Compact spaces homeomorphic to a ray of ordinals

by

John Warren Baker (Tallahassee, Flor.)

For an ordinal number ξ, we denote by $\Gamma(\xi)$ the set of ordinals not exceeding ξ provided with the interval topology. The second theorem characterizes the compact Hausdorff spaces which are homeomorphic to some $\Gamma(\xi)$ and generalizes the following well-known theorem due to Mazurkiewicz and Sierpiński (cf. [8], p. 21 or [7], p. 103): *If $X^{(a)}$ is the last nonempty derivative of a compact, countable space X and n is the number of elements in $X^{(a)}$, then X is homeomorphic to $\Gamma(\omega^{\alpha}\cdot n)$.* Theorem 4 gives a generalization of the Cantor-Bendixson Theorem restricted to compact sets.

A subset of a topological space X which is both closed and open is called *clopen*. If there is a neighborhood base for each point of X consisting of clopen sets, X is called *zero-dimensional*. If each component of X is a point, then X is *totally disconnected*. For compact Hausdorff spaces, these two concepts are equivalent [3]. If ξ is an ordinal number, $X^{(\xi)}$ denotes the *derivative* of order ξ of X (cf. [6], p. 261 or [12], p. 64). If $X^{(\lambda)}$ is finite and contains exactly n points, the pair (λ, n) is called the *characteristic* of X. The set of ordinals less than ξ with the interval topology is denoted $\Gamma_0(\xi)$; therefore, $\Gamma(\xi) = \Gamma_0(\xi+1)$. The symbol ω represents the first infinite denumerable ordinal and the symbol Ω denotes the first uncountable ordinal. By map, we mean a continuous function. *All spaces are assumed to be Hausdorff.* The predecessor of a nonlimit ordinal α is denoted by $\alpha-1$. All other notation and terminology is standard as found, for example, in Kelley's *General Topology*.

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In [10], Professor Z. Semadeni asked if a sufficient condition for a compact, dispersed space X to be homeomorphic to $\Gamma(\alpha)$ for an ordinal α is that the following condition be satisfied: for each $x \in X$ there exists a (possibly transfinite) decreasing sequence $\{U_\xi\}_{\xi<\gamma}$ of neighborhoods of x

such that $\bigcap_{\xi<\gamma} U_\xi = \{x\}$. The first example shows that this condition is not sufficient. However, if $\{U_\xi\}_{\xi<\gamma}$ is assumed to be a neighborhood base for x, Theorem 1 states there is a map of X onto an ordinal space $\Gamma(\omega^\lambda\cdot n)$ with the same characteristic system. Theorem 2 establishes a necessary and sufficient condition for this map to be a homeomorphism. Example 1 in [1] shows that the "decreasing neighborhood base" assumption cannot be omitted in Theorem 1.

The following two lemmas are needed in the proof of the first theorem. The first lemma is known (see [8], p. 21) and the proof of the second is a routine transfinite induction argument.

LEMMA 1. *For every ordinal* λ, $\Gamma(\omega^\lambda)^{(\lambda)} = \{\omega^\lambda\}$. *Therefore, if* n *is a natural number,* $\operatorname{Card}\Gamma(\pi)^{(\lambda)} \geqslant n$ *if and only if* $\pi \geqslant \omega^\lambda\cdot n$.

LEMMA 2. *Suppose* X *is a totally disconnected, compact space and* $x \in X$. *If* x *has a local neighborhood base consisting of a (possibly transfinite) decreasing sequence* $\{U_\xi\}_{\xi<\gamma}$ *of sets, then* $\{U_\xi\}_{\xi<\gamma}$ *can also be selected with each* U_ξ *clopen.*

THEOREM 1. *Let* X *be a compact dispersed space with characteristic* (λ, n). *If each point* x *in* X *has a neighborhood base consisting of a (possibly transfinite) decreasing sequence* $\{U_\alpha\}_{\alpha<\tau}$ *of sets, there is a map of* X *onto* $\Gamma(\omega^\lambda\cdot n)$.

Proof. First observe that if the theorem is true for $(\lambda, 1)$, it is also true for (λ, n) for each positive integer n. For suppose that $X^{(\lambda)}$ has exactly n points, say $x_1, x_2, \ldots, x_n$ and that the theorem has been established for $(\lambda, 1)$. We can partition X into disjoint clopen sets $U_1, U_2, \ldots, U_n$ so that $x_i \in U_i$ for each i. But there exists a map f_i of U_i onto $\big(\omega^\lambda\cdot(i-1), \omega^\lambda\cdot i\big)$ and $f = \bigcup f_i$ is a map of X onto $\Gamma(\omega^\lambda\cdot n)$.

If Y is a closed subspace of X with characteristic system $(0, 1)$, Y is homeomorphic to $\Gamma(1)$ and the theorem statement is valid for Y. Suppose the theorem has been established for each closed subset of X with characteristic system $(\gamma, 1)$ where $\gamma < \lambda$ and $\lambda \geqslant 1$. By the preceding paragraph, we may also assume the theorem has been proved for each closed subspace with characteristic (γ, m) where $\gamma < \lambda$ and m is a positive integer. Let Y be a closed subset of X with characteristic $(\lambda, 1)$ and let y_0 be the one point in $Y^{(\lambda)}$. There is a decreasing sequence $\{U_\alpha\}_{\alpha<\tau}$ of sets in Y which form a neighborhood base for y_0 and, by Lemma 2, we may assume each U_α is clopen. It is convenient to assign $U_0 = Y$ and $U_\tau = \emptyset$. Since $\lambda \geqslant 1$, τ is a limit ordinal. Suppose $W_\alpha = U_\alpha \sim U_{\alpha+1}$ has characteristic $(\lambda_\alpha, n_\alpha)$. But $W_\alpha \subset (Y \sim Y^{(\lambda)})$; hence, $\lambda_\alpha < \lambda$ and, by hypothesis, there exists a map f_α of W_α onto $\Gamma(\omega^{\lambda_\alpha}\cdot n_\alpha)$. Therefore, there is a map g_α of W_α onto

$$\Big(\sum_{\sigma<\alpha} \omega^{\lambda_\sigma}\cdot n_\sigma, \sum_{\sigma\leqslant\alpha} \omega^{\lambda_\sigma}\cdot n_\sigma\Big], \qquad (\omega^{\lambda_0}\cdot n_0 = 1)\,.$$

We define a map h_α of $Y \sim U_\alpha$ onto $\Gamma\big(\sum_{\sigma<\alpha} \omega^{\lambda_\sigma}\cdot n_\sigma\big)$ for each $\alpha \leqslant \tau$ with the property that $\alpha < \beta$ implies $h_\alpha \subset h_\beta$. Let $h_1 = g_0$. Suppose h_α has been defined for each $\alpha < \beta$ where β is a fixed ordinal and $\beta \leqslant \tau$. If β is nonlimit, $h_{\beta-1}$ is defined and we let $h_\beta = h_{\beta-1} \cup g_{\beta-1}$. Clearly, h_β is the desired map since $Y \sim U_\beta = (Y \sim U_{\beta-1}) \cup W_{\beta-1}$ where $Y \sim U_{\beta-1}$ and $W_{\beta-1}$ are clopen, disjoint subsets of Y.

Next, suppose β is a limit ordinal. Since $\mu < \alpha < \beta$ implies $h_\mu \subset h_\alpha$, $g = \bigcup_{\alpha<\beta} h_\alpha$ is a map of $Y_\beta = \bigcup_{\alpha<\beta} (Y \sim U_\alpha)$ onto $\Gamma_0\big(\sum_{\alpha<\beta} \omega^{\lambda_\alpha}\cdot n_\alpha\big)$. For if $x \in Y_\beta$, there exists $\xi < \beta$ such that $x \in Y \sim U_\xi$ and

$$g(x) = h_\xi(x) \in \Gamma\Big(\sum_{\alpha<\xi} \omega^{\lambda_\alpha}\cdot n_\alpha\Big) \subseteq \Gamma_0\Big(\sum_{\alpha<\beta} \omega^{\lambda_\alpha}\cdot n_\alpha\Big)\,.$$

Moreover, if $\delta \in \Gamma_0\big(\sum_{\alpha<\beta} \omega^{\lambda_\alpha}\cdot n_\alpha\big)$, then $\delta \in \Gamma\big(\sum_{\alpha<\xi} \omega^{\lambda_\alpha}\cdot n_\alpha\big)$ for some $\xi < \beta$ and there exists $x \in (Y \sim U_\xi) \subseteq Y_\beta$ such that $g(x) = h_\xi(x) = \delta$.

We define h_β by

$$h_\beta(x) = \begin{cases} g(x)\,, & \text{if} \quad x \in Y_\beta = Y \sim \big(\bigcap_{\alpha<\beta} U_\alpha\big)\,, \\ \sum_{\alpha<\beta} \omega^{\lambda_\alpha}\cdot n_\alpha\,, & \text{if} \quad x \in \big(\bigcap_{\alpha<\beta} U_\alpha\big) \sim U_\beta\,. \end{cases}$$

The domain of h_β is $Y \sim U_\beta$, and it is obvious that h_β is continuous at each point in $Y \sim \big(\bigcap_{\alpha<\beta} U_\alpha\big)$. Suppose $x \in \big(\bigcap_{\alpha<\beta} U_\alpha\big) \sim U_\beta$ and $\{x_\mu\}_{\mu\in M}$ is a net in $Y \sim U_\beta$ such that $x_\mu \to x$. Let $\delta < \sum_{\alpha<\beta} \omega^{\lambda_\alpha}\cdot n_\alpha$. There exists $\xi < \beta$ such that $\delta < \sum_{\alpha<\xi} \omega^{\lambda_\alpha}\cdot n_\alpha$. Thus, $\delta < h_\beta(y) < \sum_{\alpha<\beta} \omega^{\lambda_\alpha}\cdot n_\alpha$ for $y \in [U_\xi \sim (\bigcap_{\alpha<\beta} U_\alpha)]$. But for $y \in (\bigcap_{\alpha<\beta} U_\alpha) \sim U_\beta$, $h_\beta(y) = \sum_{\alpha<\beta} \omega^{\lambda_\alpha}\cdot n_\alpha$. Since $x_\mu \to x$, there exists μ_0 such that $x_\mu \in U_\xi \sim U_\beta$ for $\mu \geqslant \mu_0$. Therefore,

$$h_\beta(x_\mu) \in \Big[\delta, \sum_{\alpha<\beta} \omega^{\lambda_\alpha}\cdot n_\alpha\Big]$$

for all $\mu \geqslant \mu_0$, so $h(x_\mu) \to h(x)$. This shows h_β is continuous; consequently, h_α can be defined for each $\alpha \leqslant \tau$.

Thus, h_τ is a map of Y onto $\Gamma\big(\sum_{\alpha<\tau} \omega^{\lambda_\alpha}\cdot n_\alpha\big)$. We show $\sum_{\alpha<\tau} \omega^{\lambda_\alpha}\cdot n_\alpha \geqslant \omega^\lambda$. If λ is nonlimit, $\lambda_\sigma = \lambda - 1$ for infinitely many values of σ and, by induction on n,

$$\sum_{\alpha<\tau} \omega^{\lambda_\alpha}\cdot n_\alpha = \lim_{\gamma<\tau} \Big(\sum_{\alpha<\gamma} \omega^{\lambda_\alpha}\cdot n_\alpha\Big) \geqslant \sum_{i=1}^{n} \omega^{\lambda-1}$$

for each natural number n. Thus,

$$\sum_{\alpha<\tau} \omega^{\lambda_\alpha}\cdot n_\alpha \geqslant \sum_{i=1}^{\infty} \omega^{\lambda-1} = \omega^{\lambda}.$$

If λ is a limit ordinal and $\gamma<\lambda$, there exists $\sigma<\tau$ such that $\lambda_\sigma>\gamma$. Therefore, $\sum_{\alpha<\tau}\omega^{\lambda_\alpha}\cdot n_\alpha \geqslant \omega^{\lambda_\sigma} \geqslant \omega^{\gamma}$. Since this is true for each $\gamma<\lambda$, $\sum_{\alpha<\tau}\omega^{\lambda_\alpha}\cdot n_\alpha \geqslant \sup_{\gamma<\lambda}\omega^{\gamma} = \omega^{\lambda}$. Thus $\sum_{\alpha<\tau}\omega^{\lambda_\alpha}\cdot n_\alpha \geqslant \omega^{\lambda}$ in both cases. If $\sum_{\alpha<\tau}\omega^{\lambda_\alpha}\cdot n_\alpha > \omega^{\lambda}$, there exists $\gamma<\tau$ such that $\sum_{\alpha<\gamma}\omega^{\lambda_\alpha}\cdot n_\alpha > \omega^{\lambda}$. Since h_γ is a homeomorphism of $Y \sim U_\gamma$ onto $\Gamma\big(\sum_{\alpha<\gamma}\omega^{\lambda_\alpha}\cdot n_\alpha\big)$, by Lemma 1 $(Y\sim U_\gamma)^{(\lambda)}$ is nonempty. This is impossible; hence, $\sum_{\alpha<\tau}\omega^{\lambda_\alpha}\cdot n_\alpha = \omega^{\lambda}$. Thus h_τ is a map of Y onto $\Gamma(\omega^{\lambda})$.

It follows by transfinite induction and by the first paragraph of this proof that if Y is a closed subspace of X with characteristic (λ, n), there exists a map of Y onto $\Gamma(\omega^{\lambda}\cdot n)$. This completes the proof.

We shall say that a point x in X satisfies (D) in X if x has a neighborhood base consisting of a decreasing possibly transfinite, sequence $\{U_\alpha\}_{\alpha<\tau}$ of clopen sets with the additional property that $\big(\bigcap_{\alpha<\beta} U_\alpha\big)\sim U_\beta$ contains at most one point for each limit ordinal β with $\beta<\tau$. If each point in X satisfies (D), we say that X has *property* (D). It should be noted that every first-countable, regular space and every set of ordinals satisfies (D). Theorem 2 gives a complete characterization of compact dispersed (Hausdorff) spaces with property (D). In particular, it characterizes closed sets of ordinals which are homeomorphic.

THEOREM 2. *Let X be a compact, dispersed space with characteristic (λ, n). If X has property* (D), *it is homeomorphic to $\Gamma(\omega^{\lambda}\cdot n)$.*

Proof. The proof of this theorem is identical to the proof of Theorem 1, except "map" is replaced with "homeomorphism" throughout. Note that h_β is injective since $\big(\bigcap_{\alpha<\beta} U_\alpha\big)\sim U_\beta$ contains exactly one point.

A subset X of $\Gamma(\xi)=\Gamma_0(\xi+1)$ is well-ordered and has order type β for some $\beta\leqslant\xi+1$. Therefore, there is an injective, order preserving map φ of $\Gamma_0(\beta)$ onto X. If X is a closed subspace of $\Gamma(\xi)$, φ is also a homeomorphism. The proof that φ is a homeomorphism is omitted since the following corollary is an easy consequence of Theorem 2.

COROLLARY 1. *A closed subspace of $\Gamma(\xi)$ is homeomorphic to $\Gamma(\eta)$ for some $\eta\leqslant\xi$.*

Theorem 2 is a generalization of the previously cited theorem of Mazurkiewicz and Sierpiński. In fact, we obtain the following generalization of their theorem established by Z. Semadeni in [10].

COROLLARY 2. (Semadeni). *A first-countable, dispersed, compact space X is metrizable. In fact, X is homeomorphic to $\Gamma(\omega^{\alpha}\cdot n)$ where (α, n) is the characteristic of X and $\alpha<\Omega$.*

Proof. Since X satisfies (D), it is homeomorphic to $\Gamma(\omega^{\alpha}\cdot n)$ by Theorem 2. Now $\Omega\notin\Gamma(\omega^{\alpha}\cdot n)$ because X is first-countable. Therefore, X is denumerable and second-countable. According to the Urysohn Metrization Theorem, X is metrizable.

Every ordinal $\lambda>0$ has a unique representation of the form $\lambda=\omega^{\gamma_1}\cdot a_1+\omega^{\gamma_2}\cdot a_2+\ldots+\omega^{\gamma_k}\cdot a_k$ where k and $a_1, a_2, \ldots, a_k$ are natural numbers and $\gamma_1, \gamma_2, \ldots, \gamma_k$ is a decreasing sequence of ordinals ([11], p. 323). We follow Cantor and call this the *normal form* of λ and the ordinal a_1 its *degree*. Also, a_1 will be called the *leading coefficient*. According to Lemma 1, the characteristic system of $\Gamma(\lambda)$ is (γ_1, a_1). The following well-known corollary is an immediate consequence of these definitions, Lemma 1, and Theorem 2.

COROLLARY 3. *Suppose a and β are ordinals and (λ, n) is the characteristic of $\Gamma(\lambda)$. The following are equivalent:*

(a) *$\Gamma(a)$ is homeomorphic to $\Gamma(\beta)$.*

(b) *$\Gamma(a)$ and $\Gamma(\beta)$ both have characteristic (λ, n).*

(c) *$\omega^{\lambda}\cdot n\leqslant a<\omega^{\lambda}\cdot(n+1)$ and $\omega^{\lambda}\cdot n\leqslant\beta<\omega^{\lambda}\cdot(n+1)$.*

(d) *The normal forms of both a and β have degree λ and leading coefficient n.*

The following example illustrates that "map" cannot be replaced with "homeomorphism" in the statement of Theorem 1. This example gives a complete answer to question 8 raised by Z. Semadeni in [10] [1].

EXAMPLE 1. There exists a compact, dispersed, Hausdorff space X not homeomorphic to any set of ordinals and such that for each $x\in X$ there exists a transfinite decreasing sequence $\{U_\alpha\}_{\alpha<\tau}$ of clopen sets which form a local neighborhood base for x.

Proof. Let K be the decomposition of $\Gamma(\Omega\cdot 2)$ consisting of points and the one plural set $\{\Omega, \Omega\cdot 2\}$. Then the quotient space $X=\Gamma(\Omega\cdot 2)/K$ is a compact dispersed Hausdorff space. It is easy to see that for each $x\in X$, there is a sequence $\{U_\alpha\}_{\alpha<\tau}$ of clopen sets which form a neighborhood base for x. Since X contains two uncountable sets A and B such that $K\notin A$, $K\notin B$, but $\bar{A}\cap\bar{B}=\{K\}$, X cannot be homeomorphic to a subset of $\Gamma(a)$ for any a.

[1] The referee informed the author that Example 1 was also given by M. Katetov in an unpublished letter in 1959 to Z. Semadeni. Actually, Katetov's example was $\Omega+1+\Omega^*$ where Ω^* denotes the order type inverse to Ω which is a simpler way of describing the author's space $\Gamma(\Omega\cdot 2)/K$.

By using a (transfinite) stairstep method, we can select for each x in the space X of Example 1 a decreasing sequence $\{U_\alpha\}_{\alpha<\tau}$ of sets forming a closed neighborhood base for x which also satisfies the property that $(\bigcap_{\alpha<\beta} U_\alpha)\sim U_\beta$ contains at most one point for each $\beta<\tau$. Therefore, one cannot assume that the sets in the required neighborhood systems in Theorem 2 are merely closed instead of clopen.

The next example shows that the compactness requirement in Theorem 2 cannot be replaced by sequentially compact or by normal. This gives a negative answer to a natural refinement of question 8 in [10]. I wish to thank Professor James Keesling for suggesting this example.

Example 2. Let $X=\Gamma_0(\Omega)\times\Gamma(\omega)$. Then X is sequentially compact, dispersed, normal, and satisfies (D), but X is not homeomorphic to a subset of $\Gamma(\alpha)$ for any ordinal α.

Proof. Let $\{(\alpha_n, k_n)\}$ be an arbitrary sequence in X and let $\alpha=\sup\alpha_n$. Since $\alpha<\Omega$ and $\{(\alpha_n, k_n)\}\subset\Gamma(\alpha)\times\Gamma(\omega)$, this sequence has an accumulation point. Thus, X is countably compact. By Proposition 1 in [2], X is sequentially compact. Also, X is normal by problem 8J in [4]. As each limit ordinal less than Ω is the limit of an increasing sequence of ordinals, X has property (D).

Suppose X is homeomorphic to a subset of $\Gamma(\alpha)$ for some α. Let $\mathrm{Cl}(X)$ denote the closure of X in $\Gamma(\alpha)$. By 8J in [4], the Stone–Čech compactification βX of X is $\Gamma(\Omega)\times\Gamma(\omega)$. There is a quotient map q from βX onto $\mathrm{Cl}(X)$ with $q(x)=x$ for each $x\in X$. Suppose x and y are distinct points in βX with $q(x)=q(y)$. Let $x=(\Omega, n)$ and $y=(\Omega, m)$. Then $Y=q[\{(\delta, k)|\ k\in\{m, n\}$ and $\delta\in\Gamma(\Omega)\}]$ is a closed subset of $\mathrm{Cl}(X)$ homeomorphic to $\Gamma(\Omega\cdot 2)/\{\Omega, \Omega\cdot 2\}$. By Corollary 1, Y is homeomorphic to a ray of ordinals. This contradicts Example 1. Thus, q is a homeomorphism of $\Gamma(\Omega)\times\Gamma(\omega)$ onto a ray of ordinal numbers. Since $\Gamma(\Omega)\times\Gamma(\omega)$ does not satisfy (D), this is impossible (see 3.10 in [10]).

Proposition 1. *If ξ and η are ordinal numbers, then $\Gamma(\eta+\xi)$, $\Gamma(\xi+\eta)$, and the free (i.e., discrete) union $\Gamma(\eta)\overset{f}{\cup}\Gamma(\xi)$ are all homeomorphic. If either $\xi\cdot\omega\leqslant\eta$ or $\omega\cdot\xi\leqslant\eta$, they are all also homeomorphic to $\Gamma(\eta)$.*

Proof. Since $\Gamma(\eta+\xi)=[1, \eta]\overset{f}{\cup}[\eta+1, \eta+\xi]$ and $\Gamma(\xi+\eta)=[1, \xi]\overset{f}{\cup}[\xi+1, \xi+\eta]$, the first statement follows from Theorem 2. Let (α, k) be the characteristic of $\Gamma(\xi)$ and let (β, m) be the characteristic of $\Gamma(\eta)$. By Lemma 1, $\omega^\alpha\cdot k\leqslant\xi$ and $\omega^\beta\cdot m\leqslant\eta$. If, in addition, $\xi\cdot\omega\leqslant\eta$, then $\omega^{\alpha+1}\leqslant\xi\cdot\omega<\eta$. By Lemma 1, $\omega^{\alpha+1}\leqslant\omega^\beta\cdot m$ and $\alpha+1\leqslant\beta$. As $[\Gamma(\eta)\overset{f}{\cup}\Gamma(\xi)]^{(\beta)}=\Gamma(\omega^\beta\cdot m)^{(\beta)}\overset{f}{\cup}\Gamma(\omega^\alpha\cdot k)^{(\beta)}=\{\omega^\beta, \omega^\beta\cdot 2, \ldots, \omega^\beta\cdot m\}$, $\Gamma(\eta)\overset{f}{\cup}\Gamma(\xi)$ has characteristic (β, m) and is homeomorphic to $\Gamma(\eta)$ by Theorem 2. The proof for the case where $\omega\cdot\xi\leqslant\eta$ is similar.

Lemma 3. *Let X and Y be compact spaces. If φ is a map of X onto Y, the inclusion $Y^{(\alpha)}\subset\varphi(X^{(\alpha)})$ is satisfied for every ordinal α.*

This lemma is established in [9] and is restated here for convenience.

Let X be a compact, dispersed space with characteristic system (λ, n). An immediate consequence of Theorem 2 and Lemma 3 is that *if φ is a map of X onto a space Y which satisfies* (D), *then Y is homeomorphic to $\Gamma(\xi)$ for some $\xi\leqslant\omega^\lambda\cdot n$.* In particular, if $X=\Gamma(\alpha)$ then Y is homeomorphic to $\Gamma(\xi)$ for some $\xi\leqslant\alpha$. If $X=\Gamma(\alpha)$ and $\alpha<\Omega$, the requirement that Y satisfy (D) can be dropped as $\Gamma(\alpha)$ and its images will be denumerable, and therefore metrizable. However, this requirement cannot be dropped in general since it is known that if $\alpha\geqslant\Omega$, the continuous image of $\Gamma(\alpha)$ may not be homeomorphic to a subset of $\Gamma(\beta)$ for any β. For example, the quotient space $Y=\Gamma(\Omega)/\{\Omega, \omega\}$ (i.e., $Y=\Omega+1+\omega^*$ where ω^* denotes the order type inverse to ω) is a continuous image of $\Gamma(\Omega)$ with characteristic $(\Omega, 1)$. However, if Y is homeomorphic to a subset of some $\Gamma(\beta)$, then by Theorem 2, Y is homeomorphic to $\Gamma(\Omega)$. This is impossible as $\{\omega, \Omega\}$ is the limit of a sequence of distinct points but each of its neighborhoods are uncountable.

If X is a topological space and if ξ is the least ordinal such that $X^{(\xi)}$ is perfect, then $P=X^{(\xi)}$, $G=\bigcup_{\alpha<\xi}(X^{(\alpha)}\sim X^{(\alpha+1)})$ is a unique partition of X into a perfect set P and a dispersed set G (see [12], p. 64). Of course, either of these sets may be empty. The set P is the largest perfect subset of X and is denotes by $\mathrm{Ker}(X)$ (i.e., the *kernel* of X).

In Theorem 3, we assume that X is compact, ξ is the least ordinal such that $X^{(\xi)}$ is perfect, and $G=X\sim X^{(\xi)}$. The notation $X/X^{(\xi)}$ is used for the quotient space of X consisting of $X^{(\xi)}$ and points of $X\sim X^{(\xi)}$. The restriction of a map φ to the subset G is denoted by $\varphi|G$. Theorem 3 gives a characterization of G for spaces X in which $X/X^{(\xi)}$ satisfies (D).

Theorem 3. *If $X/X^{(\xi)}$ satisfies* (D), *there is an ordinal π and a map φ of X onto $\Gamma(\pi)$ such that $\varphi|G$ is a homeomorphism of G onto $\Gamma_0(\pi)$ and $\varphi(x)=\pi$ for $x\in X\sim G$.*

In particular,

(a) *If $X^{(\alpha)}\cap G$ is infinite for each $\alpha<\xi$, then $\pi=\omega^\xi$.*

(b) *If $X^{(\alpha)}\cap G$ is finite for some $\alpha<\xi$, then ξ is nonlimit and $\pi=\omega^{\xi-1}\cdot n+\omega^\lambda$ where $\lambda=\inf\{\alpha\colon X^{(\alpha)}\cap G$ is closed$\}$ and $n=\mathrm{Card}(X^{(\xi-1)}\cap G)$.*

Proof. Let D be the decomposition of X consisting of $X^{(\xi)}$ and the singleton subsets of $X\sim X^{(\xi)}$. Let q denote the quotient map. This decomposition of X is upper semicontinuous and the quotient space X/D (i.e., $X/X^{(\xi)}$) is a dispersed, compact, Hausdorff space satisfying (D). Clearly $(X/D)^{(\xi)}\subseteq\{X^{(\xi)}\}$.

If $X^{(\alpha)}\cap G$ is infinite for each $\alpha<\xi$, by compactness $X^{(\alpha)}\cap G$ is not closed in X for $\alpha<\xi$. Since X/D is compact, $(X/D)^{(\xi)}$ contains the one point $X^{(\xi)}$. By Theorem 2, there is a homeomorphism h of X/D onto $\Gamma(\omega^{\xi})$. Then $\varphi = h\circ q$ is a map of X onto $\Gamma(\omega^{\xi})$ and, by Lemma 3, $\varphi(x) = \omega^{\xi}$ if and only if $x \in X^{(\xi)}$.

If $X^{(\alpha)} \cap G$ is finite for some $\alpha < \xi$, then ξ is nonlimit and $\alpha = \xi - 1$. Let $m = \operatorname{Card}(X^{(\xi-1)} \cap G)$. If $\lambda = \inf\{\sigma\colon X^{(\sigma)} \cap G \text{ is closed}\}$, then $\lambda \leqslant \xi - 1$ and $X^{(\xi)} \in (X/D)^{(\lambda)}$, but $X^{(\xi)} \notin (X/D)^{(\lambda+1)}$. There is a clopen subset U of X/D with $U \cap (X/D)^{(\lambda)} = \{X^{(\xi)}\}$. Consequently, $U \cap (X^{(\xi-1)} \cap G) = \emptyset$, $G \sim U$ is closed, and $\operatorname{Card}\big(X^{(\xi-1)} \cap (G\sim U)\big) = m$. By Theorem 2, there is a homeomorphism f of $G\sim U$ onto $\Gamma(\omega^{\xi-1}\cdot m)$ and a homeomorphism g of U onto $\Gamma(\omega^{\lambda})$. By Lemma 3, $g(u) = \omega^{\lambda}$ if and only if $u = X^{(\xi)}$. Let ψ be the homeomorphism of $\Gamma(\omega^{\lambda})$ onto $(\omega^{\xi-1}\cdot m, \omega^{\xi-1}\cdot m + \omega^{\lambda}]$ defined by $\psi(\alpha) = \omega^{\xi-1}\cdot m + \alpha$. It follows that h defined by $h(y) = f(y)$ for $y \in G\sim U$ and $h(y) = \psi\circ g(y)$ for $y \in U$ is a homeomorphism of X/D onto $\Gamma(\omega^{\xi-1}\cdot m + \omega^{\lambda})$ and $h(y) = \omega^{\xi-1}\cdot m + \omega^{\lambda}$ if and only if $y = X^{(\xi)}$. Thus, the $\varphi = h\circ q$ has the required properties.

Remark 1. In case (b) of Theorem 3, it follows by Corollary 3 that $\Gamma(\pi)$ is homeomorphic to $\Gamma(\omega^{\xi-1}\cdot m)$ where $m = \operatorname{Card}(X^{(\xi-1)} \cap G)$ if $\lambda < \xi - 1$ and $m = 1 + \operatorname{Card}(X^{(\xi-1)} \cap G)$ if $\lambda = \xi - 1$. Therefore, G is homeomorphic to a subset of $\Gamma(\omega^{\xi-1}\cdot m)$. However, it follows from Lemma 3 that G is homeomorphic to $\Gamma_0(\omega^{\xi-1}\cdot m)$ if and only if $\lambda = \xi - 1$.

Remark 2. Cases (a) and (b) of Theorem 3 may be combined as follows: If $\lambda = \inf\{\alpha\colon X^{(\alpha)} \cap G \text{ is closed}\}$, $\mu = \inf\{\alpha\colon X^{(\alpha)} \cap G \text{ is finite}\}$, and $n = \operatorname{Card}(X^{(\mu)} \cap G)$, then $\pi = \omega^{\mu}\cdot n + \omega^{\lambda}$. Moreover, $\omega^{\alpha} \leqslant \pi \leqslant \omega^{\xi}$ for each $\alpha < \xi$.

Indeed, for Case (a), it is easy to see that $\mu = \lambda = \xi$, $n = 0$, and $\omega^{\mu}\cdot n + \omega^{\lambda} = \omega^{\xi}$. For Case (b), note that $\mu = \xi - 1$.

Theorem 3 is related to the Cantor-Bendixson Theorem (cf. [6], p. 253): *Every separable metric space is the union of two disjoint sets, one of which is perfect and the other is countable and dispersed.* Theorem 4 gives a generalization of this theorem for compact spaces.

THEOREM 4. *Suppose X is a first-countable, compact space and* $\operatorname{Ker}(X)$ *is a G_δ-set. Then X is the union of two disjoint sets $P = \operatorname{Ker}(X)$ and G where P is perfect and G is metrizable, countable, and dispersed.*

In fact, there is a denumerable ordinal ξ such that $P = X^{(\xi)}$ and G is homeomorphic to $\Gamma_0(\pi)$ for an ordinal π with $\omega^{\alpha} \leqslant \pi \leqslant \omega^{\xi}$ for each $\alpha < \xi$.

Proof. Let ξ be the least ordinal number such that $X^{(\xi)}$ is perfect. Since $\operatorname{Ker}(X) = X^{(\xi)}$ is a G_δ-set, there is a decreasing sequence $\{U_i\}_{i=1}^{\infty}$ of clopen sets which form a neighborhood base for $X^{(\xi)}$. By Theorem 3, there is an ordinal π with $\omega^{\alpha} \leqslant \pi \leqslant \omega^{(\xi)}$ for $\alpha < \xi$ and there is a map φ of X onto $\Gamma(\pi)$ such that $\varphi|G$ is a homeomorphism of G onto $\Gamma_0(\pi)$. Also, $\varphi(x) = \pi$ if and only if $x \in X^{(\xi)}$; hence, X/D is homeomorphic to $\Gamma(\pi)$ where D contains $X^{(\xi)}$ and singleton subsets of $X\sim X^{(\xi)}$. But $\pi < \Omega$, as X/D is first countable; consequently, G is metrizable, countable, and dispersed.

COROLLARY 4. *A perfectly normal* ([6], p. 133), *compact space X is the union of two disjoint sets P and G where P is perfect and G is metrizable, countable, and dispersed.*

There are well-known examples which establish that, in Theorem 4, the hypothesis that $\operatorname{Ker}(X)$ is a G_δ-set cannot be dropped. However, if this hypothesis is dropped, it is easy to establish the following weaker conclusion using a compactness argument and Corollary 43–1 in [12]: Suppose X is a first-countable compact space. Then X is the union of two disjoint sets $P = \operatorname{Ker}(X)$ and G where P is perfect, G is dispersed, and a point has a countable neighborhood if and only if it belongs to G.

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THE FLORIDA STATE UNIVERSITY
Tallahassee, Florida

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