U_{\alpha_2}, \ldots$ be a countable subcollection of $\{U_\alpha\}_\alpha$ which covers M, and let $U_{\beta_1}, U_{\beta_2}, \ldots$ be a countable subcollection of $\{U_\beta\}_\beta$ which covers N. Such countable subcoverings can be chosen because a Peano space has a countable base (see p. 75 of [16]). We define

$$V_{\alpha_1} = U_{\alpha_1},$$
$$V_{\alpha_n} = U_{\alpha_n} - \overline{U}_{\beta_1} \cup \overline{U}_{\beta_2} \cup \ldots \cup \overline{U}_{\beta_{n-1}},$$

for each $n > 1$. Then $M \cap U_{\alpha_n} \subset V_{\alpha_n}$ and $\mathrm{Fr}\, V_{\alpha_n} \subset L$ for each n.

Now let $G = \bigcup\limits_{n=1}^{\infty} V_{\alpha_n}$. Then $M \subset G$ and $G \cap B = \emptyset$. We assert that $\mathrm{Fr}\, G \subset L$.

To prove this, suppose that there is a point $x \in (\mathrm{Fr}\, G) - L$. Then $x \in N$, and consequently $x \in U_{\beta_n}$, for some β_n. Now it follows from the definition that $U_{\beta_n} \cap (V_{\alpha_{n+1}} \cup V_{\alpha_{n+2}} \cup \ldots) = \emptyset$. Further, $x \notin G$, and so $x \notin V_{\alpha_1} \cup V_{\alpha_2} \cup \ldots \cup V_{\alpha_n}$. But also $x \notin \mathrm{Fr}\, V_{\alpha_1} \cup \mathrm{Fr}\, V_{\alpha_2} \cup \ldots \cup \mathrm{Fr}\, V_{\alpha_n}$, for this latter set is contained in L. Thus $x \notin \overline{V}_{\alpha_1} \cup \overline{V}_{\alpha_2} \cup \ldots \cup \overline{V}_{\alpha_n}$. But now $U_{\beta_n} - \overline{V}_{\alpha_1} \cup \overline{V}_{\alpha_2} \cup \ldots \cup \overline{V}_{\alpha_n}$ is a neighbourhood of x which does not meet G at all; i.e., $x \notin \mathrm{Fr}\, G$. The contradiction shows that $\mathrm{Fr}\, G \subset L$.

Let $K = (A-G) \cup \mathrm{Fr}\, G$. Then K is a subset of L which is closed in X, and it separates $A-K$ and $B-K$ in X.

The result of theorem (2.1) was announced for a wider class of spaces in theorem (3.4) of [9].

In the proof of theorem (2.2) we shall make use of lemma (2.3) and each of the next two lemmas.

LEMMA (2.4). *Let G_1, G_2 be disjoint semi-open subsets of X. Then $L = X - G_1 \cup G_2$ is a semi-closed subset of X.*

Proof. Let $\{F_n\}_n$ be a convergent sequence of components of L such that $F = \lim F_n$ is non-degenerate. Each F_n is then contained in $X-G_1$ and so is contained in a component M_n of $X-G_1$. Let $\{M_{n_k}\}_k$ be a convergent subsequence of $\{M_n\}_n$ and let $M = \lim M_{n_k}$. Since $X-G_1$ is semi-closed and $M \supset F$, it follows that $M \subset X-G_1$. Consequently $F \cap G_1 = \emptyset$. Similarly $F \cap G_2 = \emptyset$. This proves that $F \subset L$ and so L is semi-closed.

LEMMA (2.5). *Let K, L be two semi-closed subsets of X such that $L \subset K$. Let $[L]$ be an enclosure of L such that $X-[L]$ is non-degenerate. Then $K \cup [L]$ is also a semi-closed subset of X.*

Proof. By lemma (2.1), each component of $[L]$ is the enclosure of a component of L with respect to $X-[L]$. Let $\{F_\beta\}_\beta$ be the subcollection of all components of L such that, for each β, the enclosure $[F_\beta]$ of F_β with respect to $X-[L]$ is a component of $[L]$. Let $\{E_\alpha\}_\alpha$ be the subcollection of all components of K such that, for each α, $E_\alpha \not\subset [F_\beta]-F_\beta$ for all β. For each α, let $\widetilde{E}_\alpha$ be the the union of E_α and all the sets $[F_\beta]$ for which $F_\beta \subset E_\alpha$. Then $K \cup [L] = \bigcup\limits_\alpha \widetilde{E}_\alpha$.

We claim that $\{\widetilde{E}_\alpha\}_\alpha$ is the collection of components of $K \cup [L]$. We first show that the sets in this collection are disjoint, by supposing on the contrary that $\widetilde{E}_{\alpha_1} \cap \widetilde{E}_{\alpha_2} \neq \emptyset$ for some pair $E_{\alpha_1} \neq E_{\alpha_2}$. In this case we may suppose that there is a set $F_{\beta_1} \subset E_{\alpha_1}$ such that $[F_{\beta_1}] \cap E_{\alpha_2} \neq \emptyset$. But this implies that $E_{\alpha_2} \subset [F_{\beta_1}] - F_{\beta_1}$, which contradicts the choice of E_{α_2} as a member of the collection $\{E_\alpha\}_\alpha$. Thus $\{\widetilde{E}_\alpha\}_\alpha$ is a collection of disjoint sets. This now implies that $\{\widetilde{E}_\alpha\}_\alpha$ is the collection of components of $K \cup [L]$. For if we take any one of its non-degenerate subcollections $\{\widetilde{E}_{\alpha_\gamma}\}_\gamma$ then there are two non-empty separated sets M, N whose union is $\bigcup\limits_\alpha E_{\alpha_\gamma}$. By setting M' equal to the union of M and all the $\widetilde{E}_{\alpha_\gamma}$ for which $E_{\alpha_\gamma} \subset M$, and N' equal to the union of N and all the $\widetilde{E}_{\alpha_\gamma}$ for which $E_{\alpha_\gamma} \subset N$, we obtain two non-empty disjoint sets M', N' which, by the local connectedness of X, are separated.

Since it follows immediately from the local connectedness of X that the sets $\widetilde{E}_\alpha$ are closed, we have only to consider a convergent sequence $\widetilde{E}_{\alpha_1}, \widetilde{E}_{\alpha_2}, \ldots$ of distinct components of $K \cup [L]$ in order to prove that $K \cup [L]$ is semi-closed. For such a sequence it follows from the local connectedness of X that $\liminf \widetilde{E}_{\alpha_n} \subset \liminf E_{\alpha_n}$. Thus $\lim \widetilde{E}_{\alpha_n} = \lim E_{\alpha_n}$, and from this it immediately follows that $\lim \widetilde{E}_{a_n}$ is a subset of $K \cup [L]$ in case it is non-degenerate. This proves that $K \cup [L]$ is semi-closed.

THEOREM (2.2). *Let G_1, G_2 be two disjoint semi-open subsets of X. Then there is no connected subset C of $G_1 \cup G_2$ which meets both G_1 and G_2.*

Proof. We suppose on the contrary that there is a connected subset C of $G_1 \cup G_2$ which meets both G_1 and G_2. We let $L = X - G_1 \cup G_2$. By lemma (2.4), L is a semi-closed set.

Firstly, let H be a non-degenerate component of G_1. Then H is a non-degenerate component of $X - L \cup G_2$, and $L \cup G_2$ is a semi-closed set. Thus by lemma (2.3) there is a neighbourhood U of H such that the non-empty set $C \cap G_2 \not\subset U$ and $\operatorname{Fr} U \subset \overline{H} - H$. But H is a connected subset of the semi-closed set $L \cup G_1$, and this set has closed components. Thus $\overline{H} \subset L \cup G_1$. Since $(\overline{H} - H) \cap G_1 = \emptyset$, it follows that $\overline{H} - H \subset L$. That is, $\operatorname{Fr} U \subset L$.

Now let $[L]$ be the enclosure of L with respect to C. It follows from the result of the previous paragraph that each non-degenerate component of G_1 is contained in $[L]$. Thus $T = G_1 - [L]$ is a totally disconnected set and, since it contains the non-empty set $C \cap G_1$, it is itself non-empty. Thus no component of T separates X, which is a cyclic space, and so it follows from the unicoherence of the space that $X - T$ is connected. Now let $K = L \cup G_2$. Then, by lemma (2.5), $K \cup [L]$ is a semi-closed set, because $[L]$ is an enclosure of L and $X - [L]$ is non-degenerate (because C, by supposition, is non-degenerate). But $K \cup [L] = X - T$ and so $K \cup [L]$, as a connected set, is closed. Since the complementary components of a proper closed subset of a Peano space are all non-degenerate, it now follows that T has non-degenerate components. This contradiction proves the theorem.

3. Applications. We shall now deal with several applications of theorems (2.1) and (2.2), these being similar to the corresponding applications of theorems I and II, as we mentioned in § 1.

The *graph function* $g_f\colon X \to X \times Y$ of a function $f\colon X \to Y$ is defined by $g_f(x) = (x, f(x))$. A function $f\colon X \to Y$ is called a *connectivity function* if its graph function $g_f\colon X \to X \times Y$ preserves connectedness.

It is easily shown that *if $f\colon X \to Y$ is a connectivity function, where X is a Peano space and Y is a regular space, then $f^{-1}(F)$ is semi-closed in X whenever F is closed in Y* (cf. theorem (3.1) of [5]). However, since this property does not characterize connectivity functions, the notion of a pseudo-continuous function was defined in [8], [9] and [11]. A function $f\colon X \to Y$ on a Peano space X is said to be *pseudo-continuous* if $f^{-1}(F)$ is semi-closed in X whenever F is closed in Y.

The following theorem is then a consequence of theorem (2.2).

THEOREM (3.1). *Let $f\colon X \to Y$ be a pseudo-continuous function, where X is a cyclic, unicoherent Peano space and Y is a completely normal space. Then f preserves connectedness.*

Proof. Suppose there is a connected set C in X such that $f(C)$ is not connected. Then there are two disjoint open sets U, V in Y such that $f(C) \subset U \cup V$ and $U \cap f(C) \neq \emptyset \neq V \cap f(C)$, because Y is completely normal. But now $f^{-1}(U), f^{-1}(V)$ are by definition two disjoint semi-open sets in X such that $C \subset f^{-1}(U) \cup f^{-1}(V)$ and $C \cap f^{-1}(U) \neq \emptyset \neq C \cap f^{-1}(V)$. This contradicts theorem (2.2).

In [11] the following proposition is proved: *if $f\colon X \to Y$ is a pseudo-continuous function on a Peano space X, then the graph function $g_f\colon X \to X \times Y$ of f is also pseudo-continuous.* This and theorem (3.1) immediately imply the following.

THEOREM (3.2). *Let $f\colon X \to Y$ be a pseudo-continuous function, where X is a cyclic, unicoherent Peano space and $X \times Y$ is a completely normal space. Then f is a connectivity function.*

A function $f\colon X \to Y$ is said to be *peripherally continuous* if for each point p in X and for each pair of neighbourhoods V and W of p and $f(p)$, respectively, there is a neighbourhood U of p such that $U \subset V$ and $f(\operatorname{Fr} U) \subset W$. A set E in a space X is called *quasi-closed* if each point in $X - E$ has a base of neighbourhoods whose frontiers lie in $X - E$. The complement of a quasi-closed set is called *quasi-open*.

The following characterization of peripherally continuous functions was mentioned in § 1: *a function $f\colon X \to Y$ is peripherally continuous if and only if $f^{-1}(F)$ is a quasi-closed set in X whenever F is a closed set in Y* (see [22]). Since quasi-closed sets are semi-closed, this means that a peripherally continuous function is automatically pseudo-continuous. The following theorem, the proof of which uses theorem (3.1), is a converse to this.

THEOREM (3.3). *Let $f\colon X \to Y$ be a pseudo-continuous function, where X is a cyclic, unicoherent Peano space and Y is a regular and completely normal space. Then f is peripherally continuous.*

Proof. Let V, W be arbitrary neighbourhoods of $p, f(p)$, respectively, p being any point in X. Let W_1 be a neighbourhood of $f(p)$ such that $\overline{W}_1 \subset W$ and let $L = f^{-1}(\operatorname{Fr} W_1)$, so that L is a semi-closed set. Let p belong to a component H of $X - L$.

If in the first case $H = \{p\}$, then there is by lemma (2.3) a neighbourhood U of p such that $U \subset V$ and $\operatorname{Fr} U \subset L$; i.e., $f(\operatorname{Fr} U) \subset \operatorname{Fr} V_1 \subset V$. This shows that f is peripherally continuous at p.

Suppose in the second case that $H \neq \{p\}$, so that H is non-degenerate. Then $f(H) \subset V_1$, because f preserves connectedness, by theorem (3.1). Thus $f(\overline{H}) \subset \overline{V}_1$, again because f preserves connectedness. We now use a technique introduced in the second half of the proof of lemma (2.3). We refer to the paragraph starting "Secondly, ..." in that proof and use practically the same notation as there. Let V_p be a neighbourhood of p such that $\overline{V}_p$ is compact and $V_p \subset V$. Let M_p be the union of $X-V_p$ and all the components of $X-H$ that meet $X-V_p$. Because $X-H$ is actually a semi-closed set, it follows, as in the proof of lemma (2.3), that M_p is closed. Let $U_p = X - M_p$. Then, as in the last paragraph of the proof of lemma (2.3), $\mathrm{Fr}\, U_p \subset \overline{H}$. Putting $U = U_p$, this means that $f(\mathrm{Fr}\, U) \subset \overline{V}_1 \subset V$; i.e., f is again peripherally continuous at p. This completes the proof.

Notice that theorem (3.3) is an improvement on theorem (2.4) of [9] in the case where the domain space is a cyclic, unicoherent Peano space, because the function f is not assumed to be connectedness-preserving in the hypotheses of theorem (3.3).

The above three theorems, which concern cyclic, unicoherent Peano spaces, are direct or indirect consequences of theorem (2.2). We shall now show how the theorems (2.1) and (2.2) can be used to prove new results for semi-closed sets and semi-open sets in cubes and spheres.

Let I^n denote the set of points $x = (x_1, x_2, \ldots, x_n)$ in Euclidean n-space defined by $-1 \leqslant x_i \leqslant 1$ for each i. Let A_i and B_i be the subsets of I^n defined by $x_i = -1$ and $x_i = 1$, respectively. Then we have the following theorem:

THEOREM (3.4). *Let $L_1, L_2, \ldots, L_n$ be semi-closed sets in I^n such that L_i disconnects A_i and B_i in I^n for each i. Then $\bigcap_{i=1}^{n} L_i \neq \emptyset$.*

Proof. By theorem (2.1), there is for each i a closed set K_i in L_i which separates $A_i - K_i$ and $B_i - K_i$ in I^n. By a simple modification of the proof of proposition D, p. 40 of [12] (the modification referred to is carried out in the proof of theorem (2.3) of [22]) we have $\bigcap_{i=1}^{n} K_i \neq \emptyset$, and this proves the theorem.

This result is a generalization of theorem (2.3) of [22], in which the same conclusion is deduced under the hypothesis that $L_1, L_2, \ldots, L_n$ are quasi-closed sets.

By virtue of theorem (3.3) and theorem 5 of [6], it is of course true that any pseudo-continuous function $f\colon I^n \to I^n$ has a fixed point, for $n \geqslant 2$. However, using theorem (3.4), this property can be proved directly without referring to peripherally continuous functions at all (c.f., the proof of the Fixed-Point Theorem of [22], which is deduced from theorem (2.3) of that paper).

From the fixed point theorem, it follows that there is no pseudo-

continuous retraction $r\colon I^n \to \partial I^n$ for $n \geqslant 2$, where $\partial I^n = \bigcup_{i=1}^{n} A_i \cup B_i$. This result can also be deduced directly from theorem (3.4) as follows. Let C_i be the set of points $x = (x_1, x_2, \ldots, x_n)$ in ∂I^n defined by $x_i = 0$. Then C_i separates A_i and B_i in ∂I^n. Thus by theorem (3.1), $r^{-1}(C_i)$ is a semi-closed set which disconnects A_i and B_i in I^n. Theorem (3.4) now tells us that $\bigcap_{i=1}^{n} r^{-1}(C_i) \neq \emptyset$, but this is impossible, because $\bigcap_{i=1}^{n} C_i = \emptyset$. The contradiction proves what was required.

Let S^n denote the set of points $x = (x_1, x_2, \ldots, x_{n+1})$ in Euclidean $(n+1)$-space such that $\sum_{i=1}^{n+1} x_i^2 = 1$. Let T denote the antipodal map on S^n defined by $T(x) = -x$. In [10] the following theorem was proved, and we shall use it to prove theorem (3.6).

THEOREM (3.5). *If $L_1, L_2, \ldots, L_n$ are self-antipodal semi-closed sets in S^n and each L_i disconnects $x, -x$ in S^n for all $x \notin L_i$, then $\bigcap_{i=1}^{n} L_i \neq \emptyset$.*

The Lusternik–Schnirelmann theorem is given in, among other places, theorem (21.2), p. 138 of [14]. It is stated there for a covering of S^n by $n+1$ closed sets. It holds, equivalently, for a covering of S^n by $n+1$ open sets. We now prove it for a covering of S^n by $n+1$ semi-open sets.

THEOREM (3.6). *If S^n is covered by $n+1$ semi-open sets, then at least one of these sets contains a pair of antipodal points, where $n \neq 1$.*

Proof. The theorem is true for $n = 0$, so suppose $n > 1$, in which case S^n is a cyclic, unicoherent Peano continuum. Let $G_1, G_2, \ldots, G_{n+1}$ be $n+1$ semi-open sets which cover S^n such that $G_i \cap TG_i = \emptyset$ for each i. Then $TG_{n+1} \subset \bigcup_{i=1}^{n} G_i$ and $G_{n+1} \subset \bigcup_{i=1}^{n} TG_i$. Thus $G_1, G_2, \ldots, G_n, TG_1, TG_2, \ldots$ $\ldots, TG_n$ is a covering of S^n. Now let $L_i = S^n - G_i \cup TG_i$ for each $i \leqslant n$, so that $\bigcap_{i=1}^{n} L_i = \emptyset$. By lemma (2.4), L_i is a semi-closed set. Further, by theorem (2.2), L_i disconnects $x, -x$ in S^n for all $x \notin L_i$. Thus, by theorem (3.5), $\bigcap_{i=1}^{n} L_i \neq \emptyset$. This contradiction proves the theorem.

Notice that this theorem does not hold for $n = 1$, because S^1 can be expressed as the union of two disjoint 0-dimensional sets neither of which contains a pair of antipodal points. However, S^1 is not unicoherent.

The following example shows that the Lusternik–Schnirelmann theorem does not hold for a covering of S^n by $n+1$ semi-closed sets, where $n > 0$.

EXAMPLE (3.1). *There is a covering of S^n by $n+1$ disjoint 0-dimensional sets no one of which contains a pair of antipodal points, for $n>0$.*

Let $E^n = \{(x_1, x_2, \ldots, x_{n+1}) \in S^n:\ x_{n+1} > 0\}$. We show the existence of such a covering by induction. We have already mentioned that S^1 can be covered by two disjoint 0-dimensional sets neither of which contains a pair of antipodal points; in fact, $E^1 \cup \{(1, 0)\}$ can be expressed as the union of two disjoint 0-dimensional sets A and B, and $A \cup TB$, $B \cup TA$ is the required covering of S^1. Suppose now that S^{n-1}, for $n>1$, has a covering by n disjoint 0-dimensional sets $A_1, A_2, \ldots, A_n$ no one of which contains a pair of antipodal points. By theorem III 3, p. 32 of [12], E^n has a covering by $n+1$ disjoint 0-dimensional sets $B_1, B_2, \ldots, B_{n+1}$. Let

$$\begin{aligned} L_1 &= A_1 \cup B_1 \cup TB_2 ,\\ L_2 &= A_2 \cup B_2 \cup TB_3 ,\\ &\ldots\ldots\ldots\ldots\ldots\ldots ,\\ L_n &= A_n \cup B_n \cup TB_{n+1} ,\\ L_{n+1} &= \emptyset \cup B_{n+1} \cup TB_1 . \end{aligned}$$

Then $L_1, L_2, \ldots, L_{n+1}$ are disjoint sets no one of which contains a pair of antipodal points, and L_{n+1} is evidently 0-dimensional. To see that L_r is 0-dimensional, for $r \leqslant n$, let $E^n = \bigcup_{i=1}^{\infty} K_i$, where each K_i is compact. Then

$$L_r = (A_r \cap S^{n-1}) \cup (B_r \cap K_1) \cup (B_r \cap K_2) \cup \ldots$$
$$\ldots \cup (TB_{r+1} \cap TK_1) \cup (TB_{r+1} \cap TK_2) \cup \ldots ,$$

and each set in parentheses is a relatively closed 0-dimensional subset of the subspace L_r. Thus by theorem II 2, p. 18 of [12], L_r is 0-dimensional.

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