As $\mathfrak{h}$ is order-preserving, $\mathfrak{h}(\pi_{\mathfrak{F}}(t)) < q$ on $\mathfrak{F}$. By the definition of $A$ and the relation of $s$ to $t$, there is $t' \subseteq t$ and range $t' \cap y = \emptyset$, and such that $\mathfrak{h}(\pi_{\mathfrak{F}}(t')) \supseteq q$ on $\mathfrak{F}$. In particular, $(t, y), (t', y) \in \bar{X}(a)$ and $(t', y) \subseteq (t, y)$. But look; $(t, y) \in \mathcal{A}$, so by definition we must have that $\mathfrak{h}(\pi_{\mathfrak{F}}(t)) \supseteq q$ on $\mathfrak{F}$. Since $h \upharpoonright a = h_a$, this contradicts our earlier inequality. Hence $\mathcal{T} = \mathcal{T}(\mathfrak{F})$ cannot be $Q$-embeddable.

Suppose now that for some $F, G \in \mathcal{S}^\omega$, $F \neq G$, we have $\mathcal{T}(F) \cong \mathcal{T}(G)$. Pick $\alpha < \omega_1$ such that $F \upharpoonright \alpha \neq G \upharpoonright \alpha$. Let $A = \{ a \in a \upharpoonright \alpha \mid a = \bigcup a \upharpoonright \alpha \} \cap \{ \pi_{\mathfrak{F}}'' \mathcal{T}(\mathcal{G}) \}$. Pick $\alpha_0 < \omega_1$ such that $F \upharpoonright \alpha_0 \neq G \upharpoonright \alpha_0$. Let $A = \{ a \in a \upharpoonright \alpha_0 \mid a = \bigcup a \upharpoonright \alpha_0 \} \cap \{ \pi_{\mathfrak{F}}'' \mathcal{T}(\mathcal{G}) \}$.

Clearly, $A$ is closed and unbounded in $\omega_1$. By $\diamond$, there is $a \in A$ such that $h \upharpoonright a = h_a$. Thus Case II applied in constructing $\mathcal{T}(\mathcal{F})$ from $\mathcal{T}(\mathcal{F})$ and $\mathcal{T}(\mathcal{G})$ from $\mathcal{T}(\mathcal{G})$. This means that the map $\pi_{\mathfrak{F}}^{-1} \cdot h : \mathcal{T}(\mathcal{F}) \to \mathcal{T}(\mathcal{G})$ does not extend to an isomorphism of $\mathcal{T}(\mathcal{F})$ and $\mathcal{T}(\mathcal{G})$, which is absurd, since $\pi_{\mathfrak{F}}^{-1} \cdot h \circ \pi_{\mathfrak{F}}$ extends it. Thus $\mathcal{T}(\mathcal{F})$ and $\mathcal{T}(\mathcal{G})$ are not isomorphic. The proof is complete.

References


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Shapes of finite-dimensional compacta

by

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1. Introduction. The results of this paper deal with shapes of finite-dimensional compact metric spaces (see [4] for definitions concerning the concept of shape). In Theorem 1 below we give a characterization of shapes of finite-dimensional compact metric spaces (i.e., compacta) in terms of embeddings in Euclidean $n$-space $E^n$. In an earlier paper the author obtained a characterization of shapes of compacta (with no dimensional restriction) in terms of embeddings in the Hilbert cube $I^8$. In a sense the results obtained here are motivated by [3], and to some extent the general structure of the proof of Theorem 1 is a modification of the argument used in [3], but the present paper does not involve any infinite-dimension topology. For the sake of completeness we give a short summary of the infinite-dimensional characterization at the end of the Introduction. We use the notation $\text{Sh}(X) = \text{Sh}(Y)$ to indicate that compacta $X$ and $Y$ have the same shape.

**Theorem 1.** Let $X, Y$ be compacta such that $\dim X, \dim Y \leq m$.

(a) For any integer $n \geq 2m + 2$ there exist copies $X', Y' \subseteq E^n$ (of $X, Y$ respectively) such that $\text{Sh}(X') = \text{Sh}(Y')$, then $E^n \setminus X'$ and $E^n \setminus Y'$ are homeomorphic.

(b) For any integer $n \geq 3m + 3$ there exist copies $X', Y' \subseteq E^n$ (of $X, Y$ respectively) such that $E^n \setminus X'$ and $E^n \setminus Y'$ are homeomorphic, then $\text{Sh}(X') = \text{Sh}(Y')$.

We remark that a similar result holds for embeddings of $X$ and $Y$ in the $n$-sphere $S^n$.

For prerequisites we will need some elementary facts concerning the piecewise-linear topology of $E^n$ plus an isotopy extension theorem from [11]. We also use a characterization of dimension in terms of mappings onto polyhedra in $E^n$ (see [14], p. 111). As for techniques we remark that part (a) of Theorem 1 is the most difficult part of the proof. Roughly the idea is to construct a sequence $(h_i)_{i=1}^\infty$ of homeomorphisms

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of $E^n$ onto itself such that the sequence $\{h_1 \circ h_{i-1} \circ \ldots \circ h_2 \circ h_1\}$ of left products converges pointwise (on $E^n \setminus X$) to define a homeomorphism of $E^n \setminus X$ onto $E^n \setminus Y$. This is the idea that was used in [8].

We now make some comments concerning the infinite-dimensional characterization obtained in [8]. We represent the Hilbert cube $Q$ by $igcap_{i=1}^\infty I_i$, where each $I_i$ is the closed interval $[0, 1]$, and the pseudo-interior of $Q$ is $s = \bigcap_{i=1}^\infty I'_i$, where each $I'_i$ is the open interval $(0, 1)$. The characterization obtained in [8] is as follows:

**Theorem 2.** Let $X, Y \subseteq s$ be compacta. Then $Sh(X) = Sh(Y)$ iff $Q \setminus X$ and $Q \setminus Y$ are homeomorphic.

The condition "$X, Y \subseteq s$" in Theorem 2 is crucial and in general cannot be replaced by the weaker condition "$X, Y \subseteq Q". Also it follows from [2] that if $X, Y \subseteq s$ are any two compacta, then $s \setminus X$ and $s \setminus Y$ are homeomorphic to $s$. However the characterization is generally applicable to compacta, since any compactum can be embedded in $s$. We remark that the proof of Theorem 2 given in [8] is non-elementary and uses some recent developments in the theory of infinite-dimensional manifolds modeled on $Q$ (see [7] for a summary). The proof we give here of Theorem 2 is a bit more complicated since there are some infinite-dimensional techniques used in the proof of Theorem 2 which have no finite-dimensional analogues.

The author is grateful to Morton Brown for suggesting something on the order of Theorem 1, in the sense that he felt "shaped" for finite-dimensional compacta should mean "homeomorphic complements" (in an appropriate setting in Euclidean space). The author also wishes to thank R. D. Anderson for making some valuable comments on the manuscript.

2. Definitions and notation. For any topological space $X$ and any set $A \subseteq X$ we let $Bd(x, A), Int_A(x)$, and $Cl_A(x)$ denote, respectively, the topological boundary, interior, and closure of $A$ in $X$. When no ambiguity results we will suppress the subscript $X$. If $X$ is another space and $f : X \to Y$ is a function, then $f \mid A$ will denote the restriction of $f$ to $A$.

All homeomorphisms will be onto and we use the notation $X \cong Y$ to indicate that spaces $X$ and $Y$ are homeomorphic. The identity homeomorphism of $X$ onto itself will be denoted by $id_X$ and by a map we will mean a continuous function. If $(Y, d)$ is a metric space and $f, g : X \to Y$ are maps, then we use

$$d(f, g) = \sup \{d(f(x), g(x)) \mid x \in X\} \quad (\text{if it exists})$$

for the distance between $f$ and $g$. In the sequel we will indiscriminately use $d$ to denote the metric of any space under consideration.

For products $X \times Y$ we use $p_X : X \times Y \to X$ to denote projection. In Euclidean space $E^n$ and any integer $m \leq n$ we use $p_m : E^n \to E^m$ to denote projection onto the first $m$ coordinates, i.e.

$$p_m(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_m), \quad \text{for all } (a_1, a_2, \ldots, a_n) \in E^n.$$

We use $I$ to denote the unit interval $[0, 1]$ and by a homotopy we mean a continuous function $F : X \times I \to Y$. The leftee of $F$ are the maps $F_t : X \to Y$, defined by $F_t(x) = F(x, t)$, for all $(x, t) \in X \times I$. For a map $G : X \times I \to Y \times I$ we will also use the notation $G_t : X \to Y$ to denote the map defined by $G_t(x) = p_X \circ G(x, t)$, for all $(x, t) \in X \times I$. If $B \subseteq Y$ and $f, g : X \to Y$ are maps satisfying $f(X), g(X) \subseteq B$, then we use the notation $f \simeq g$ (in $B$) to mean that there exists a homotopy $F : X \times I \to Y$ such that $F_0 = f, F_1 = g$, and $F(X \times I) \subseteq B$.

In Euclidean space $E^n$ and any $\varepsilon > 0$ we let

$$E^n_\varepsilon = \{x \in E^n \mid \|x\| \leq \varepsilon\},$$

$$\varepsilon B^n = \{x \in E^n \mid \|x\| = \varepsilon\}.$$

For any integer $m \leq n$ we will use $E^m \times 0 \subseteq E^n$ to indicate the Euclidean subspace of $E^n$ defined by

$$E^m \times 0 = \{(x_1, x_2, \ldots, x_m) \in E^n \mid x_{m+1} = x_{m+2} = \ldots = x_n = 0\}. \quad \text{For example,}\quad E^3 \times 0 \subseteq E^4.$$

By a polyhedron we will mean a (locally-finite) union of linear cells contained in some Euclidean space $E^n$ and by a topological polyhedron we will mean any space homeomorphic to a polyhedron. Generally we will use notation and results from [16] concerning elementary piecewise linear (PL) topology, including such concepts as PL maps, derived and regular neighborhoods, etc.

If $X$ is a polyhedron and $Y$ is a PL manifold (i.e. a polyhedron which is a manifold), a concordance $F$ of $X$ in $Y$ is a PL embedding $F : X \times I \to Y \times I$ such that $F(X \times 0) \subseteq Y \times 0$ and $F(X \times I) \subseteq Y \times I$. The concordance $F$ is allowable if $F^{-1}(Y \times 0) = X \times 0$, $F^{-1}(Y \times I) = X \times I$, and $F^{-1}(\partial Y \times I) = X \times I$, $X_0$ being a closed subpolyhedron of $X$. The following result will be needed in the proof of Theorem 1.

**Lemma 2.1 (Hudson [11]).** Let $X$ be a compact polyhedron, $Y$ be a PL manifold, and let $F : X \times I \to Y \times I$ be an allowable concordance which satisfies $F(x) \cap (\partial Y) = \emptyset$, for all $t \in I$. If $\dim X \leq \dim Y - 3$, then there exists a PL homeomorphism $h : Y \to Y$ which satisfies $h \circ F_3 = F_3$ and $h|\partial Y = id$.

As an easy consequence of Lemma 2.1 we get the following corollary (which will be more immediately useful to us).
COROLLARY 2.2. Let $X$ be a compact polyhedron such that $\dim X \leq n$, $Y$ be an open subset of $E^{n+2}$, and let $f, g: X \to Y$ be PL embeddings such that $f \simeq g$ (in $Y$). Then there exists a PL homeomorphism $h: E^{n+3} \to E^{n+3}$ such that $h|E^{n+3} \cap Y = \text{id}$ and $h \circ f = g$.

Proof. For $n = 0$ the result is trivial and we therefore assume $n \geq 1$, in which case $\dim Y < \dim X$. Let $F: X \times I \to Y$ be a map such that $F_0 = f$ and $F_1 = g$. Using Lemma 4.2 of [10] (which is concerned with approximating maps by PL maps) we can replace $F$ by a PL map $G: X \times I \to Y$ such that $G_0 = f$ and $G_1 = g$. If we note that $2\dim (X \times I) + 1 \leq \dim (Y \times I)$, then we can use standard procedures for general position to modify $G$ to obtain an allowable concordance $H: X \times I \to Y \times I$ such that $H_0 = f$ and $H_1 = g$. Let $P \subset Y$ be a compact polyhedron such that $H(P) \subset Y$, for all $t \in I$, and let $N \subset Y$ be a regular neighborhood of $P$. Then $H: X \times I \to D_0 \times I \times \{0\} \subset D_0 \times \{0\}$, for all $t \in I$. (Here the combinatorial boundary of $N$ coincides with the topological boundary of $N$.) Thus Lemma 2.1 implies that there exists a PL homeomorphism $h': N \to Y\setminus P$ such that $h' \circ f = g$ and $h'|N = \text{id}$. Extend $h'$ to a PL homeomorphism $h: E^{n+3} \to E^{n+3}$ by defining $h|E^{n+3} \setminus N = \text{id}$.

We will also need the following result on regular neighborhoods which follows from Theorem 2.1 on page 65 of [10].

LEMMA 2.3. Let $X$ be a compact polyhedron in the interior of a PL manifold $Y$ and let $N_1, N_2$ be two regular neighborhoods of $X$ in the interior of $Y$. If $U \subset Y$ is an open set containing $N_1 \cup N_2$, then there exists a PL homeomorphism $h: Y \to Y$ such that $h|X \cup (Y \setminus U) = \text{id}$ and $h|N_1 = N_1$.

3. Embedding compacta in $E^n$. If $X$ is any compactum satisfying $\dim X \leq n$, then it is well-known that $X$ can be embedded in $E^{n+1}$. In Proposition 3.4 below we prove that $X$ can be embedded into $E^{n+1}$ in a "nice" way which will be useful in the sequel. This "niceness" condition is described in the following definition.

DEFINITION 3.1. Let $X \subset E^n$ be a compactum which satisfies $\dim X \leq n$, for some $m > 0$. Then we say that $X$ is in standard position if there exist sequences $\{P_i\}_{i=1}^m$ and $\{N_i\}_{i=1}^m$ such that the following properties are satisfied:

1. each $P_i$ is a compact polyhedron in $E^n$ satisfying $\dim P_i \leq m$;
2. each $N_i$ is a regular neighborhood of $P_i$ in $E^n$;
3. each $N_i \cap \overline{Cint}(N_i) = \text{id}$;
4. $X = \bigcap_{i=1}^m N_i$.

We remark that this condition does not necessarily imply tameness. For example if $X \subset E^n$ is the wild arc of Artin–Fox (as described on page 177 of [9]), then it is easily verified that $X$ is in standard position. On the other hand if $X \subset E^n$ ($n \geq 4$) is any arc such that $E^n \setminus X$ is not simply connected (such arcs exist from [9]), then it can be verified that $X$ is not in standard position. We omit the details because these observations are not needed in the sequel. One obvious fact which will be needed in the sequel is the following: If $X \subset E^n$ is a compactum in standard position, then $X \times 0 \subset E^{n+1}$ is in standard position, for all $m > 0$.

In Proposition 3.4 below we show that every compactum of dimension less than or equal to $n$ can be embedded into $E^{n+1}$ in standard position. The following characterization of dimension will be needed in the proof of Proposition 3.4.

LEMMA 3.2. ([14, p. 111]. A compactum $X \subset E^n$ satisfies dim $X \leq m$ if and only if there exists a polyhedron $P \subset E^n$ satisfying dim $P < m$ and a map $f: X \to P$ such that $f(X) = P$ and $d(f, \text{id}) < \epsilon$.

We will also need a convergence procedure for sequences of embeddings of compacta into complete metric spaces. Various forms of this type of convergence procedure are known and have been used occasionally (for example see Lemma 2.1 of [1]). It is for this reason that we state the result with no proof. For notation let $(X, d)$ be a metric space and let $X \subset Y$ be a compactum. Then for any embedding $f: X \to Y$ and any $\delta > 0$ let

$$(f, \delta) = \text{glb} \{d(f(x), f(y)) \leq \delta \} \quad x, y \in X \quad, d(x, y) \geq \delta,$$

which is clearly a positive number. (Here glb means greatest lower bound.)

LEMMA 3.3. Let $(Y, d)$ be a complete metric space and let $X \subset Y$ be a compactum. Moreover let

$$(f_i, \delta_i) \in \text{glb} \{d(f_i(x), f_i(y)) \leq \delta_i \} \quad x, y \in X \quad, d(x, y) \geq \delta_i,$$

be a sequence of embeddings such that

$d(f_i, \text{id}) < \min(3^{-i}, (3^{-i})^{\epsilon}, (f_{i-1} \circ \cdots \circ f_1)^{-1}(2^{-i})),\quad i > 1.$

Then the sequence $(f_i)_{i=1}^\infty$ converges pointwise to an embedding of $X$ into $Y$.

PROPOSITION 3.4. Let $X \subset E^{n+1}$ be a compactum such that $\dim X \leq n$. Then there exists an embedding $f: X \to E^{n+1}$ such that $f(X)$ is in standard position.

Proof. We will apply Lemma 3.3 with $Y = E^{n+1}$. To do this we will inductively construct sequences $\{P_i\}_{i=1}^\infty$, $\{N_i\}_{i=1}^\infty$, and $\{f_i\}_{i=1}^\infty$ which satisfy

1. each $P_i$ is a compact polyhedron in $E^{n+1}$ such that $\dim P_i \leq n$;
2. each $N_i$ is a regular neighborhood of $P_i$ in $E^{n+1}$ such that $N_{i+1} \subset \text{Int}(N_i)$.

$X = \bigcap_{i=1}^\infty N_i$. 

We conclude by noting that the embedding $f$ is in standard position.
(3) \( \{f_i\}_{i=1}^{\infty} \) is a sequence of embeddings:
\[ X \xrightarrow{h} \text{Int}(N_1), f_1(X) \xrightarrow{h} \text{Int}(N_2), f_1 \circ f_2(X) \xrightarrow{h} \text{Int}(N_3), \ldots \]

(4) \( d(f_i, \text{id}) < \min(\alpha_i, 3^{-i} \cdot \epsilon(f_{i+1}, \ldots, f_1, 2^{-i})) \) for all \( i > 1 \).

(5) If \( \delta_1 = d(f_1, \ldots, f_i(X), E^{n+1}\setminus\text{Int}(N_1)) \), then \( d(f_i, \text{id}) < \delta_i / 2 \)
for all \( i, j > 0 \), and

(6) \( d(f_i, \ldots, f_1(X), x) < 1/2^{i-1} \) for all \( i > 0 \) and \( x \in N_i \).

To start the induction we now construct \( P_1, N_1 \), and \( f_1 \). Using Lemma 3.2 there exists a polyhedron \( P_1 \subset E^{n+1} \) satisfying \( \dim P_1 \leq n \) and a map \( g_1 \): \( X \to P_1 \) such that \( g_1(X) = P_1 \) and \( d(g_1(x), \text{id}) < 1/2 \). Choose a regular neighborhood \( N_1 \) of \( P_1 \) in \( E^{n+1} \) such that there exists a retraction \( r_1 \): \( N_1 \to P_1 \) satisfying \( d(r_1, \text{id}) < 1/2 \). It is well-known that any continuous function of \( X \) into \( E^{n+1} \) can be approximated by an embedding. Thus there exists an embedding \( f_1 \): \( X \to \text{Int}(N_1) \) such that \( d(f_1, \text{id}) < 1/2 \). This implies that \( d(f_i(X), x) < 1 \) for all \( x \in N_i \). This completes the construction of \( P_1, N_1 \), and \( f_1 \).

For the inductive step let us now assume that \( \{P_i\}_{i=1}^{n}, \{N_i\}_{i=1}^{n}, \) and \( \{f_i\}_{i=1}^{n} \) have been constructed so that conditions (1)-(6) are satisfied. We will construct \( P_{n+1}, N_{n+1} \), and \( f_{n+1} \) so that \( \{P_i\}_{i=1}^{n+1}, \{N_i\}_{i=1}^{n+1}, \) and \( \{f_i\}_{i=1}^{n+1} \) satisfy conditions (1)-(6). To simplify notation let
\[ \epsilon_i = \min(\alpha_i, 3^{-i} \cdot \epsilon(f_{i+1}, \ldots, f_1, 2^{-i})) \]
for \( 2 \leq i \leq n+1 \).

Using Lemma 3.2 there exists a polyhedron \( P_{n+1} \subset E^{n+1} \) satisfying \( \dim P_{n+1} \leq n \) and a map \( g_{n+1} \): \( f_{n+1} \ldots f_1(X) \to P_{n+1} \) such that \( g_{n+1} \circ f_{n+1} \circ \ldots \circ f_1(X) = P_{n+1} \), and
\[ d(g_{n+1}, \text{id}) < \min(\epsilon_{n+1} / 2, \delta_{n+1} / 2^{n+1}, \delta_{n+1} / 2^{n+2}, \ldots, \delta_{n} / 2^{n+1}) \]
Since \( d(g_{n+1}, \text{id}) < \delta_n \) we have \( P_{n+1} \subset \text{Int}(N_n) \). Thus we can choose a regular neighborhood \( N_{n+1} \) of \( P_{n+1} \) in \( E^{n+1} \) such that \( N_{n+1} \subset \text{Int}(N_n) \) and for which there exists a retraction \( r_{n+1} \): \( N_{n+1} \to P_{n+1} \) satisfying \( d(r_{n+1}, \text{id}) < 1/2^{n+1} \). Now let \( f_{n+1} \): \( f_{n+1} \ldots f_1(X) \to \text{Int}(N_{n+1}) \) be an embedding satisfying
\[ d(f_{n+1}, g_{n+1}) < \min(\epsilon_{n+1} / 2, \delta_{n+1} / 2^{n+1}, \delta_{n+1} / 2^{n+2}, \ldots, \delta_{n} / 2^{n+1}) \]
It then follows that
\[ d(f_{n+1}, \text{id}) < \min(\epsilon_{n+1}, \delta_{n+1} / 2, \delta_{n+1} / 2^{n+1}, \delta_{n+1} / 2^{n+2}, \ldots, \delta_{n} / 2^{n+1}) \]
If \( x \in N_{n+1} \), then \( d(g_{n+1} \circ f_{n+1} \ldots f_1(X), x) = d(P_{n+1}, x) < 1/2^{n+1} \). Since \( d(f_{n+1} \circ g_{n+1}, \text{id}) < \epsilon_{n+1} / 2 < 1/2^{n+1} \), it follows that \( d(f_{n+1} \ldots f_1(X), x) < 1/2^{n} \).
Thus \( \{P_i\}_{i=1}^{n+1}, \{N_i\}_{i=1}^{n+1}, \) and \( \{f_i\}_{i=1}^{n+1} \) satisfy properties (1)-(6). Thus we have inductively constructed the desired sequences.

It follows from (4) and Lemma 3.3 that the sequence \( \{f_i\}_{i=1}^{\infty} \) converges to an embedding \( f : X \to E^{n+1} \). From (3) and (6) we have \( f(X) \subset \text{Int}(N_i) \) for all \( i > 0 \). Thus all we need to do is show that \( f \cap N_i = f(X) \). To see this, choose any \( x \in N_i \) and use (6) to conclude that \( d(f_i \ldots f_1(X), x) < 1/2^{i-1} \) for all \( i > 0 \). Since we have \( d(f_i, \text{id}) < \epsilon_i \), for all \( i > 1 \), it then follows that \( x \in f(X) \).

The following result will be useful in the proof of part (b) of Theorem 1.

**Lemma 3.5.** Let \( Y \subset E^n \) be a compactum in standard position such that \( n \geq 2 \dim Y + 1 \) and let \( X \subset E^n \) be a compactum such that \( \dim X + \dim Y < n \). Also let \( U \subset E^n \) be an open set containing \( Y, A \subset X \) be closed, and let \( f : X \to U \) be a map such that \( f(A) \cap X = \emptyset \). Then there exists a map \( g : X \to U \) such that \( g(X) \cap Y = \emptyset \) and \( g \cap A = f(A) \).

**Proof.** Since \( X \) is in standard position there exists a compact polyhedron \( P \subset U \) such that \( \dim P \leq \dim X \) and a regular neighborhood \( N \) of \( P \) such that \( \chi(N) \subset U \setminus f(A) \). There are standard techniques for approximating maps by maps into polyhedra (for example see pp. 69, 70 of [12]). Since \( \dim(X \setminus A) \leq \dim X \) we can use these techniques to find a map \( f' : X \to U \) such that \( f'(X \setminus A) \subset X' \). Let \( P' \) be the intersection of this locally compact polyhedron with \( N \). Then we have \( f'(X \setminus N) \subset P' \) and \( \dim P' < \dim X \). Lemma 4 on page 97 of [10] implies that there exists a PL homeomorphism \( h : N \to N \) such that \( h(Bd(N)) = \emptyset \) and \( h(Bd(N \setminus A)) \subset h(Bd(N)) \) is in general position with respect to \( P \), i.e.
\[ \dim h(Bd(N \setminus A)) \cap P' \leq \dim P' + \dim A - n . \]
But \( \dim P' + \dim A < n \), hence \( \dim h(Bd(N \setminus A)) \cap P' \leq -1 \), which implies that \( h(Bd(N \setminus A)) \cap P = \emptyset \). Extend \( h \) to a homeomorphism \( h : E^n \to E^n \) so that \( h(E^n \setminus A) = \emptyset \). Thus \( h \circ f' : X \to E^n \) is a map satisfying \( h \circ f'(X) \cap A = \emptyset \). Now let \( N \subset \text{Int}(N) \) be a regular neighborhood of \( P \) such that \( N \cap f'(X) = \emptyset \) and let \( N \subset \text{Int}(N) \) be a regular neighborhood of \( P \) such that \( \chi(N) \subset N \). Using Lemma 2.3 there exists a homeomorphism \( h' : E^n \to E^n \) such that \( h'(N) = N \) and \( h'(E^n \setminus N) = \emptyset \). Then \( g = h' \circ h \circ f' : X \to U \) fulfill our requirements.

4. Relative shape. In this section we define a relative notion of shape which will be needed in the proof of Theorem 1. This apparatus was also employed in [8] to prove Theorem 2 as cited at the end of our Introduction.

Consider compacta \( X, Y \) contained in a space \( W \) and let \( C \subset W \) be a neighborhood of \( X \). Let \( \{f_i\}_{i=1}^{\infty} \) be a sequence of maps \( f_i : G \to W \) such that...
(1) each $f_k$ is homotopic to the inclusion of $G$ in $W$ (we will incorrectly write this as $f_k \simeq \text{id}_G$).

(2) for each neighborhood $V \subset W$ of $Y$ there exists a neighborhood $U \subset W$ of $X$ such that $f_k: U \to f_k(U)$ (in $V$) for almost all integers $k$ and $l$. Then we define $f_k = (f_k, X, Y) \in (W)$ and write $f = (f_1, X, Y, G)$.

We will agree to identify relative fundamental sequences $f = (f_k, X, Y, G)$ and $g = (g_k, X, Y, H)$ provided that there exists a neighborhood $U \subset G \cap H$ of $X$ such that $f_k: U \to g_k(U)$, for almost all $k$.

Now choose compacta $X, Y, Z$ in a space $W$ and relative fundamental sequences $f = (f_k, X, Y, G)$ and $g = (g_k, X, Y, H)$ (in $W$). It is clear that there exists a neighborhood $G \subset C$ of $X$ and an integer $k_1 > 0$ large enough so that

$$g \cdot f = (g_k \cdot f_k, G_1, X, Z, G_2) \quad (k \geq k_1)$$

is a relative fundamental sequence (in $W$). Because of the agreement made above on the identification of relative fundamental sequences it follows that the composition $g \cdot f$ is well-defined.

If $X, Y$ are compact in $W$ and $f = (f_k, X, Y)$ is a fundamental sequence (in $W$), then $(f_k, X, Y, G)$ uniquely defines a relative fundamental sequence (in $W$), for any neighborhood $G$ of $X$. We also see that if $X \subset W$ is a compactum and $G$ is any neighborhood of $X$, then $(f_k, X, Y, G)$ uniquely defines a relative fundamental sequence (in $W$).

We denote this sequence by $g_k f$ (when no ambiguity results) and call it the "identity relative fundamental sequence" from $X$ to $X$.

If $X, Y$ are compact in $W$ and $f = (f_k, X, Y, G), g = (g_k, X, Y, H)$ are relative fundamental sequences (in $W$), then we write $f \simeq g$ if for each neighborhood $V \subset W$ of $Y$ there exists a neighborhood $U \subset G \cap H$ of $X$ such that

$$f_k: U \simeq g_k: U \quad (k \geq k_1)$$

for almost all integers $k$.

Now let $X, Y$ be compact in $W$ and assume that there exist relative fundamental sequences $f = (f_k, X, Y, G)$ and $g = (g_k, X, Y, H)$ (in $W$) such that $g \cdot f \simeq g \cdot f_k$ and $f \cdot g \simeq f_k \cdot g_k$. Then we say that $X$ and $Y$ have the same relative shape (in $W$).

We emphasize the fact that the notion of relative shape depends on $W$ and the positioning of $X$ and $Y$ in $W$.

5. The main lemma. In Lemma 5.1 below we establish what amounts to the inductive step in the proof of part (b) of Theorem 1. This is the only place that it becomes necessary to get deeply involved with the apparatus of Section 4.

**Lemma 5.1.** For any integer $n \geq 0$ let $W \subset \mathbb{E}^{n+2}$ be an open set and let $X, Y \subset W$ be compacta such that $X$ is in standard position, dim $X \leq n$, and $X, Y$ have the same relative shape (in $W$). If $W \subset W$ is any neighborhood of $Y$, then there exists a $PL$ homeomorphism $\Phi: \mathbb{E}^{n+2} \to \mathbb{E}^{n+2}$ such that $\Phi(X)$ is in standard position, $\Phi|\mathbb{E}^{n+2} \cap W = \text{id}$, and $\Phi(X)$ have the same relative shape (in $W$).

**Proof.** Since $X$ is in standard position we can find sequences $(P_k)_{k=1}^\infty$ and $(N_k)_{k=1}^\infty$ which satisfy properties (1)–(4) of Definition 3.1. Choose neighborhoods $\delta W \subset W$ of $X$, $H \subset W$ of $Y$, and relative fundamental sequences $f = (f_k, X, Y, G)$ and $g = (g_k, X, Y, H)$ (in $W$) such that $g \cdot f \simeq g_k \cdot f_k$ and $f \cdot g \simeq f_k \cdot g_k$ (in $W$).

Now choose an integer $n_0 > 0$ and an integer $i_0 > 0$ such that $N_{n_0} \subset \delta W$ and

$$f_k|N_{n_0} \simeq f_k|N_{n_0} \quad (k \simeq k_1)$$

for all integers $k_1, l \geq m$. Since $f_{n_0}|P_{n_0}: P_{n_0} \to H \cap W_1$ is a map and dim $P_{n_0} \leq n$, we can find a PL embedding $\varphi: P_{n_0} \to \mathbb{E}^{n+2}$ which is as close to $f_{n_0}|P_{n_0}$ as we like. We can therefore choose $\varphi$ close enough to $f_{n_0}|P_{n_0}$ so that $\varphi|P_{n_0} \subset H \subset W_1$ and $\varphi \simeq f_{n_0}|P_{n_0}$ (in $H \cap W_1$) (for example we can use the straight-line homotopy joining $\varphi$ to $f_{n_0}|P_{n_0}$). Using Corollary 2.3 we can extend $\varphi$ to a PL homeomorphism $\Phi: \mathbb{E}^{n+2} \to \mathbb{E}^{n+2}$ satisfying $\Phi|\mathbb{E}^{n+2} \cap W = \text{id}$.

Since $P_{n_0} \subset W_1$ is a neighborhood of $P_{n_0}$, we can find a regular neighborhood $N_{n_0} \subset P_{n_0}$ such that $N_{n_0} \cap \Phi^{-1}(H \cap W_1)$. Using Lemma 2.3 it follows that there exists a PL homeomorphism $\Phi: \mathbb{E}^{n+2} \to \mathbb{E}^{n+2}$ satisfying $\Phi|\mathbb{E}^{n+2} \cap W = \text{id}$ and $\Phi|P_{n_0} \simeq f_{n_0}|P_{n_0}$ (in $H \cap W$). Also it follows that $\Phi(X)$ is in standard position, since $\Phi$ is PL.

Since $\Phi$ is a regular neighborhood of $P_{n_0}$ there exists a retraction $r: N_{n_0} \to P_{n_0}$ such that $r \simeq \text{id}$ (in $N_{n_0}$). Thus the two smaller triangles in the following diagram commute (where we use $|$ for the appropriate restriction):
This will be needed below. Thus all that remains to be done is prove that \( \Phi(X) \) and \( \Phi(Y) \) have the same relative shape (in \( W_k \)).

In order to do this choose an integer \( n_k \geq n_0 \) and a neighborhood \( H' \subset H \cap W_k \) of \( X \) such that \( g_k \mid H' \simeq g_1 \mid H' \) (in \( N_k \)), for all integers \( k, l \geq n_k \). Using the fact that \( f \circ g \simeq id_Y \) (in \( W_k \)) we can find an integer \( n_k \geq n_0 \) and a neighborhood \( H' \subset H \) of \( Y \) such that \( f_k \circ g_k \mid H' \simeq id_H \) (in \( H' \)), for all \( k \geq n_k \). Put \( G = \Phi(X) \) (in \( X \)) and for all \( k \geq n_k \) define \( f_k : G \rightarrow W_k \) by \( f_k = f_k \circ g^{-1} \) and define \( g_k : H' \rightarrow W_k \) by \( g_k = \Phi \circ g_k \mid H' \). For all \( k \geq n_k \) put \( f' = (f_k \circ \Phi(X), Y, G') \) and \( g' = (g_k \circ \Phi(X), H') \). We will prove that \( f' \simeq g' \) are fundamental sequences (in \( W_k \)) which satisfy \( g' \circ f' \simeq id_{\varepsilon_{\alpha_k}} \) (in \( W_k \)) and \( f' \simeq g' \circ id_{\varepsilon_{\alpha_k}} \) (in \( W_k \)).

To see that \( f' \) is a relative fundamental sequence (in \( W_k \)) we first note that
\[
f'_k = f_k \circ \Phi^{-1} \simeq \Phi \circ \Phi^{-1} \simeq id_{\varepsilon_{\alpha_k}} \quad \text{(in } W_k) \]
\[
\text{since } k \geq n_k. \text{ Now choose a neighborhood } V \subset W_k \text{ of } Y. \text{ Since } f \text{ is a relative fundamental sequence (in } W_k \text{) there exists a neighborhood } U \subset X_k \text{ of } X \text{ and an integer } n_k \geq n_k \text{ such that } f_k \mid U \simeq f_k \mid V \text{ (in } W_k) \text{, for all } k, l \geq n_k. \text{ This obviously implies that } f'_k \mid \Phi(U) \simeq f'_k \mid \Phi(U) \text{ (in } W_k) \text{, for all } k, l \geq n_k. \text{ Thus } f' \text{ is a relative fundamental sequence (in } W_k \text{).}
\]

To see that \( g' \) is a relative fundamental sequence (in \( W_k \)) we note that
\[
g'_k = \Phi \circ g_k \mid H' \simeq f_k \circ g_k \mid H' \text{ (in } W_k) \simeq f_k \circ g_k \mid H' \text{ (in } W_k) \simeq id_{\varepsilon_{\alpha_k}} \text{ (in } W_k),
\]
\[
\text{for all } k \geq n_k. \text{ Now choose a neighborhood } U \subset G' \text{ of } \Phi(X). \text{ Then } \Phi^{-1}(U) \text{ is a neighborhood of } X \text{ and there exists a neighborhood } V \subset H' \text{ and an integer } n_k \geq n_k \text{ such that } g_k \mid V \simeq g_k \mid V \text{ (in } \Phi^{-1}(U)), \text{ for all } k, l \geq n_k. \text{ It is then clear that}
\]
\[
g'_k \mid V = \Phi \circ g_k \mid V \simeq \Phi \circ g_k \mid V \text{ (in } U) = g'_k \mid V,
\]
\[
\text{for all } k, l \geq n_k. \text{ Thus } g' \text{ is a relative fundamental sequence (in } W_k). \text{ To see that } f' \circ g' \simeq id_{\varepsilon_{\alpha_k}} \text{ (in } W_k) \text{, choose a neighborhood } V \subset C' \text{ of } X \text{ and there exists an integer } n_k \geq n_k \text{ and a neighborhood } U \subset \Phi^{-1}(U) \text{ of } X \text{ such that } g_k \mid f_k \mid U \simeq id_{\varepsilon_{\alpha_k}} \text{ (in } \Phi^{-1}(U)), \text{ for all } k \geq n_k. \text{ Clearly } \Phi(U) \subset U \text{ is a neighborhood of } \Phi(X).
\]
\[
\text{and also}
\]
\[
g'_k \circ f'_k \mid \Phi(U') = \Phi \circ g_k \circ f_k \circ \Phi^{-1} \mid \Phi(U')
\]
\[
\simeq \Phi \circ id_{\varepsilon_{\alpha_k}} \circ \Phi^{-1} \mid \Phi(U') \quad \text{(in } U) = id_{\varepsilon_{\alpha_k}},
\]
\[
\text{for all } k \geq n_k. \text{ Thus } f' \circ g' \simeq id_{\varepsilon_{\alpha_k}} \text{ (in } W_k) \text{ and we are done.}
\]

\section{Proof of Theorem 1}

Let \( X', Y' \subset E^{m+1} \times 0 \subset E^m \) be copies of \( X, Y \), respectively which are in standard position in \( E^{m+1} \times 0 \) and let \( h : E^m \times X' \rightarrow E^m \times Y' \) be a homeomorphism. We must prove that \( Sh(X') = Sh(Y') \) (which implies that \( Sh(X) = Sh(Y) \)). Choose a number \( t_0 > 0 \) such that \( X' \cup Y' \subset Int(B^m_{t_0}) \). Then choose \( t_0 \in (0, t_1) \) such that \( Int(B^m_{t_0+1}) \times 0 \subset E^m \) contains \( X' \cup Y' \). Since \( h(B^m_{t_0}) \) is bicollared in \( E^m \) we can use the Generalized Schoenflies Theorem of \([6]\) to write
\[
E^m \setminus h(B^m_{t_0}) = A \cup B,
\]
\[
\text{where } A \text{ is the bounded component (which is homeomorphic to } Int(B^m_{t_0}). \text{ This proof now splits into cases and we first treat the case in which } X' \subset A \text{ and } h(Int(B^m_{t_0}), X') = A \cup Y'. \text{ In this case it is clear that } t_0 \text{ can be chosen large enough so that } B^m_{t_0+1} \times 0 \subset A.
\]

Let \( r : E^m \rightarrow E^m \times 0 \) be a retraction and define a homotopy
\[
F : (B^m_{t_0+1} \times 0) \times I \rightarrow E^m \text{ by}
\]
\[
F(x_1, ..., x_{m+1}, 0, 0, 0, ..., t) = (x_1, ..., x_{m+1}, t, 0, 0, ...,)
\]
\[
\text{for all } (x_1, ..., x_{m+1}, 0, 0, 0, ...,) \in B^m_{t_0+1} \times 0 \text{ and } t \in I. \text{ For each integer } k \geq 0 \text{ let } f_k : B^m_{tk+1} \times 0 \rightarrow B^m_{tk+1} \times 0 \text{ and } g_k : B^m_{tk+1} \times 0 \rightarrow B^m_{tk+1} \times 0 \text{ be defined by}
\]
\[
f_k = r \circ h \circ F_{tk+1} \text{ and } g_k = r \circ \iota \circ F_{tk+1}.
\]

We will show that \( f = \{f_k, X', Y'\} \) and \( g = \{g_k, X', Y'\} \) are fundamental sequences which satisfy \( g \circ f \simeq id_{\varepsilon_{\alpha_k}} \) and \( f \circ g \simeq id_{\varepsilon_{\alpha_k}} \) (in \( B^m_{tk+1} \times 0 \)), where \( id_{\varepsilon_{\alpha_k}} \) and \( id_{\varepsilon_{\alpha_k}} \) are the identity fundamental sequences of \( X' \) and \( Y' \), respectively. Then from \([5]\) it will follow that \( Sh(X') = Sh(Y') \).

To see that \( f \) is a fundamental sequence let \( V \subset B^m_{tk+1} \times 0 \) be an open set containing \( X' \). Since \( r^{-1}(V) \) is an open set containing \( X' \) it follows that \( h^{-1}(r^{-1}(V) \setminus X') \cup X' \) is an open set in \( E^m \) containing \( X' \). Thus there exists an open set \( U \subset B^m_{tk+1} \times 0 \) containing \( X' \) and a number \( e > 0 \) such that \( E^m \times [0, e] \subset h^{-1}(r^{-1}(V) \setminus X') \cup X' \). This implies that \( F_{tk+1} \mid U \simeq F_{tk+1} \mid U \) (in \( h^{-1}(r^{-1}(V) \setminus X') \), for all integers \( k, l \geq 1/e \). Since this homotopy takes place in the complement of \( X' \) we have \( h \circ F_{tk+1} \mid U \simeq h \circ F_{tk+1} \mid U \) (in \( r^{-1}(V) \)), for all \( k, l \geq 1/e \). Then applying \( r \) to this homotopy we have \( r \circ h \circ F_{tk+1} \mid U \simeq r \circ h \circ F_{tk+1} \mid U \) (in \( V \)), which means precisely that \( f_k \mid U \simeq f_k \mid U \).
cellular and we have $\text{Sh}(X') = \text{Sh}(\text{(point)}) = \text{Sh}(Y')$. Thus we have treated all cases in which $Y' \subset A$.

On the other hand let us now assume that $Y' \not\subset A$, hence $Y' = B \neq \emptyset$. Note that we have either $\text{h}(\text{Int}(B_{\emptyset}^n) \cap X') = B' \neq \emptyset$ or $\text{h}(\text{Int}(B_{\emptyset}^n) \cap X') = B' \neq \emptyset$.

If $\text{h}(\text{Int}(B_{\emptyset}^n) \cap X') = B' \neq \emptyset$, then it follows that $B \cap Y' = \emptyset$, a contradiction. Thus we must have $\text{h}(\text{Int}(B_{\emptyset}^n) \cap X') = B' \neq \emptyset$, hence $\text{h}(\text{Int}(B_{\emptyset}^n) \cap X') = B' \neq \emptyset$.

Choose any $p \in A$ and let us: $\text{Cl}(B) \cap \text{Cl}(A \setminus \{p\})$ be a homeomorphism such that $u \mid B \cap (A \setminus \{p\}) \neq \emptyset$. Then $u \mid B \cap (A \setminus \{p\}) \neq \emptyset$ is a homeomorphism and we can extend $u \mid B \cap (A \setminus \{p\}) \neq \emptyset$ to a homeomorphism $u': E^n \setminus \text{Int}(B_{\emptyset}^n) \to \text{Cl}(B)$. Define $h': E^n \setminus \text{Int}(B_{\emptyset}^n) \to (E^n \setminus \{p\}) \cup (Y' \cap B)$ by setting $h'(E^n \setminus \text{Int}(B_{\emptyset}^n)) = \emptyset$.

Then $h'$ is a homeomorphism and the first case that we treated above implies that $\text{Sh}(X') = \text{Sh}(p) \cup (Y' \cap B)$. Since $\text{h}(\text{Int}(B_{\emptyset}^n) \cap X') = B' \neq \emptyset$ it follows that $\text{h}(\text{Int}(B_{\emptyset}^n) \cap X') = B' \neq \emptyset$. This implies that $Y' \not\subset A$ is cellular, hence $\text{Sh}(Y' \cap A) = \text{Sh}(\text{(point)})$. Now decomposing $Y'$ we get

$$\text{Sh}(Y') = \text{Sh}(Y' \cap A) \cup (Y' \cap B) = \text{Sh}(\text{(point)}) \cup (Y' \cap B) = \text{Sh}(X')$$

as we observed above. This completes the proof of (b) of Theorem 1.

For the proof of (a) of Theorem 1 we choose $X', Y' \subset E^n$ (n ≥ 2m+2) to be copies of $X, Y$ respectively which are in standard position. We will prove that $E^n \setminus X'$ and $E^n \setminus Y'$ are homeomorphic. The procedure will be to use Lemma 3.1 to inductively construct sequences $\{U_{i,1}\}_{i=0}^\infty$, and $\{V_{i,1}\}_{i=0}^\infty$ of open subsets of $E^n$ and a sequence $(h_{i,1})_{i=0}^\infty$ of homeomorphisms of $E^n$ onto itself such that

1. $X' = \bigcap_{i=0}^\infty U_i$ and $U_{i+1} \subset U_i$ for all $i > 0$,
2. $Y' = \bigcap_{i=0}^\infty V_i$ and $V_{i+1} \subset V_i$ for all $i > 0$,
3. $h_{i,1-1} \cdot \ldots \cdot h_{i,1}(X') \subset V_{i+1}$ for all $i > 0$,
4. $h_{i,1}(E^n \setminus Y') = \emptyset$ for all $i > 0$ and $j > 0$,
5. $h_{i,1} \cdot \ldots \cdot h_{i}(U_1) \cap Y' \subset Y'$ for all $i > 0$,
6. $h_{i,1} \cdot \ldots \cdot h_{i}(U_1) \cap Y' \subset Y'$ for all $i > 0$ and $j > 0$.

For the time being we assume that these sequences have been constructed and we show how to prove that $E^n \setminus X'$ and $E^n \setminus Y'$ are homeomorphic. Choose any $x \in E^n \setminus X'$ and consider the sequence $(h_1 \cdot \ldots \cdot h_1(x))_{i=1}$. If $f$ is chosen large enough so that $x \in U_1$, then it follows from (6) that

$h_1 \cdot \ldots \cdot h_1(x) = h_2 \cdot \ldots \cdot h_1(x)$.
for all \( i \geq 2j \). From (5) it follows that \( h_i \circ \ldots \circ h_j(x) \in Y' \), for all \( i \geq 2j \).

Thus it makes sense to define a function \( h : E^m \to E^n \) by

\[
h(x) = \lim_{i \to \infty} h_i \circ \ldots \circ h_j(x).
\]

Since each \( h_i \) is 1-1 it follows that \( h \) is 1-1. If \( x \in E^m \) and \( U \subset E^m \) is a compact neighborhood of \( x \), then there exists an integer \( j \) so that \( U \cap V_j = \emptyset \). From (6) it follows that \( h(U) = h_j \circ \ldots \circ h_1(U) \), hence \( h(U) \) is a neighborhood of \( h(x) \). This implies that \( h \) is open. To see that \( h \) is continuous and onto choose a compactum \( V \subset E^m \) and choose \( j > 0 \) such that \( V \cap V_j = \emptyset \). It follows from (3) that \( V \times h_i^{-1} \circ \ldots \circ h_1(E^m) \) and it follows from (4) that \( V \subset h(E^m \times X') \) and \( h^{-1}(V) = (h_{k+1} \circ \ldots \circ h_1)^{-1}(V) \). Thus \( h \) is our desired homeomorphism.

We now turn to the construction of the necessary sequences. For each integer \( k \) consider the following statement.

\( P_k \): There exist collections \( \{ U_i \}_{i=1}^{k} \) and \( \{ V_i \}_{i=1}^{k} \) of open subsets of \( E^m \) and a collection \( \{ h_i \}_{i=1}^{k} \) of homeomorphisms of \( E^m \) onto itself such that \( X' \subset U_i \) and \( Y' \subset V_i \), for \( 1 \leq i \leq k \), and

1. \( U_i \subset U_i \) for \( 1 \leq i \leq k \), and \( U_i \subset \{ x \in E^m | d(x', x) < 1/i \} \), for \( 1 \leq i \leq k \),
2. \( V_i \subset V_i \) for \( 1 \leq i \leq k \), and \( V_i \subset \{ x \in E^m | d(x', x) < 1/i \} \), for \( 1 \leq i \leq k \),
3. \( h_i(U) \subset V_i \) for \( 1 \leq i \leq k \), and \( h_i(U) \subset \{ x \in E^m | d(x', x) < 1/i \} \), for \( 1 \leq i \leq k \),
4. \( h_i(E^m \times X') = \text{id} \) for \( 1 \leq j < k \) and \( 2j-1 < i \leq 2k \),
5. \( h_i(E^m \times X') = \text{id} \) for \( 1 \leq i \leq k \), and \( h_{2k} \circ \ldots \circ h_1(U) \subset X' \),
6. \( h_i(E^m \times X') = \text{id} \) for \( 1 \leq j < k \) and \( 2j-1 < i \leq 2k \),
7. \( h_{2k} \circ \ldots \circ h_1(X') \) is in standard position, and
8. \( h_{2k} \circ \ldots \circ h_1(X') \) and \( Y' \) have the same relative shape (in \( h_{2k} \circ \ldots \circ h_1(U) \)).

We will prove that \( P_k \) is true for all \( k \). Moreover in the inductive step from \( P_k \) to \( P_{k+1} \) we will construct the necessary collections of open sets and homeomorphisms for \( P_{k+1} \) by adding appropriately constructed \( U_{2k+1}, V_{2k+1}, h_{2k+1}, \) and \( h_{2k+2} \) to the given collections \( \{ U_i \}_{i=1}^{k+1}, \{ V_i \}_{i=1}^{k+1}, \) and \( \{ h_i \}_{i=1}^{k+1} \) for \( P_k \). Once this is done we will be finished with the proof.

For \( k = 1 \) let

\[
V_1 = \{ x | x \in E^m, d(x', x) < 1 \}
\]

and use the assumption that \( S(X') = S(Y') \) to conclude that \( X' \) and \( Y' \) have the same relative shape (in \( E^m \)). Then Lemma 5.1 implies the existence of a PL homeomorphism \( h : E^m \to E^m \) such that \( h(X') \subset V_1 \), \( h(Y') \) is in standard position, and \( h(Y') \) and \( Y' \) have the same relative shape (in \( V_1 \)). Then put

\[
U_1 = h^{-1}(V_1) \cap \{ x \in E^m | d(x', x) < 1 \}
\]

and once more use Lemma 5.1 to obtain a PL homeomorphism \( g_1 : E^m \to E^m \) such that \( g_1(Y') \subset h(U_1) \), \( g_1(Y') \) is in standard position, \( g_1(E^m \times V_1) = \text{id} \), and \( h(Y') \) and \( g_1(Y') \) have the same relative shape (in \( h(U_1) \)). Then define \( h_2 = g_1 \) and note that this implies that \( P_1 \) is true.

For the inductive step assume that we have collections \( \{ U_i \}_{i=1}^{k+1}, \{ V_i \}_{i=1}^{k+1}, \) and \( \{ h_i \}_{i=1}^{k+1} \) for which \( P_k \) is true. Let

\[
V_{k+1} = V_k \cap h_k \circ \ldots \circ h_2 \circ h_1(U_k) \cap \{ x \in E^m | d(x', x) < 1/(k+1) \}
\]

and use (8) and Lemma 5.1 to get a PL homeomorphism \( h_{2k+1} : E^m \to E^m \) such that \( h_{2k+1} \circ \ldots \circ h_1(X') \subset V_{k+1} \), \( h_{2k+1} \circ \ldots \circ h_1(X') \) is in standard position, \( h_{2k+1} \circ \ldots \circ h_1(X') \) and \( Y' \) have the same relative shape (in \( V_{k+1} \)), and \( h_{2k+1} \circ \ldots \circ h_1(U_k) = \text{id} \). Then put

\[
U_{k+1} = U_k \cap (h_{2k+1} \circ \ldots \circ h_1(U_{k+1})) \cap \{ x \in E^m | d(x', x) < 1/(k+1) \}
\]

and once more (as in the construction of \( h_2 \)) use Lemma 5.1 to construct a PL homeomorphism \( h_{2k+2} : E^m \to E^m \) such that \( h_{2k+2} \circ \ldots \circ h_1(V_{k+1}) = \text{id} \), \( h_{2k+2} \circ \ldots \circ h_1(U_{k+1}) \subset X' \), \( h_{2k+2} \circ \ldots \circ h_1(X') \) is in standard position, and \( h_{2k+2} \circ \ldots \circ h_1(Y') \) and \( Y' \) have the same relative shape (in \( h_{2k+2} \circ \ldots \circ h_1(U_{k+1}) \)). This completes the inductive step and the proof of the theorem.

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