As h is order-preserving, $h(\pi_F(t)) <_Q n$. By the definition of A and the relation of s to t, there is $t' \in T'_{F\restriction\alpha}$ such that $t' \supset t$ and range$(t') \cap y = \emptyset$, and such that $h(\pi_F(t')) \geqslant_Q n$. In particular, $(t, y), (t', y) \in X(\alpha)$ and $(t', y) \geqslant_\alpha (t, y)$. But look, $(t, y) \in \Delta_n^\alpha$, so by definition we must have that $h_\alpha(\pi_F(t)) \geqslant_Q n$. Since $h \restriction \alpha = h_\alpha$, this contadicts our earlier inequality. Hence $T(F)$ cannot be Q-embeddable.

Suppose now that for some $F, G \in 2^{\omega_1}$, $F \neq G$, we have $T(F) \cong T(G)$. Let h: $\pi_F{}''T(F) \cong \pi_G{}''T(G)$. Pick $\alpha_0 < \omega_1$ such that $F \restriction \alpha_0 \neq G \restriction \alpha_0$. Let

$$A = \{\alpha \in \omega_1 |\ \alpha = \bigcup \alpha > \alpha_0 \cdot \& \cdot [\pi_F{}''T(F)] \restriction \alpha = \pi'_{F\restriction\alpha}{}''T'_{F\restriction\alpha} \cdot \& \cdot$$

$$\cdot \& \cdot [\pi_G{}''T(G)] \restriction \alpha = \pi'_{G\restriction\alpha}{}''T'_{G\restriction\alpha} \cdot \& \cdot h \restriction \alpha \colon \pi'_{F\restriction\alpha}{}''T'_{F\restriction\alpha} \cong \pi'_{G\restriction\alpha}{}''T'_{G\restriction\alpha}\}\,.$$

Clearly, A is closed and unbounded in ω_1. By $\diamond$, there is $\alpha \in A$ such that $h \restriction \alpha = h_\alpha$. Thus Case II applied in constructing $T_{F\restriction\alpha}$ from $T'_{F\restriction\alpha}$ and $T_{G\restriction\alpha}$ from $T'_{G\restriction\alpha}$. This means that the map $\pi'^{-1}_{G\restriction\alpha} \cdot h_\alpha \cdot \pi'_{F\restriction\alpha}$ does not extend to an isomorphism of $T_{F\restriction\alpha}$ and $T_{G\restriction\alpha}$, which is absurd, since $\pi_G^{-1} \cdot h \cdot \pi_F$ extends it. Thus $T(F)$ and $T(G)$ are not isomorphic. The proof is complete.

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Shapes of finite-dimensional compacta

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1. Introduction. The results of this paper deal with shapes of finite-dimensional compact metric spaces (see [4] for definitions concerning the concept of shape). In Theorem 1 below we give a characterization of shapes of finite-dimensional compact metric spaces (i.e. *compacta*) in terms of embeddings in Euclidean n-space E^n. In an earlier paper the author obtained a characterization of shapes of compacta (with no dimensional restriction) in terms of embeddings in the Hilbert cube [8]. In a sense the results obtained here are motivated by [8], and to some extent the general structure of the proof of Theorem 1 is a modification of the argument used in [8], but the present paper does not involve any infinite-dimensional topology. For the sake of completeness we give a short summary of the infinite-dimensional characterization at the end of the Introduction. We use the notation $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ to indicate that compacta X and Y have the same shape.

THEOREM 1. *Let X, Y be compacta such that* $\dim X$, $\dim Y \leqslant m$.

(a) *For any integer $n \geqslant 2m+2$ there exist copies $X', Y' \subset E^n$ (of X, Y respectively) such that if* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$, *then $E^n \setminus X'$ and $E^n \setminus Y'$ are homeomorphic.*

(b) *For any integer $n \geqslant 3m+3$ there exist copies $X', Y' \subset E^n$ (of X, Y respectively) such that if $E^n \setminus X'$ and $E^n \setminus Y'$ are homeomorphic, then* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

We remark that a similar result holds for embeddings of X and Y in the n-sphere S^n.

For prerequisites we will need some elementary facts concerning the piecewise-linear topology of E^n plus an isotopy extension theorem from [11]. We also use a characterization of dimension in terms of mappings onto polyhedra in E^n (see [14], p. 111). As for techniques we remark that part (a) of Theorem 1 is the most difficult part of the proof. Roughly the idea is to construct a sequence $\{h_i\}_{i=1}^\infty$ of homeomorphisms

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of E^n onto itself such that the sequence $\{h_i \circ h_{i-1} \circ \ldots \circ h_1\}_{i=1}^{\infty}$ of left products converges pointwise (on $E^n \backslash X'$) to define a homeomorphism of $E^n \backslash X'$ onto $E^n \backslash Y'$. This is the idea that was used in [8].

We now make some comments concerning the infinite-dimensional characterization obtained in [8]. We represent the Hilbert cube Q by $\prod_{i=1}^{\infty} I_i$, where each I_i is the closed interval $[0,1]$, and the *pseudo-interior* of Q is $s = \prod_{i=1}^{\infty} I_i^{\circ}$, where each I_i° is the open interval $(0,1)$. The characterization obtained in [8] is as follows:

THEOREM 2. *Let $X, Y \subset s$ be compacta. Then* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ *iff $Q \backslash X$ and $Q \backslash Y$ are homeomorphic.*

The condition "$X, Y \subset s$" in Theorem 2 is crucial and in general cannot be replaced by the weaker condition "$X, Y \subset Q$". Also it follows from [2] that if $X, Y \subset s$ are any two compacta, then $s \backslash X$ and $s \backslash Y$ are homeomorphic to s. However the characterization is generally applicable to compacta, since any compactum can be embedded in s. We remark that the proof of Theorem 2 given in [8] is non-elementary and uses some recent developments in the theory of infinite-dimensional manifolds modeled on Q (see [7] for a summary). The proof we give here of Theorem 1 is a bit more complicated since there are some infinite-dimensional techniques used in the proof of Theorem 2 which have no finite-dimensional analogues.

The author is grateful to Morton Brown for suggesting something on the order of Theorem 1, in the sense that he felt "shape" for finite-dimensional compacta should mean "homeomorphic complements" (in an appropriate setting in Euclidean space). The author also wishes to thank R. D. Anderson for making some valuable comments on the manuscript.

2. Definitions and notation. For any topological space X and any set $A \subset X$ we let $\mathrm{Bd}_X(A)$, $\mathrm{Int}_X(A)$, and $\mathrm{Cl}_X(A)$ denote, respectively, the topological *boundary*, *interior*, and *closure* of A in X. When no ambiguity results we will suppress the subscript X. If Y is another space and $f\colon X \to Y$ is a function, then $f|A$ will denote the *restriction* of f to A.

All homeomorphisms will be onto and we use the notation $X \cong Y$ to indicate that spaces X and Y are homeomorphic. The identity homeomorphism of X onto itself will be denoted by id_X and by a *map* we will mean a continuous function. If (Y, d) is a metric space and $f, g\colon X \to Y$ are maps, then we use

$$d(f,g) = \sup\{d(f(x), g(x)) \mid x \in X\} \quad \text{(if it exists)}$$

for the distance between f and g. In the sequel we will indiscriminately use d to denote the metric of any space under consideration.

For products $X \times Y$ we use $p_X\colon X \times Y \to X$ to denote projection. In Euclidean space E^n and any integer $m \leqslant n$ we use $p_m\colon E^n \to E^m$ to denote projection onto the first m coordinates, i.e.

$$p_m(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_m), \quad \text{for all } (x_1, x_2, \ldots, x_n) \in E^n.$$

We use I to denote the unit interval $[0,1]$ and by a *homotopy* we mean a continuous function $F\colon X \times I \to Y$. The *levels* of F are the maps $F_t\colon X \to Y$, defined by $F_t(x) = F(x,t)$, for all $(x,t) \in X \times I$. For a map $G\colon X \times I \to Y \times I$ we will also use the notation $G_t\colon X \to Y$ for the map defined by $G_t(x) = p_X \circ G(x,t)$, for all $(x,t) \in X \times I$. If $B \subset Y$ and $f, g\colon X \to Y$ are maps satisfying $f(X), g(X) \subset B$, then we use the notation $f \simeq g$ (in B) to mean that there exists a homotopy $F\colon X \times I \to Y$ such that $F_0 = f$, $F_1 = g$, and $F(X \times I) \subset B$.

In Euclidean space E^n and any $\varepsilon > 0$ we let

$$B_\varepsilon^n = \{x \in E^n \mid \|x\| \leqslant \varepsilon\},$$

$$\partial B_\varepsilon^n = \{x \in E^n \mid \|x\| = \varepsilon\}.$$

For any integer $m \leqslant n$ we will use $E^m \times 0 \subset E^n$ to indicate the Euclidean subspace of E^n defined by

$$E^m \times 0 = \{(x_1, x_2, \ldots, x_n) \in E^n \mid x_{m+1} = x_{m+2} = \ldots = x_n = 0\}.$$

By a *polyhedron* we will mean a (locally-finite) union of linear cells contained in some Euclidean space E^n and by a *topological polyhedron* we will mean any space homeomorphic to a polyhedron. Generally we will use notation and results from [10] concerning elementary piecewise linear (PL) topology, including such concepts as PL maps, derived and regular neighborhoods, etc.

If X is a polyhedron and Y is a PL manifold (i.e. a polyhedron which is a manifold), a *concordance* F of X in Y is a PL embedding $F\colon X \times I \to Y \times I$ such that $F(X \times 0) \subset Y \times 0$ and $F(X \times 1) \subset Y \times 1$. The concordance F is *allowable* if $F^{-1}(Y \times 0) = X \times 0$, $F^{-1}(Y \times 1) = X \times 1$, and $F^{-1}(\partial Y \times I) = X_0 \times I$, X_0 being a closed subpolyhedron of X. The following result will be needed in the proof of Theorem 1.

LEMMA 2.1 (Hudson [11]). *Let X be a compact polyhedron, Y be a* PL *manifold, and let $F\colon X \times I \to Y \times I$ be an allowable concordance which satisfies $F_t(X) \cap (\partial Y) = \emptyset$, for all $t \in I$. If $\dim X \leqslant \dim Y - 3$, then there exists a* PL *homeomorphism $h\colon Y \to Y$ which satisfies $h \circ F_0 = F_1$ and* $h|\partial Y = \mathrm{id}$.

As an easy consequence of Lemma 2.1 we get the following corollary (which will be more immediately useful to us).

Corollary 2.2. *Let X be a compact polyhedron such that* $\dim X \leqslant n$, *Y be an open subset of* E^{2n+2}, *and let* $f, g\colon X \to Y$ *be* PL *embeddings such that* $f \simeq g$ *(in Y). Then there exists a* PL *homeomorphism* $h\colon E^{2n+2} \to E^{2n+2}$ *such that* $h|E^{2n+2}\setminus Y = \mathrm{id}$ *and* $h \circ f = g$.

Proof. For $n = 0$ the result is trivial and we therefore assume $n \geqslant 1$, in which case $\dim Y - \dim X \geqslant 3$. Let $F\colon X \times I \to Y$ be a map such that $F_0 = f$ and $F_1 = g$. Using Lemma 4.2 of [10] (which is concerned with approximating maps by PL maps) we can replace F by a PL map $G\colon X \times I \to Y$ such that $G_0 = f$ and $G_1 = g$. If we note that $2\dim(X\times I)+1 \leqslant \dim(Y\times I)$, then we can use standard procedures for general positioning to modify G to obtain an allowable concordance $H\colon X\times I \to Y \times I$ such that $H_0 = f$ and $H_1 = g$. Let $P \subset Y$ be a compact polyhedron such that $H_t(X) \subset P$, for all $t \in I$, and let $N \subset Y$ be a regular neighborhood of P. Then $H\colon X \times I \to \mathrm{Int}(N) \times I$ is an allowable concordance which satisfies $H_t(X) \cap (\partial N) = \emptyset$, for all $t \in I$. (Here the combinatorial boundary of N coincides with the topological boundary of N.) Thus Lemma 2.1 implies that there exists a PL homeomorphism $h'\colon N \to N$ such that $h' \circ f = g$ and $h'|\partial N = \mathrm{id}$. Then extend h' to a PL homeomorphism $h\colon E^{2n+2} \to E^{2n+2}$ by defining $h|E^{2n+2}\setminus N = \mathrm{id}$.

We will also need the following result on regular neighborhoods which follows from Theorem 2.1 on page 65 of [10].

Lemma 2.3. *Let X be a compact polyhedron in the interior of a* PL *manifold Y and let N_1, N_2 be two regular neighborhoods of X in the interior of Y. If $U \subset Y$ is an open set containing $N_1 \cup N_2$, then there exists a* PL *homeomorphism* $h\colon Y \to Y$ *such that* $h|X \cup (Y\setminus U) = \mathrm{id}$ *and* $h(N_1) = N_2$.

3. Embedding compacta in E^n. If X is any compactum satisfying $\dim X \leqslant n$, then it is well-known that X can be embedded in E^{2n+1}. In Proposition 3.4 below we prove that X can be embedded into E^{2n+1} in a "nice" way which will be useful in the sequel. This "niceness" condition is described in the following definition.

Definition 3.1. Let $X \subset E^n$ be a compactum which satisfies $\dim X \leqslant m$, for some $m > 0$. Then we say that X is in *standard position* iff there exist sequences $\{P_i\}_{i=1}^\infty$ and $\{N_i\}_{i=1}^\infty$ such that the following properties are satisfied

(1) each P_i is a compact polyhedron in E^n satisfying $\dim P_i \leqslant m$,
(2) each N_i is a regular neighborhood of P_i in E^n,
(3) each $N_{i+1} \subset \mathrm{Int}(N_i)$, and
(4) $X = \bigcap_{i=1}^{\infty} N_i$.

We remark that this condition does not necessarily imply tameness. For example if $X \subset E^3$ is the wild arc of Artin–Fox (as described on page 177 of [9]), then it is easily verified that X is in standard position. On the other hand if $X \subset E^n$ $(n \geqslant 4)$ is any arc such that $E^n\setminus X$ is not simply connected (such arcs exist from [3]), then it can be verified that X is not in standard position. We omit the details because these observations are not needed in the sequel. One obvious fact which will be needed in the sequel is the following: If $X \subset E^n$ is a compactum in standard position, then $X \times 0 \subset E^{n+m}$ is in standard position, for all $m \geqslant 0$.

In proposition 3.4 below we show that every compactum of dimension less than or equal to n can be embedded into E^{2n+1} in standard position. The following characterization of dimension will be needed in the proof of Proposition 3.4.

Lemma 3.2. ([14], p. 111). *A compactum $X \subset E^n$ satisfies $\dim X \leqslant m$ iff for each $\varepsilon > 0$ there exists a polyhedron $P \subset E^n$ satisfying $\dim P \leqslant m$ and a map $f\colon X \to P$ such that $f(X) = P$ and $d(f, \mathrm{id}) < \varepsilon$.*

We will also need a convergence procedure for sequences of embeddings of compacta into complete metric spaces. Various forms of this type of convergence procedure are known and have been used occasionally (for example see Lemma 2.1 of [1]). It is for this reason that we state the result with no proof. For notation let (Y, d) be a metric space and let $X \subset Y$ be a compactum. Then for any embedding $f\colon X \to Y$ and any $\delta > 0$ let

$$\varepsilon(f, \delta) = \mathrm{glb}\{d(f(x), f(y))|\ x, y \in X \text{ and } d(x, y) \geqslant \delta\},$$

which is clearly a positive number. (Here glb means greatest lower bound.)

Lemma 3.3. *Let (Y, d) be a complete metric space and let $X \subset Y$ be a compactum. Moreover let*

$$X \xrightarrow{f_1} Y,\ f_1(X) \xrightarrow{f_2} Y,\ f_2 \circ f_1(X) \xrightarrow{f_3} Y,\ \ldots$$

be a sequence of embeddings such that

$$d(f_i, \mathrm{id}) < \min\big(3^{-i}, (3^{-i})\cdot\varepsilon(f_{i-1}\circ \ldots \circ f_1, 2^{-i})\big),$$

for all $i > 1$. Then the sequence $\{f_i \circ f_{i-1} \circ \ldots \circ f_1\}_{i=1}^\infty$ converges pointwise to an embedding of X into Y.

Proposition 3.4. *Let $X \subset E^{2n+1}$ be a compactum such that $\dim X \leqslant n$. Then there exists an embedding $f\colon X \to E^{2n+1}$ such that $f(X)$ is in standard position.*

Proof. We will apply Lemma 3.3 (with $Y = E^{2n+1}$). To do this we will inductively construct sequences $\{P_i\}_{i=1}^\infty$, $\{N_i\}_{i=1}^\infty$, and $\{f_i\}_{i=1}^\infty$ which satisfy

(1) each P_i is a compact polyhedron in E^{2n+1} such that $\dim P_i \leqslant n$,

(2) each N_i is a regular neighborhood of P_i in E^{2n+1} such that $N_{i+1} \subset \mathrm{Int}(N_i)$,

(3) $\{f_i\}_{i=1}^{\infty}$ is a sequence of embeddings:

$$X \xrightarrow{f_1} \mathrm{Int}(N_1),\quad f_1(X) \xrightarrow{f_2} \mathrm{Int}(N_2),\quad f_2\circ f_1(X) \xrightarrow{f_3} \mathrm{Int}(N_3),\ \ldots,$$

(4) $d(f_i, \mathrm{id}) < \min\big(3^{-i}, (3^{-i})\cdot\varepsilon(f_{i-1}\circ\ldots\circ f_1, 2^{-i})\big)$, for all $i > 1$,

(5) if $\delta_i = d\big(f_i\circ\ldots\circ f_1(X), E^{2n+1}\backslash\mathrm{Int}(N_i)\big)$, then $d(f_{i+j}, id) < \delta_i/2^j$, for all $i, j > 0$, and

(6) $d\big(f_i\circ\ldots\circ f_1(X), x\big) < 1/2^{i-1}$, for all $i > 0$ and $x \in N_i$.

To start the induction we now construct P_1, N_1 and f_1. Using Lemma 3.2 there exists a polyhedron $P_1 \subset E^{2n+1}$ satisfying $\dim P_1 \leqslant n$ and a map $g_1: X\to P_1$ such that $g_1(X) = P_1$ and $d(g_1, \mathrm{id}) < 1/2$. Choose a regular neighborhood N_1 of P_1 in E^{2n+1} such that there exists a retraction $r_1: N_1\to P_1$ satisfying $d(r_1, \mathrm{id}) < 1/2$. It is well-known that any continuous function of X into E^{2n+1} can be approximated by an embedding. Thus there exists an embedding $f_1: X\to\mathrm{Int}(N_1)$ such that $d(f_1, g_1) < 1/2$. This implies that $d\big(f_1(X), x\big) < 1$, for all $x\in N_1$. This completes the construction of P_1, N_1, and f_1.

For the inductive step let us now assume that $\{P_i\}_{i=1}^{m}$, $\{N_i\}_{i=1}^{m}$, and $\{f_i\}_{i=1}^{m}$ have been constructed so that conditions (1)–(6) are satisfied. We will construct P_{m+1}, N_{m+1}, and f_{m+1} so that $\{P_i\}_{i=1}^{m+1}$, $\{N_i\}_{i=1}^{m+1}$, and $\{f_i\}_{i=1}^{m+1}$ satisfy conditions (1)–(6). To simplify notation let

$$\varepsilon_i = \min\big(3^{-i}, (3^{-i})\cdot\varepsilon(f_{i-1}\circ\ldots\circ f_1, 2^{-i})\big), \quad \text{for} \quad 2\leqslant i\leqslant m+1\,.$$

Using Lemma 3.2 there exists a polyhedron $P_{m+1}\subset E^{2n+1}$ satisfying $\dim P_{m+1}\leqslant n$ and a map $g_{m+1}: f_m\circ\ldots\circ f_1(X)\to P_{m+1}$ such that $g_{m+1}\circ f_m\circ \circ\ldots\circ f_1(X) = P_{m+1}$ and

$$d(g_{m+1}, \mathrm{id}) < \min(\varepsilon_{m+1}/2, \delta_1/2^{m+1}, \delta_2/2^m, \ldots, \delta_m/2^2)\,.$$

Since $d(g_{m+1}, \mathrm{id}) < \delta_m$ we have $P_{m+1}\subset\mathrm{Int}(N_m)$. Thus we can choose a regular neighborhood N_{m+1} of P_{m+1} in E^{2n+1} such that $N_{m+1}\subset\mathrm{Int}(N_m)$ and for which there exists a retraction $r_{m+1}: N_{m+1}\to P_{m+1}$ satisfying $d(r_{m+1}, \mathrm{id}) < 1/2^{m+1}$. Now let $f_{m+1}: f_m\circ\ldots\circ f_1(X)\to\mathrm{Int}(N_{m+1})$ be an embedding satisfying

$$d(f_{m+1}, g_{m+1}) < \min(\varepsilon_{m+1}/2, \delta_1/2^{m+1}, \delta_2/2^m, \ldots, \delta_m/2^2)\,.$$

It then follows that

$$d(f_{m+1}, \mathrm{id}) < \min(\varepsilon_{m+1}, \delta_1/2^m, \delta_2/2^{m-1}, \ldots, \delta_m/2)\,.$$

If $x\in N_{m+1}$, then $d\big(g_{m+1}\circ f_m\circ\ldots\circ f_1(X), x\big) = d(P_{m+1}, x) < 1/2^{m+1}$. Since $d(f_{m+1}, g_{m+1}) < \varepsilon_{m+1} < 1/2^{m+1}$, it follows that $d\big(f_{m+1}\circ\ldots\circ f_1(X), x\big) < 1/2^m$. Thus $\{P_i\}_{i=1}^{m+1}$, $\{N_i\}_{i=1}^{m+1}$, and $\{f_i\}_{i=1}^{m+1}$ satisfy properties (1)–(6). Thus we have inductively constructed the desired sequences.

It follows from (4) and Lemma 3.3 that the sequence $\{f_i\circ\ldots\circ f_1\}_{i=1}^{\infty}$ converges to an embedding $f: X\to E^{2n+1}$. From (3) and (5) we have $f(X)\subset\mathrm{Int}(N_i)$, for all $i > 0$. Thus all we need to do is show that $\bigcap_{i=1}^{\infty} N_i = f(X)$. To see this choose any $x\in\bigcap_{i=1}^{\infty} N_i$ and use (6) to conclude that $d\big(f_i\circ\ldots\circ f_1(X), x\big) < 1/2^{i-1}$, for all $i > 0$. Since we have $d(f_i, \mathrm{id}) < 3^{-i}$, for all $i > 1$, it then follows that $x\in f(X)$.

The following result will be useful in the proof of part (b) of Theorem 1.

Lemma 3.5. *Let $Y\subset E^n$ be a compactum in standard position such that $n\geqslant 2\dim Y+1$ and let X be a compactum such that $\dim X+\dim Y < n$. Also let $U\subset E^n$ be an open set containing Y, $A\subset X$ be closed, and let $f: X\to U$ be a map such that $f(A)\cap Y = \emptyset$. Then there exists a map $g: X\to U$ such that $g(X)\cap Y = \emptyset$ and $g|A = f|A$.*

Proof. Since Y is in standard position there exists a compact polyhedron $P\subset U$ such that $\dim P\leqslant\dim Y$ and a regular neighborhood N of P such that $Y\subset N\subset U\backslash f(A)$. There are standard techniques for approximating maps by maps into polyhedra (for example see pp. 69, 70 of [12]). Since $\dim(X\backslash A)\leqslant\dim X$ we can use these techniques to find a map $f': X\to U$ such that $f'(X\backslash A)$ lies in a locally compact polyhedron of dimension equal to or less than $\dim X$ and $f'|A = f|A$. Let P' be the intersection of this locally compact polyhedron with N. Then we have $f'(X)\cap N\subset P'$ and $\dim P'\leqslant\dim X$. Lemma 4 on page 97 of [10] implies that there exists a PL homeomorphism $h_1: N\to N$ such that $h_1|\mathrm{Bd}(N) = \mathrm{id}$ and $h_1\big(P'\backslash\mathrm{Bd}(N)\big)$ is in *general position with respect to* P, i.e.

$$\dim\big(h_1(P'\backslash\mathrm{Bd}(N))\cap P\big)\leqslant\dim P'+\dim P-n\,.$$

But $\dim P'+\dim P < n$, hence $\dim\big(h_1(P'\backslash\mathrm{Bd}(N))\cap P\big)\leqslant -1$, which implies that $h_1\big(P'\backslash\mathrm{Bd}(N)\big)\cap P = \emptyset$. Extend h_1 to a homeomorphism $h: E^n\to E^n$ so that $h|E^n\backslash N = \mathrm{id}$. Thus $h\circ f': X\to E^n$ is a map satisfying $h\circ f'(X)\cap P = \emptyset$ and $h\circ f'|A = f|A$.

Now let $N_1\subset\mathrm{Int}(N)$ be a regular neighborhood of P such that $N_1\cap h\circ f'(X) = \emptyset$ and let $N_2\subset\mathrm{Int}(N)$ be a regular neighborhood of P such that $Y\subset N_2$. Using Lemma 2.3 there exists a homeomorphism $h': E^n\to E^n$ such that $h'(N_1) = N_2$ and $h'|E^n\backslash N = \mathrm{id}$. Then $g = h'\circ h\circ f': X\to U$ fulfills our requirements.

4. Relative shape. In this section we define a relative notion of shape which will be needed in the proof of Theorem 1. This apparatus was also employed in [8] to prove Theorem 2 as cited at the end of our Introduction.

Consider compacta X, Y contained in a space W and let $G\subset W$ be a neighborhood of X. Let $\{f_k\}_{k=k_1}^{\infty}$ be a sequence of maps $f_k: G\to W$ such that

(1) each f_k is homotopic to the inclusion of G in W (we will incorrectly write this as $f_k \simeq \mathrm{id}_G$),

(2) for each neighborhood $V \subset W$ of Y there exists a neighborhood $U \subset G$ of X such that $f_k|U \simeq f_l|U$ (in V), for almost all integers k and l.

Then we define $\{f_k, X, Y, G\}$ to be a *relative fundamental sequence* (in W) and we write $\underline{f} = \{f_k, X, Y, G\}$. We will agree to identify relative fundamental sequences $\underline{f} = \{f_k, X, Y, G\}$ and $\underline{g} = \{g_k, X, Y, H\}$ provided that there exists a neighborhood $U \subset G \cap H$ of X such that $f_k|U = g_k|U$, for almost all k.

Now choose compacta X, Y, Z in a space W and relative fundamental sequences $\underline{f} = \{f_k, X, Y, G\}$ and $\underline{g} = \{g_k, Y, Z, H\}$ (in W). It is clear that there exists a neighborhood $G_1 \subset G$ of X and an integer $k_1 > 0$ large enough so that

$$\underline{g} \circ \underline{f} = \{g_k \circ f_k|G_1, X, Z, G_1\} \qquad (k \geqslant k_1)$$

is a relative fundamental sequence (in W). Because of the agreement made above on the identification of relative fundamental sequences it follows that the composition $\underline{g} \circ \underline{f}$ is well-defined.

If X, Y are compacta in W and $\underline{f} = \{f_k, X, Y\}$ is a fundamental sequence (in W), then $\{f_k|G, X, Y, G\}$ uniquely defines a relative fundamental sequence (in W), for any neighborhood G of X. We also see that if $X \subset W$ is a compactum and G is any neighborhood of X, then $\{\mathrm{id}_G, X, X, G\}$ uniquely defines a relative fundamental sequence (in W). We denote this sequence by $\underline{\mathrm{id}}_X$ (when no ambiguity results) and call it the *identity relative fundamental sequence* from X to X.

If X, Y are compacta in W and $\underline{f} = \{f_k, X, Y, G\}$, $\underline{g} = \{g_k, X, Y, H\}$ are relative fundamental sequences (in W), then we write $\underline{f} \simeq \underline{g}$ iff for each neighborhood $V \subset W$ of Y there exists a neighborhood $U \subset G \cap H$ of X such that

$$f_k|U \simeq g_k|U \qquad (\text{in } V),$$

for almost all integers k.

Now let X, Y be compacta in W and assume that there exist relative fundamental sequences $\underline{f} = \{f_k, X, Y, G\}$ and $\underline{g} = \{g_k, Y, X, H\}$ (in W) such that $\underline{g} \circ \underline{f} \simeq \underline{\mathrm{id}}_X$ and $\underline{f} \circ \underline{g} \simeq \underline{\mathrm{id}}_Y$. Then we say that X and Y have the same *relative shape* (in W).

We emphasize the fact that the notion of relative shape depends on W and the positioning of X and Y in W.

5. The main lemma. In Lemma 5.1 below we establish what amounts to the inductive step in the proof of part (b) of Theorem 1. This is the only place that it becomes necessary to get deeply involved with the apparatus of Section 4.

LEMMA 5.1. *For any integer $n > 0$ let $W \subset E^{2n+2}$ be an open set and let $X, Y \subset W$ be compacta such that X is in standard position, $\dim X \leqslant n$, and X, Y have the same relative shape (in W). If $W_1 \subset W$ is any neighborhood of Y, then there exists a PL homeomorphism $\Phi\colon E^{2n+2} \to E^{2n+2}$ such that $\Phi(X)$ is in standard position, $\Phi|E^{2n+2} \setminus W = \mathrm{id}$, and $\Phi(X)$, Y have the same relative shape (in W_1).*

Proof. Since X is in standard position we can find sequences $\{P_i\}_{i=1}^{\infty}$ and $\{N_i\}_{i=1}^{\infty}$ which satisfy properties (1)–(4) of Definition 3.1. Choose neighborhoods $G \subset W$ of X, $H \subset W$ of Y, and relative fundamental sequences $\underline{f} = \{f_k, X, Y, G\}$ and $\underline{g} = \{g_k, Y, X, H\}$ (in W) such that $\underline{g} \circ \underline{f} \simeq \underline{\mathrm{id}}_X$ and $\underline{f} \circ \underline{g} \simeq \underline{\mathrm{id}}_Y$ (in W).

Now choose an integer $n_1 > 0$ and an integer $i_1 > 0$ such that $N_{i_1} \subset G$ and

$$f_k|N_{i_1} \simeq f_l|N_{i_1} \qquad (\text{in } H \cap W_1),$$

for all integers $k, l \geqslant n_1$. Since $f_{n_1}|P_{i_1}\colon P_{i_1} \to H \cap W_1$ is a map and $\dim P_{i_1} \leqslant n$, we can find a PL embedding $\varphi\colon P_{i_1} \to E^{2n+2}$ which is as close to $f_{n_1}|P_{i_1}$ as we like. We can therefore choose φ close enough to $f_{n_1}|P_{i_1}$ so that $\varphi(P_{i_1}) \subset H \cap W_1$ and $\varphi \simeq f_{n_1}|P_{i_1}$ (in $H \cap W_1$) (for example we can use the straight-line homotopy joining φ to $f_{n_1}|P_{i_1}$). Using Corollary 2.2 we can extend φ to a PL homeomorphism $\Phi_1\colon E^{2n+2} \to E^{2n+2}$ satisfying $\Phi_1|E^{2n+2} \setminus W = \mathrm{id}$.

Since $\Phi_1^{-1}(H \cap W_1)$ is a neighborhood of P_{i_1} we can find a regular neighborhood N of P_{i_1} such that $N \subset \Phi_1^{-1}(H \cap W_1)$. Using Lemma 2.3 it follows that there exists a PL homeomorphism $\alpha\colon E^{2n+2} \to E^{2n+2}$ such that $\alpha(N_{i_1}) = N$ and $\alpha|P_{i_1} \cup (E^{2n+2} \setminus W) = \mathrm{id}$.

Then $\Phi = \Phi_1 \circ \alpha\colon E^{2n+2} \to E^{2n+2}$ is a PL homeomorphism satisfying $\Phi|E^{2n+2} \setminus W = \mathrm{id}$ and $\Phi|P_{i_1} \simeq f_{n_1}|P_{i_1}$ (in $H \cap W_1$). Also it follows that $\Phi(X)$ is in standard position, since Φ is PL.

Since N_{i_1} is a regular neighborhood of P_{i_1} there exists a retraction $r\colon N_{i_1} \to P_{i_1}$ such that $r \simeq \mathrm{id}$ (in N_{i_1}). Thus the two smaller triangles in the following diagram homotopy commute (where we use $|$ for the appropriate restriction):

$$\begin{tikzcd}
 & N_{i_1} \arrow[dl, "f_{n_1}|"'] \arrow[dr, "\Phi|"] \arrow[dd, "r"] & \\
H \cap W_1 & & H \cap W_1 \\
 & P_{i_1} \arrow[ul, "f_{n_1}|"] \arrow[ur, "\Phi|"'] &
\end{tikzcd}$$

That is $\Phi| \simeq (\Phi|P_{i_1}) \circ r$ (in $H \cap W_1$) and $f_{n_1}|N_{i_1} \simeq (f_{n_1}|P_{i_1}) \circ r$ (in $H \cap W_1$). We can now use this to prove that $f_{n_1}|N_{i_1} \simeq \Phi|$ (in $H \cap W_1$) as follows:

$$f_{n_1}|N_{i_1} \simeq f_{n_1} \circ r \ (\text{in } H \cap W_1) \simeq \Phi \circ r \ (\text{in } H \cap W_1) \simeq \Phi| \ (\text{in } H \cap W_1).$$

This will be needed below. Thus all that remains to be done is prove that $\Phi(X)$, Y have the same relative shape (in W_1).

In order to do this choose an integer $n_2 \geqslant n_1$ and a neighborhood $H_1' \subset H \cap W_1$ of Y such that $g_k|H_1' \simeq g_l|H_1'$ (in N_{i_1}), for all integers $k, l \geqslant n_2$. Using the fact that $\underline{f} \circ \underline{g} \simeq \underline{\mathrm{id}}_Y$ (in W) we can find an integer $n_3 \geqslant n_2$ and a neighborhood $H' \subset H_1'$ of Y such that $f_k \circ g_k|H' \simeq \mathrm{id}_{H'}$ (in H_1'), for all $k \geqslant n_3$. Put $G' = \Phi(N_{i_1})$ and for all $k \geqslant n_3$ define $f_k'\colon G' \to W_1$ by $f_k' = f_k \circ \Phi^{-1}|$ and define $g_k'\colon H' \to W_1$ by $g_k' = \Phi \circ g_k|H'$. For all $k \geqslant n_3$ put $\underline{f}' = \{f_k', \Phi(X), Y, G'\}$ and $\underline{g}' = \{g_k', Y, \Phi(X), H'\}$. We will prove that $\underline{f}', \underline{g}'$ are relative fundamental sequences (in W_1) which satisfy $\underline{g}' \circ \underline{f}' \simeq \underline{\mathrm{id}}_{\Phi(X)}$ (in W_1) and $\underline{f}' \circ \underline{g}' \simeq \underline{\mathrm{id}}_Y$ (in W_1).

To see that $\underline{f}'$ is a relative fundamental sequence (in W_1) we first note that

$$f_k' = f_k \circ \Phi^{-1}| \simeq \Phi \circ \Phi^{-1}| \text{ (in } W_1) = \mathrm{id}_{G'},$$

since $k \geqslant n_3$. Now choose a neighborhood $V \subset W_1$ of Y. Since $\underline{f}$ is a relative fundamental sequence (in W) there exists a neighborhood $U \subset N_{i_1}$ of X and an integer $n_4 \geqslant n_3$ such that $f_k|U \simeq f_l|U$ (in V), for all $k, l \geqslant n_4$. This obviously implies that $f_k'|\Phi(U) \simeq f_l'|\Phi(U)$ (in V), for all $k, l \geqslant n_4$. Thus $\underline{f}'$ is a relative fundamental sequence (in W_1).

To see that $\underline{g}'$ is a relative fundamental sequence (in W_1) we note that

$$g_k' = \Phi \circ g_k|H' \simeq f_{n_1} \circ g_k|H' \text{ (in } W_1) \simeq f_k \circ g_k|H' \text{ (in } W_1) \simeq \mathrm{id}_{H'} \text{ (in } W_1),$$

for all $k \geqslant n_3$. Now choose a neighborhood $U \subset G'$ of $\Phi(X)$. Then $\Phi^{-1}(U)$ is a neighborhood of X and there exists a neighborhood $V \subset H'$ and an integer $n_4 \geqslant n_3$ such that $g_k|V \simeq g_l|V$ (in $\Phi^{-1}(U)$), for all $k, l \geqslant n_4$. It is then clear that

$$g_k'|V = \Phi \circ g_k|V \simeq \Phi \circ g_l|V \text{ (in } U) = g_l'|V,$$

for all $k, l \geqslant n_4$. Thus $\underline{g}'$ is a relative fundamental sequence (in W_1).

To see that $\underline{f}' \circ \underline{g}' \simeq \underline{\mathrm{id}}_Y$ (in W_1) choose a neighborhood $V \subset H'$ of Y. Since $\underline{f} \circ \underline{g} \simeq \underline{\mathrm{id}}_Y$ (in W) we can find a neighborhood $V' \subset V$ of Y and an integer $n_4 \geqslant n_3$ such that $f_k \circ g_k|V' \simeq \mathrm{id}_{V'}$ (in V), for all $k \geqslant n_4$. Then we have

$$f_k' \circ g_k'|V' = (f_k \circ \Phi^{-1}) \circ (\Phi \circ g_k)|V' = f_k \circ g_k|V' \simeq \mathrm{id}_{V'} \text{ (in } V),$$

for all $k \geqslant n_4$. This implies that $\underline{f}' \circ \underline{g}' \simeq \underline{\mathrm{id}}_Y$ (in W_1).

To see that $\underline{g}' \circ \underline{f}' \simeq \underline{\mathrm{id}}_{\Phi(X)}$ (in W_1), choose a neighborhood $U \subset G'$ of $\Phi(X)$. Then $\Phi^{-1}(U)$ is a neighborhood of X and there exists an integer $n_4 \geqslant n_3$ and a neighborhood $U' \subset \Phi^{-1}(U)$ of X such that $g_k \circ f_k|U' \simeq \mathrm{id}_{U'}$ (in $\Phi^{-1}(U)$), for all $k \geqslant n_4$. Clearly $\Phi(U') \subset U$ is a neighborhood of $\Phi(X)$ and also

$$g_k' \circ f_k'|\Phi(U') = \Phi \circ g_k \circ f_k \circ \Phi^{-1}|\Phi(U')$$
$$\simeq \Phi \circ \mathrm{id}_{U'} \circ \Phi^{-1}|\Phi(U') \text{ (in } U) = \mathrm{id}_{\Phi(U')},$$

for all $k \geqslant n_4$. Thus $\underline{g}' \circ \underline{f}' \simeq \underline{\mathrm{id}}_{\Phi(X)}$ (in W_1) and we are done.

6. Proof of Theorem 1. We will first establish part (b) of Theorem 1. Let $X', Y' \subset E^{2m+1} \times 0 \subset E^n$ be copies of X, Y, respectively which are in standard position in $E^{2m+1} \times 0$ and let $h\colon E^n \backslash X' \to E^n \backslash Y'$ be a homeomorphism. We must prove that $\mathrm{Sh}(X') = \mathrm{Sh}(Y')$ (which implies that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$). Choose a number $t_1 > 0$ such that $X' \cup Y' \subset \mathrm{Int}(B_{t_1}^n)$. Then choose $t_2 \in (0, t_1)$ such that $\mathrm{Int}(B_{t_2}^{2m+1}) \times 0 \subset E^n$ contains $X' \cup Y'$. Since $h(\partial B_{t_1}^n)$ is bicollared in E^n we can use the Generalized Schoenflies Theorem of [6] to write

$$E^n \backslash h(\partial B_{t_1}^n) = A \cup B,$$

where A is the bounded component (which is homeomorphic to $\mathrm{Int}(B_{t_1}^n)$). The proof now splits into cases and we first treat the case in which $Y' \subset A$ and $h(\mathrm{Int}(B_{t_1}^n) \backslash X') = A \backslash Y'$. In this case it is clear that t_1 can be chosen large enough so that $B_{t_2}^{2m+1} \times 0 \subset A$.

Let $r\colon E^n \to B_{t_2}^{2m+1} \times 0$ be a retraction and define a homotopy $F\colon (B_{t_2}^{2m+1} \times 0) \times I \to E^n$ by

$$F_t(x_1, \ldots, x_{2m+1}, 0, 0, 0, \ldots) = (x_1, \ldots, x_{2m+1}, t, 0, 0, \ldots),$$

for all $(x_1, \ldots, x_{2m+1}, 0, 0, 0, \ldots) \in B_{t_2}^{2m+1} \times 0$ and $t \in I$. For each integer $k > 0$ let $f_k\colon B_{t_2}^{2m+1} \times 0 \to B_{t_2}^{2m+1} \times 0$ and $g_k\colon B_{t_2}^{2m+1} \times 0 \to B_{t_2}^{2m+1} \times 0$ be defined by

$$f_k = r \circ h \circ F_{1/k}, \quad g_k = r \circ h^{-1} \circ F_{1/k}.$$

We will show that $\underline{f} = \{f_k, X', Y'\}$ and $\underline{g} = \{g_k, Y', X'\}$ are fundamental sequences which satisfy $\underline{g} \circ \underline{f} \simeq \underline{\mathrm{id}}_{X'}$ and $\underline{f} \circ \underline{g} \simeq \underline{\mathrm{id}}_{Y'}$ (in $B_{t_2}^{2m+1} \times 0$), where $\underline{\mathrm{id}}_{X'}$ and $\underline{\mathrm{id}}_{Y'}$ are the identity fundamental sequences of X' and Y', respectively. Then from [5] it will follow that $\mathrm{Sh}(X') = \mathrm{Sh}(Y')$.

To see that $\underline{f}$ is a fundamental sequence let $V \subset B_{t_2}^{2m+1} \times 0$ be an open set containing Y'. Since $r^{-1}(V)$ is an open set containing Y' it follows that $h^{-1}(r^{-1}(V) \backslash Y') \cup X'$ is an open set in E^n containing X'. Thus there exists an open set $U \subset B_{t_2}^{2m+1} \times 0$ containing X' and a number $\varepsilon > 0$ such that $F(U \times [0, \varepsilon]) \subset h^{-1}(r^{-1}(V) \backslash Y') \cup X'$. This implies that $F_{1/k}|U \simeq F_{1/l}|U$ (in $h^{-1}(r^{-1}(V) \backslash Y')$), for all integers $k, l \geqslant 1/\varepsilon$. Since this homotopy takes place in the complement of X' we have $h \circ F_{1/k}|U \simeq h \circ F_{1/l}|U$ (in $r^{-1}(V) \backslash Y'$), for all $k, l \geqslant 1/\varepsilon$. Then applying r to this homotopy we have $r \circ h \circ F_{1/k}|U \simeq r \circ h \circ F_{1/l}|U$ (in V), which means precisely that $f_k|U \simeq f_l|U$

(in V), for all integers $k, l \geqslant 1/\varepsilon$. Thus f is a fundamental sequence. Similarly g is a fundamental sequence.

To see that $g \circ f \simeq \mathrm{id}_{X'}$ (in $B_{t_2}^{2m+1} \times 0$) let $U \subset B_{t_2}^{2m+1} \times 0$ be an open set containing X'. Since $h(r^{-1}(U)\backslash X') \cup Y'$ is an open set in E^n containing Y' we can find an open set $V \subset h(r^{-1}(U)\backslash X') \cup Y'$ containing Y' such that

$$F_{1/k} \circ r | V \simeq \mathrm{id}_V \text{ (in } h(r^{-1}(U)\backslash X') \cup Y'),$$

for almost all k. Since $X' \subset \mathrm{Int}(B_{t_2}^{2m+1}) \times 0$ we can clearly find a compact polyhedron $P \subset B_{t_2}^{2m+1}$ such that

$$X' \subset \mathrm{Int}(P) \times 0 \subset P \times 0 \subset h^{-1}(V \backslash Y') \cup X'.$$

Then $P \times 0$ is a closed neighborhood of X' (in $B_{t_2}^{2m+1} \times 0$) and we will prove that $g_k \circ f_k \simeq \mathrm{id}_{P\times 0}$ (in U), for almost all integers k. This will be sufficient to establish that $g \circ f \simeq \mathrm{id}_{X'}$.

Since $P \times 0 \subset h^{-1}(V\backslash Y') \cup X'$ we can choose $\varepsilon_1 > 0$ so that $F((P\times 0)\times[0, \varepsilon_1]) \subset h^{-1}(V\backslash Y') \cup X'$. Choose an integer $n_1 \geqslant 1/\varepsilon_1$ so that

$$F_{1/k} \circ r | V \simeq \mathrm{id}_V \text{ (in } h(r^{-1}(U)\backslash X') \cup Y'),$$

for all $k \geqslant n_1$. Note that $h \circ F_{1/k}(P\times 0) \subset V$, for all $k \geqslant n_1$. Thus for each integer $k \geqslant n_1$ we can define a homotopy $G\colon (P\times 0)\times I \to h(r^{-1}(U)\backslash X') \cup Y'$ such that $G_0 = h \circ F_{1/k} | P \times 0$ and $G_1 = F_{1/k} \circ r \circ h \circ F_{1/k} | P \times 0$. Note that $G_0(P\times 0) \cap Y' = \emptyset$ and $G_1(P \times 0) \cap Y' = 0$. Since $\dim(P\times I) + \dim Y' < 3m+3 \leqslant n$ we can use Lemma 3.5 to obtain a homotopy $H\colon (P\times 0)\times I \to h(r^{-1}(U)\backslash X')$ such that $H_0 = G_0$ and $H_1 = G_1$. Thus for all $k \geqslant n_1$ we have

$$\begin{aligned} g_k \circ f_k | P \times 0 &= r \circ h^{-1} \circ F_{1/k} \circ r \circ h \circ F_{1/k} | P \times 0 \\ &\simeq r \circ h^{-1} \circ h \circ F_{1/k} | P \times 0 \text{ (in } U) = r \circ F_{1/k} | P \times 0. \end{aligned}$$

All we need to do now is verify that $r \circ F_{1/k} | P \times 0 \simeq \mathrm{id}_{P\times 0}$ (in U), for almost all values of k. Note that $h^{-1}(V\backslash Y') \cup X' \subset r^{-1}(U)$. Since we had chosen $\varepsilon_1 > 0$ so that $F((P\times 0)\times[0, \varepsilon_1]) \subset h^{-1}(V\backslash Y') \cup X'$, we have $r \circ F((P\times 0)\times[0, \varepsilon_1]) \subset U$. But this implies that $r \circ F_{1/k} | P \times 0 \simeq \mathrm{id}_{P\times 0}$ (in U), for all $k \geqslant n_1$. This completes the proof that $g \circ f \simeq \mathrm{id}_{X'}$. Similary one can prove that $f \circ g \simeq \mathrm{id}_{Y'}$. Thus $\mathrm{Sh}(X') = \mathrm{Sh}(Y')$.

Now returning to cases assume that $Y' \subset A$ and $h(\mathrm{Int}(B_{t_1}^n)\backslash X') \neq A\backslash Y'$. Then we must have $h(\mathrm{Int}(B_{t_1}^n)\backslash X') = B$, the unbounded component of $E^n \backslash h(\partial B_{t_1}^n)$. But appealing to the Generalized Schoenflies Theorem of [6] this implies that X' is cellular. By [13] we have $\mathrm{Sh}(X') = \mathrm{Sh}(\{\text{point}\})$. Also we must have $h(E^n\backslash B_{t_1}^n) = A \backslash Y'$. Then Y' is also cellular and we have $\mathrm{Sh}(X') = \mathrm{Sh}(\{\text{point}\}) = \mathrm{Sh}(Y')$. Thus we have treated all cases in which $Y' \subset A$.

On the other hand let us now assume that $Y' \not\subset A$, hence $Y' \cap B \neq \emptyset$. Note that we have either $h(\mathrm{Int}(B_{t_1}^n)\backslash X') = B\backslash Y'$ or $h(E^n\backslash B_{t_1}^n) = B \backslash Y'$. If $h(E^n \backslash B_{t_1}^n) = B\backslash Y'$, then it follows that $B \cap Y' = \emptyset$, a contradiction. Thus we must have $h(\mathrm{Int}(B_{t_1}^n)\backslash X') = B\backslash Y'$, hence $h(B_{t_1}^n \backslash X') = \mathrm{Cl}(B)\backslash Y'$. Choose any $p \in A$ and let $u\colon \mathrm{Cl}(B) \to \mathrm{Cl}(A)\backslash\{p\}$ be a homeomorphism such that $u|\mathrm{Bd}(B) = \mathrm{id}$. Then $u \circ h | B_{t_1}^n\backslash X'\colon B_{t_1}^n \backslash X' \to \mathrm{Cl}(A)\backslash(\{p\} \cup u(Y' \cap B))$ is a homeomorphism and we can extend $u \circ h | \partial B_{t_1}^n = h | \partial B_{t_1}^n$ to a homeomorphism $u'\colon E^n \backslash \mathrm{Int}(B_{t_1}^n) \to \mathrm{Cl}(B)$. Define $h'\colon E^n\backslash X' \to E^n \backslash (\{p\} \cup u(Y' \cap B))$ by setting

$$h' | B_{t_1}^n \backslash X' = u \circ h | B_{t_1}^n \backslash X', \qquad h' | E^n \backslash B_{t_1}^n = u'.$$

Then h' is a homeomorphism and the first case that we treated above implies that $\mathrm{Sh}(X') = \mathrm{Sh}(\{p\} \cup u(Y' \cap B))$. Since $h(\mathrm{Int}(B_{t_1}^n)\backslash X') = B\backslash Y'$ it follows that $h(E^n \backslash \mathrm{Int}(B_{t_1}^n)) = \mathrm{Cl}(A)\backslash Y'$. This implies that $Y' \cap A$ is cellular, hence $\mathrm{Sh}(Y' \cap A) = \mathrm{Sh}(\{\text{point}\})$. Now decomposing Y' we get

$$\mathrm{Sh}(Y') = \mathrm{Sh}((Y' \cap A) \cup (Y' \cap B)) = \mathrm{Sh}(\{\text{point}\} \cup (Y' \cap B)) = \mathrm{Sh}(X'),$$

as we observed above. This completes the proof of (b) of Theorem 1.

For the proof of (a) of Theorem 1 we choose $X', Y' \subset E^n$ ($n \geqslant 2m+2$) to be copies of X, Y respectively which are in standard position. We will prove that $E^n\backslash X'$ and $E^n \backslash Y'$ are homeomorphic. The procedure will be to use Lemma 5.1 to inductively construct sequences $\{U_i\}_{i=1}^\infty$ and $\{V_i\}_{i=1}^\infty$ of open subsets of E^n and a sequence $\{h_i\}_{i=1}^\infty$ of homeomorphisms of E^n onto itself such that

(1) $X' = \bigcap_{i=1}^\infty U_i$ and $U_{i+1} \subset U_i$ for all $i > 0$,

(2) $Y' = \bigcap_{i=1}^\infty V_i$ and $V_{i+1} \subset V_i$ for all $i > 0$,

(3) $h_{2i-1} \circ \ldots \circ h_1(X') \subset V_i$ for all $i > 0$,

(4) $h_i | E^n \backslash V_j = \mathrm{id}$ for all $i > 2j-1$ and $j > 0$,

(5) $h_{2i} \circ \ldots \circ h_1(U_i) \supset Y'$ for all $i > 0$, and

(6) $h_i | E^n \backslash h_{2j} \circ \ldots \circ h_1(U_j) = \mathrm{id}$ for all $i > 2j$ and $j > 0$.

For the time being we assume that these sequences have been constructed and we show how to prove that $E^n\backslash X'$ and $E^n\backslash Y'$ are homeomorphic. Choose any $x \in E^n \backslash X'$ and consider the sequence $\{h_i \circ \ldots \circ h_1(x)\}_{i=1}^\infty$. If j is chosen large enough so that $x \notin U_j$, then it follows from (6) that

$$h_i \circ \ldots \circ h_1(x) = h_{2j} \circ \ldots \circ h_1(x),$$

for all $i \geqslant 2j$. From (5) it follows that $h_i \circ \ldots \circ h_1(x) \notin Y'$, for all $i \geqslant 2j$. Thus it makes sense to define a function $h\colon E^n \backslash X' \to E^n \backslash Y'$ by

$$h(x) = \lim_{i\to\infty} h_i \circ \ldots \circ h_1(x)\,.$$

Since each h_i is 1—1 it follows that h is 1—1. If $x \in E^n \backslash X'$ and $U \subset E^n \backslash X'$ is a compact neighborhood of x, then there exists an integer j so that $U \cap U_j = \emptyset$. From (6) it follows that $h(U) = h_{2j} \circ \ldots \circ h_1(U)$, hence $h(U)$ is a neighborhood of $h(x)$. This implies that h is open. To see that h is continuous and onto choose a compactum $V \subset E^n \backslash Y'$ and choose $j > 0$ such that $V \cap V_j = \emptyset$. It follows from (3) that $V \subset h_{2j-1} \circ \ldots \circ h_1(E^n \backslash X')$ and it follows from (4) that $V \subset h(E^n \backslash X')$ and $h^{-1}(V) = (h_{2j-1} \circ \ldots \circ h_1)^{-1}(V)$. Thus h is our desired homeomorphism.

We now turn to the construction of the necessary sequences. For each integer $k > 0$ consider the following statement.

P_k: There exist collections $\{U_i\}_{i=1}^k$ and $\{V_i\}_{i=1}^k$ of open subsets of E^n and a collection $\{h_i\}_{i=1}^{2k}$ of homeomorphisms of E^n onto itself such that $X' \subset U_i$ and $Y' \subset V_i$, for $1 \leqslant i \leqslant k$, and

(1) $U_{i+1} \subset U_i$, for $1 \leqslant i \leqslant k-1$, and $U_i \subset \{x \in E^n \,|\, d(X', x) < 1/i\}$, for $1 \leqslant i \leqslant k$,

(2) $V_{i+1} \subset V_i$, for $1 \leqslant i \leqslant k-1$, and $V_i \subset \{x \in E^n \,|\, d(Y', x) < 1/i\}$, for $1 \leqslant i \leqslant k$,

(3) $h_{2i-1} \circ \ldots \circ h_1(U_i) \subset V_i$ for $1 \leqslant i \leqslant k$,

(4) $h_i | E^n \backslash V_j = \mathrm{id}$ for $1 \leqslant j \leqslant k$ and $2j-1 < i \leqslant 2k$,

(5) $h_{2i} \circ \ldots \circ h_1(U_i) \supset V_{i+1}$, for $1 \leqslant i < k$, and $h_{2k} \circ \ldots \circ h_1(U_k) \supset Y'$,

(6) $h_i | E^n \backslash h_{2j} \circ \ldots \circ h_1(U_j) = \mathrm{id}$ for $1 \leqslant j < k$ and $2j < i \leqslant 2k$,

(7) $h_{2k} \circ \ldots \circ h_1(X')$ is in standard position, and

(8) $h_{2k} \circ \ldots \circ h_1(X')$, Y' have the same relative shape (in $h_{2k} \circ \ldots \circ h_1(U_k)$).

We will prove that P_k is true for all k. Moreover in the inductive step from P_k to P_{k+1} we will construct the necessary collections of open sets and homeomorphisms for P_{k+1} by adding appropriately constructed U_{k+1}, V_{k+1}, h_{2k+1}, and h_{2k+2} to the given collections $\{U_i\}_{i=1}^k$, $\{V_i\}_{i=1}^k$, and $\{h_i\}_{i=1}^{2k}$ for P_k. Once this is done we will be finished with the proof.

For $k = 1$ let

$$V_1 = \{x \in E^n |\ d(Y', x) < 1\}$$

and use the assumption that $\mathrm{Sh}(X') = \mathrm{Sh}(Y')$ to conclude that X' and Y' have the same relative shape (in E^n). Then Lemma 5.1 implies the existence of a PL homeomorphism $h_1\colon E^n \to E^n$ such that $h_1(X') \subset V_1$, $h_1(X')$ is in standard position, and $h_1(X')$, Y' have the same relative shape (in V_1). Then put

$$U_1 = h_1^{-1}(V_1) \cap \{x \in E^n |\ d(X', x) < 1\}$$

and once more use Lemma 5.1 to obtain a PL homeomorphism $g_2\colon E^n \to E^n$ such that $g_2(Y') \subset h_1(U_1)$, $g_2(Y')$ is in standard position, $g_2 | E^n \backslash V_1 = \mathrm{id}$, and $h_1(X')$, $g_2(Y')$ have the same relative shape (in $h_1(U_1)$). Then define $h_2 = g_2^{-1}$ and note that this implies that P_1 is true.

For the inductive step assume that we have collections $\{U_i\}_{i=1}^k$, $\{V_i\}_{i=1}^k$, and $\{h_i\}_{i=1}^{2k}$ for which P_k is true. Let

$$V_{k+1} = V_k \cap h_{2k} \circ \ldots \circ h_1(U_k) \cap \{x \in E^n |\ d(Y', x) < 1/(k+1)\}$$

and use (8) and Lemma 5.1 to get a PL homeomorphism $h_{2k+1}\colon E^n \to E^n$ such that $h_{2k+1} \circ \ldots \circ h_1(X') \subset V_{k+1}$, $h_{2k+1} \circ \ldots \circ h_1(X')$ is in standard position, $h_{2k+1} \circ \ldots \circ h_1(X')$, Y' have the same relative shape (in V_{k+1}), and $h_{2k+1} | E^n \backslash h_{2k} \circ \ldots \circ h_1(U_k) = \mathrm{id}$. Then put

$$U_{k+1} = U_k \cap (h_{2k+1} \circ \ldots \circ h_1)^{-1}(V_{k+1}) \cap \{x \in E^n |\ d(X', x) < 1/(k+1)\}$$

and once more (as in the construction of h_2) use Lemma 5.1 to construct a PL homeomorphism $h_{2k+2}\colon E^n \to E^n$ such that $h_{2k+2} | E^n \backslash V_{k+1} = \mathrm{id}$, $h_{2k+2} \circ \ldots \circ h_1(U_{k+1}) \supset Y'$, $h_{2k+2} \circ \ldots \circ h_1(X')$ is in standard position, and $h_{2k+2} \circ \ldots \circ h_1(X')$, Y' have the same relative shape (in $h_{2k+2} \circ \ldots \circ h_1(U_{k+1})$). This completes the inductive step and the proof of the theorem.

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