Note on a theorem of J. Baumgartner

by

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J. Baumgartner has proved that if $V = L$ (the Axiom of Constructibility) is assumed, then there is an Aronszajn tree order embeddable in the reals but not in the rationals. We extend this result to show that, under a weaker assumption than $V = L$, there are $2^{\aleph_1}$ non-isomorphic such trees.

We wish to thank Richard Laver for introducing us to the topic here discussed.

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1. Introduction. We work in Zermelo–Fraenkel set theory (including Choice), denoted by ZFC, and use the usual notations and conventions. $\boldsymbol{R}$ denotes the real numbers (as an ordered set) and $\boldsymbol{Q}$ denotes the rationals.

A *tree* is a poset $T = \langle T, \leqslant_T \rangle$ such that for any $x \in T$, $\mathrm{pr}(x) = \{y \in T \mid y <_T x\}$ is well-ordered by $\leqslant_T$. The order-type of $\mathrm{pr}(x)$ is the *height* of x in T, $\mathrm{ht}(x)$. For each ordinal α we set $T_\alpha = \{x \in T \mid \mathrm{ht}(x) = \alpha\}$ and $T|\alpha = \bigcup_{\beta<\alpha} T_\beta$. T_α is the αth *level* of T. A *branch* of T is a maximal totally ordered subset of T; if it has order-type α it is an α-*branch*. An *antichain* of T is a pairwise incomparable subset of T.

Let $\lambda \leqslant \omega_1$. A tree T is a λ-*tree* if:

(i) $(\forall \alpha < \lambda)(T_\alpha \neq \emptyset)$ & $T_\lambda = \emptyset$;

(ii) $(\forall \alpha < \lambda)(|T_\alpha| \leqslant \aleph_0)$ & $|T_0| = 1$;

(iii) $(\forall \alpha < \lambda)(\forall x \in T_\alpha)[(|\{y \in T_{\alpha+1} | x <_T y\}| = \aleph_0) \vee (\alpha+1 = \lambda)]$;

(iv) $(\forall \alpha < \beta < \lambda)(\forall x \in T_\alpha)(\exists y \in T_\beta)(x <_T y)$;

(v) $(\forall \alpha = \bigcup \alpha < \lambda)(\forall x, y \in T_\alpha)\big(\mathrm{pr}(x) = \mathrm{pr}(y) \rightarrow x = y\big)$.

An ω_1-tree is *Aronszajn* if it has no ω_1-branches. It is *Souslin* if in addition it has no uncountable antichains.

THEOREM 1.

1. (Aronszajn) *There is an Aronszajn tree.*

2. (Gaifman–Specker) *There are* $2^{\aleph_1}$ *non-isomorphic Aronszajn trees.*

3. (Jensen) *If* $V = L$, *there is a Souslin tree.*

4. (Jech) *If* $V = L$, *there are* $2^{\aleph_1}$ *non-isomorphic Souslin trees.*

For proofs of 1 and 3 we refer the reader to [5]; a proof of 2 may be found in [3], whilst 4 is proved in [4]. It is also proved in [5] that the existence of a Souslin tree is not provable in ZFC alone.

2. The theorem. Let T be an ω_1-tree, $X = \langle X, \leqslant_X\rangle$ a poset. We say T is *X-embeddable* if there is $f\colon T \to X$ such that $x <_T y \to f(x) <_X f(y)$. We then say *f embeds T in X*.

THEOREM 2. *Let T be an ω_1-tree.*

1. *T is X-embeddable iff T is Aronszajn and there are antichains A_n, $n<\omega$, of T such that $T = \bigcup_{n<\omega} A_n$.*

2. *If T is $\boldsymbol{R}$-embeddable, then T is Aronszajn and such that every uncountable subset of T contains an uncountable antichain of T.*

Proof. 1. If f embeds T in $\boldsymbol{Q}$ then T is clearly Aronszajn. Also, $A_q = \{x \in T \mid f(x) = q\}$ is an antichain of T for each $q \in \boldsymbol{Q}$, and $T = \bigcup_{q\in Q} A_q$. The converse is trivial.

2. Let $U \subset T$ be uncountable. Then U inherits a tree structure from T. Let $U^* = \bigcup_{\alpha<\omega_1} U_{\alpha+1}$. It T is $\boldsymbol{R}$-embeddable, so is U, whence U^* is $\boldsymbol{Q}$-embeddable. By 1, U^* is the union of countably many antichains, one of which must be uncountable.

Now, the Aronszajn trees constructed in ZFC are all $\boldsymbol{Q}$-embeddable. By the above theorem, no Souslin tree can be $\boldsymbol{R}$-embeddable. The question arises, therefore, as to whether there can be Aronszajn trees $\boldsymbol{R}$-embeddable but not $\boldsymbol{Q}$-embeddable. That such trees cannot be constructed in ZFC follows from the following result, proved in [2]:

THEOREM 3 (Baumgartner). *If* ZFC *is consistent, so is* ZFC+ *"Every Aronszajn tree is $\boldsymbol{Q}$-embeddable".*

However, the following was announced in [1]:

THEOREM 4 (Baumgartner). *Assume $V = L$. Then there is an Aronszajn tree which is $\boldsymbol{R}$-embeddable but not $\boldsymbol{Q}$-embeddable.*

Baumgartner's proof, as outlined to us by Richard Laver, involved a forcing construction over initial segments of L. At the cost of some messy combinatorics, we have adapted this argument to deduce Baumgartner's conclusion from an assumption $\diamondsuit$, weaker than $V = L$, which is due to R. B. Jensen. This approach allows us to extend Theorem 4 along the lines suggested by Theorem 1.

Let λ be a limit ordinal. A set $A \subset \lambda$ is *stationary* if it intersects every closed unbounded subset of λ.

Axiom $\diamondsuit$. There is a sequence $\langle S_\alpha \mid \alpha < \omega_1\rangle$ such that $S_\alpha \subset \alpha$ for each α and such that whenever $S \subset \omega_1$, then $\{\alpha \in \omega_1 \mid S \cap \alpha = S_\alpha\}$ is stationary in ω_1.

THEOREM 5 (Jensen). *If $V = L$, then $\diamondsuit$ holds.*

Proof. By induction, define $\langle S_\alpha, C_\alpha\rangle$ as the least pair (under the canonical well-order of L) of subsets of α such that C_α is closed and unbounded in α and $\gamma \in C_\alpha \to S_\alpha \cap \gamma \neq S_\gamma$. If no such pair exists, set $S_\alpha = C_\alpha = \emptyset$. Assumption that $\langle S_\alpha \mid \alpha < \omega_1\rangle$ is not as required now leads speedily to a contradiction. Q.E.D.

We are now ready to prove our theorem. The proof was inspired by arguments in [4].

THEOREM 6. Assume $\diamondsuit$. *Then there are* $2^{\aleph_1}$ *non-isomorphic Aronszajn trees $\boldsymbol{R}$-embeddable but not $\boldsymbol{Q}$-embeddable.*

Proof. By $\diamondsuit$ there is a sequence $\langle h_\alpha \mid \alpha < \omega_1\rangle$ such that $h_\alpha\colon \alpha \to \alpha$ for each α, and whenever $h\colon \omega_1 \to \omega_1$, then $\{\alpha \in \omega_1 \mid h \upharpoonright \alpha = h_\alpha\}$ is stationary in ω_1. We fix this sequence for the rest of this proof.

By induction on $\alpha < \omega_1$, for each $f \in 2^\alpha$ we shall construct an $(\alpha+1)$-tree T_f consisting of sequences of distinct integers of lengths $\leqslant\alpha$. The ordering will be sequence extension. If $s \in T_f$ and $\gamma < \text{length}(s)$, then $s \upharpoonright \gamma \in T_f$, whence the height of any s in T_f is its length. If $f, g \in \bigcup_{\alpha<\omega_1} 2^\alpha$ and $f \subset g$, then T_g will be an end-extension of T_f. Hence for each $F \in 2^{\omega_1}$, $T(F) = \bigcup_{\alpha<\omega_1} T_{F\upharpoonright\alpha}$ will be an ω_1-tree. It will automatically be $\boldsymbol{R}$-embeddable. For, given any $X \subset \omega$, let $f_X \in 2^\omega$ be defined by $f_X(n) = 1$ iff $n \in X$. Then the map $h\colon T(F) \to \boldsymbol{R}$ defined by $h(s) = f_{\text{range}(s)}$ embeds $T(F)$ in $\boldsymbol{R}$. We shall ensure that no $T(F)$ is $\boldsymbol{Q}$-embeddable, and that $F, G \in 2^{\omega_1}$ & $F \neq G$ implies $T(F) \not\cong T(G)$.

As the induction proceeds, we define one-one maps $\pi_f\colon T_f \to \omega_1 - \omega$ so that $s \subset t \to \pi_f(s) < \pi_f(t)$, and so that $f \subset g \to \pi_f \subset \pi_g$.

By Q, we shall mean ω endowed with a dense linear order $<_Q$.

We shall carry out the construction so as to preserve the following conditions:

(Ξ) If $f \in \bigcup_{\alpha<\omega_1} 2^\alpha$ and $s \in T_f$, then $|\omega - \text{range}(s)| = \aleph_0$.

(Ψ) If $f \in \bigcup_{\alpha<\omega_1} 2^\alpha$ and $s \in T_f$ and $x \in [\omega]^{<\omega} - \text{range}(s)$, there is $s' \supset s$ on each higher level of T_f such that $\text{range}(s') \cap x = \emptyset$.

Let $T_\emptyset = \{\emptyset\}$. Suppose $\alpha < \omega_1$, $f \in 2^{\alpha+1}$, $T_{f\upharpoonright\alpha}$, $\pi_{f\upharpoonright\alpha}$ are defined, and that $T_{f\upharpoonright\alpha}$ satisfies (Ξ) and (Ψ). To obtain T_f, for each $s \in \omega^\alpha \cap T_{f\upharpoonright\alpha}$, add all one-point extensions of s by distinct integers. This is possible by (Ξ), which guarantees the existence of $\aleph_0$ such. Clearly, T_f satisfies (Ξ) and (Ψ). To obtain π_f from $\pi_{f\restriction\alpha}$, extend the latter arbitrarily, except for ensuring that if $g \in 2^{\alpha+1}$ and $g \neq f$, then $\operatorname{range}(\pi_g) \ncong \operatorname{range}(\pi_f)$. Since $2^{\aleph_0} = \aleph_1$ is a trivial consequence of $\diamond$, this causes no trouble.

Suppose now that $\alpha = \bigcup \alpha < \omega_1$, $f \in 2^\alpha$, $T_{f\restriction\gamma}, \pi_{f\restriction\gamma}$ are defined for all $\gamma < \alpha$, and that each $T_{f\restriction\gamma}$ satisfies (Ξ) and (Ψ). Let $T'_f = \bigcup_{\gamma<\alpha} T_{f\restriction\gamma}$, an α-tree. We must decide which α-branches of T'_f to extend in order to obtain T_f. Let $\pi'_f = \bigcup_{\gamma<\alpha} \pi_{f\restriction\gamma}$. Then π'_f induces an α-tree isomorphic to T'_f whose elements are countable ordinals. Also, $s <_{T'_f} t$ implies $\pi'_f(s) < \pi'_f(t)$. There are three cases to consider.

Case I. h_α embeds $\pi'_f{}''T'_f$ in $\boldsymbol{Q}$. (We understand this to imply that $\operatorname{domain}(h_\alpha) = \pi'_f{}''T'_f$.)

Then $h_\alpha \cdot \pi'_f$ embeds T'_f in $\boldsymbol{Q}$. (Note that f is uniquely determined by α here. For suppose $g \in 2^\alpha$, $g \neq f$. Since $\alpha = \bigcup \alpha$ there is $\gamma < \alpha$ with $f\restriction\gamma+1 \neq g\restriction\gamma+1$. By construction, $\operatorname{range}(\pi_{f\restriction\gamma+1}) \neq \operatorname{range}(\pi_{g\restriction\gamma+1})$. So, as π'_f and π'_g are order-preserving, $\operatorname{range}(\pi'_f) \neq \operatorname{range}(\pi'_g)$. But $\operatorname{range}(\pi'_f) = \operatorname{domain}(h_\alpha)$. Thus $\operatorname{range}(\pi'_g) \neq \operatorname{domain}(h_\alpha)$, whence h_α does not embed $\pi'_g{}''T'_g$ in $\boldsymbol{Q}$.) Let

$$X(\alpha) = \{(s, x) \mid s \in T'_f \;\&\; x \in [\omega]^{<\omega} \;\&\; \operatorname{range}(s) \cap x = \emptyset\}.$$

For $(s, x), (t, y) \in X(\alpha)$, say $(s, x) \leqslant_\alpha (t, y)$ iff $s \subset t \;\&\; x \subset y$. This defines a partial order on $X(\alpha)$.

Recall that if P is a poset, a set $U \subset P$ is *cofinal* if

$$(\forall p \in P)(\exists q \in U)(p \leqslant_P q).$$

For each $n \in \omega$, set

$$\Delta^\alpha_n = \{(s, x) \in X(\alpha) \mid h_\alpha \cdot \pi'_f(s) \geqslant_Q n \text{ or else } (\forall (t, y) \in X(\alpha))[(t, y) \geqslant_\alpha (s, x) \to h_\alpha \cdot \pi'_f(t) <_Q n]\}.$$

Clearly, each Δ^α_n is cofinal in $X(\alpha)$.

Let $s \in T'_f$, $x \in [\omega]^{<\omega} - \operatorname{range}(s)$. Let $\langle \alpha_n \mid n < \omega\rangle$ be cofinal in α with $\alpha_0 = \operatorname{length}(s)$. By (Ψ) we can find $s'_0 \in T'_f$, $s'_0 \supset s$, such that $\operatorname{length}(s'_0) \geqslant \alpha_0$ and $\operatorname{range}(s'_0) \cap x = \emptyset$. Since $(s'_0, x) \in X(\alpha)$ and Δ^α_0 is cofinal, we can find a pair $(s_0, x_0) \geqslant_\alpha (s'_0, x)$ in Δ^α_0. Let $m_0 \in \omega - [\operatorname{range}(s_0) \cup x_0]$, by (Ξ). Let $x'_1 = x_0 \cup \{m_0\}$. By (Ψ) we can find $s'_1 \in T'_f$, $s'_1 \supset s_0$, such that $\operatorname{length}(s'_1) \geqslant \alpha_1$ and $\operatorname{range}(s'_1) \cap x'_1 = \emptyset$. Since $(s'_1, x'_1) \in X(\alpha)$ and Δ^α_1 is cofinal, we can find $(s_1, x_1) \geqslant_\alpha (s'_1, x'_1)$ in Δ^α_1. Pick $m_1 \in \omega - [\operatorname{range}(s_1) \cup x_1]$, set $x'_2 = x_1 \cup \{m_1\}$, and proceed inductively.

Let $s(x) = \bigcup_{n<\omega} s_n$. Then $s(x)$ is an α-sequence of distinct integers which defines an α-branch of T'_f. Also, $s(x) \supset s$ and $\operatorname{range}(s(x)) \cap x = \emptyset$.

Finally, since $\{m_0, m_1, \ldots\} \cap \operatorname{range}(s(x)) = \emptyset$, we have $|\omega - \operatorname{range}(s(x))| = \aleph_0$. We may thus let the αth level of T_f consist of one such $s(x)$ for each pair (s, x) as above. Then T_f is an $(\alpha+1)$-tree satisfying (Ξ) and (Ψ). Extend π'_f to π_f on T_f arbitrarily.

Case II. For some $g \in 2^\alpha$ with $g \neq f$, h_α: $\pi'_f{}''T'_f \cong \pi'_g{}''T'_g$. As before, f and g are uniquely determined by α. Let $s \in T'_f$, $x \in [\omega]^{<\omega} - \operatorname{range}(s)$. Take $\langle \alpha_n \mid n < \omega\rangle$ as before. By (Ψ) we can find $s_0 \in T'_f$, $s_0 \supset s$, such that $\operatorname{length}(s_0) \geqslant \alpha_0$ and $\operatorname{range}(s_0) \cap x = \emptyset$. Let $m_0 \in \omega - [\operatorname{range}(s_0) \cup x]$. Let $x_0 = x \cup \{m_0\}$. By (Ψ) again we can find $s_1 \in T'_f$, $s_1 \supset s_0$, such that $\operatorname{length}(s_1) \geqslant \alpha_1$ and $\operatorname{range}(s_1) \cap x_0 = \emptyset$. Let $m_1 \in \omega - [\operatorname{range}(s_1) \cup x_0]$, put $x_1 = x_0 \cup \{m_1\}$, and proceed inductively. This yields an α-sequence $s(x) \supset s$ which determines an α-branch of T'_f. Also, $\operatorname{range}(s(x)) \cap x = \emptyset$ and $|\omega - \operatorname{range}(s(x))| = \aleph_0$. Let the αth level of T_f consist of one such $s(x)$ for each such pair s, x. Similarly for T_g. The only cause for concern now is if $\pi'^{-1}_g \cdot h_\alpha \cdot \pi'_f$ extends to an isomorphism of T_f and T_g. If it does, pick any distinct α-branch t of T'_f with $|\omega - \operatorname{range}(t)| = \aleph_0$ and put t into the αth level of T_f. To find such a t, proceed much as before, but miss the (countably many) branches which already extend. Thus $\pi'^{-1}_g \cdot h_\alpha \cdot \pi'_f$ cannot now extend to an isomorphism of T_f and T_g.

In either case, T_f and T_g are $(\alpha+1)$-trees satisfying (Ξ) and (Ψ). Extend π'_f and π'_g to π_f and π_g arbitrarily.

Case III. Neither of cases I or II occurs. As in Case II, add one $s(x)$ for each pair s, x to obtain T_f, and extend π'_f arbitrarily.

For $F \in 2^{\omega_1}$, set $T(F) = \bigcup_{\alpha<\omega_1} T_{F\restriction\alpha}$, an ω_1-tree embeddable in $\boldsymbol{R}$. Let $\pi_F = \bigcup_{\alpha<\omega_1} \pi_{F\restriction\alpha}$. Then π_F: $T(F) \to \omega_1 - \omega$ and $s <_{T(F)} t \to \pi_F(s) < \pi_F(t)$. Also, π_F induces a tree isomorphic to $T(F)$ whose elements are countable ordinals.

Suppose $T(F)$ is $\boldsymbol{Q}$-embeddable. Then there is an h which embeds $\pi_F{}''T(F)$ in $\boldsymbol{Q}$. Let

$$\begin{aligned} A = \{\alpha \in \omega_1 \mid\; & \alpha = \bigcup \alpha \cdot \& \cdot [\pi_F{}''T(F)]\restriction\alpha = \pi'_{F\restriction\alpha}{}''T'_{F\restriction\alpha} \cdot \& \cdot \\ & \cdot \& \cdot h\restriction\alpha \text{ embeds } \pi'_{F\restriction\alpha}{}''T'_{F\restriction\alpha} \text{ in } \boldsymbol{Q} \cdot \& \cdot \\ & \cdot \& \cdot (\forall s \in T'_{F\restriction\alpha})(\forall x \in [\omega]^{<\omega} - \operatorname{range}(s))(\forall n >_Q h(\pi_F(s))) \\ & [(\exists t \in T(F))(t \supset s \;\&\; \operatorname{range}(t) \cap x = \emptyset \;\&\; h(\pi_F(t)) \geqslant_Q n) \\ & \to (\exists t \in T'_{F\restriction\alpha})(t \supset s \;\&\; \operatorname{range}(t) \cap x = \emptyset \;\&\; h(\pi_F(t)) \geqslant_Q n)]\}. \end{aligned}$$

It is easily seen that A is closed and unbounded in ω_1. Hence by $\diamond$ there is $\alpha \in A$ such that $h\restriction\alpha = h_\alpha$. Thus, by the definition of A, Case I applied in constructing $T_{F\restriction\alpha}$ from $T'_{F\restriction\alpha}$. Let $s \in \omega^\alpha \cap T(F)$. Let $n = h(\pi_F(s))$. By construction, let $(t, y) \in \Delta^\alpha_n$ be such that $\operatorname{range}(s) \cap y = \emptyset$ and $t \subset s$.

As h is order-preserving, $h(\pi_F(t)) <_Q n$. By the definition of A and the relation of s to t, there is $t' \in T'_{F\restriction\alpha}$ such that $t' \supset t$ and range$(t') \cap y = \emptyset$, and such that $h(\pi_F(t')) \geqslant_Q n$. In particular, $(t, y), (t', y) \in X(\alpha)$ and $(t', y) \geqslant_\alpha (t, y)$. But look, $(t, y) \in \Delta_n^\alpha$, so by definition we must have that $h_\alpha(\pi_F(t)) \geqslant_Q n$. Since $h \restriction \alpha = h_\alpha$, this contadicts our earlier inequality. Hence $T(F)$ cannot be Q-embeddable.

Suppose now that for some $F, G \in 2^{\omega_1}$, $F \neq G$, we have $T(F) \cong T(G)$. Let $h\colon \pi_F{}'' T(F) \cong \pi_G{}'' T(G)$. Pick $\alpha_0 < \omega_1$ such that $F \restriction \alpha_0 \neq G \restriction \alpha_0$. Let

$$A = \{\alpha \in \omega_1 |\ \alpha = \bigcup \alpha > \alpha_0 \cdot \& \cdot [\pi_F{}'' T(F)] \restriction \alpha = \pi'_{F\restriction\alpha}{}'' T'_{F\restriction\alpha} \cdot \& \cdot$$

$$\cdot \& \cdot [\pi_G{}'' T(G)] \restriction \alpha = \pi'_{G\restriction\alpha}{}'' T'_{G\restriction\alpha} \cdot \& \cdot h \restriction \alpha\colon \pi'_{F\restriction\alpha}{}'' T'_{F\restriction\alpha} \cong \pi'_{G\restriction\alpha}{}'' T'_{G\restriction\alpha}\}.$$

Clearly, A is closed and unbounded in ω_1. By $\diamondsuit$, there is $\alpha \in A$ such that $h \restriction \alpha = h_\alpha$. Thus Case II applied in constructing $T_{F\restriction\alpha}$ from $T'_{F\restriction\alpha}$ and $T_{G\restriction\alpha}$ from $T'_{G\restriction\alpha}$. This means that the map $\pi'^{-1}_{G\restriction\alpha} \cdot h_\alpha \cdot \pi'_{F\restriction\alpha}$ does not extend to an isomorphism of $T_{F\restriction\alpha}$ and $T_{G\restriction\alpha}$, which is absurd, since $\pi_G^{-1} \cdot h \cdot \pi_F$ extends it. Thus $T(F)$ and $T(G)$ are not isomorphic. The proof is complete.

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Shapes of finite-dimensional compacta

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1. Introduction. The results of this paper deal with shapes of finite-dimensional compact metric spaces (see [4] for definitions concerning the concept of shape). In Theorem 1 below we give a characterization of shapes of finite-dimensional compact metric spaces (i.e. *compacta*) in terms of embeddings in Euclidean n-space E^n. In an earlier paper the author obtained a characterization of shapes of compacta (with no dimensional restriction) in terms of embeddings in the Hilbert cube [8]. In a sense the results obtained here are motivated by [8], and to some extent the general structure of the proof of Theorem 1 is a modification of the argument used in [8], but the present paper does not involve any infinite-dimensional topology. For the sake of completeness we give a short summary of the infinite-dimensional characterization at the end of the Introduction. We use the notation $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ to indicate that compacta X and Y have the same shape.

THEOREM 1. *Let X, Y be compacta such that $\dim X, \dim Y \leqslant m$.*

(a) *For any integer $n \geqslant 2m+2$ there exist copies $X', Y' \subset E^n$ (of X, Y respectively) such that if $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$, then $E^n \setminus X'$ and $E^n \setminus Y'$ are homeomorphic.*

(b) *For any integer $n \geqslant 3m+3$ there exist copies $X', Y' \subset E^n$ (of X, Y respectively) such that if $E^n \setminus X'$ and $E^n \setminus Y'$ are homeomorphic, then* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

We remark that a similar result holds for embeddings of X and Y in the n-sphere S^n.

For prerequisites we will need some elementary facts concerning the piecewise-linear topology of E^n plus an isotopy extension theorem from [11]. We also use a characterization of dimension in terms of mappings onto polyhedra in E^n (see [14], p. 111). As for techniques we remark that part (a) of Theorem 1 is the most difficult part of the proof. Roughly the idea is to construct a sequence $\{h_i\}_{i=1}^{\infty}$ of homeomorphisms

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