On boundedness and discontinuity of additive functions

by

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Let f be a real-valued function defined on the n-dimensional Euclidean space R_n; f is said to be *additive* if it satisfies the functional equation

(1) $$f(x+y) = f(x)+f(y)$$

for each $x, y \in R_n$. R. Ger and Marek Kuczma [6] have introduced the following classes of sets: A set $T \subset R_n$ belongs to the class $\mathfrak{B}$ if and only if each additive function upper-bounded on T is continuous. A set $T \subset R_n$ belongs to the class $\mathfrak{C}$ if and only if each additive function bounded (bilaterally) on T is continuous. It is known that $\mathfrak{B} \neq \mathfrak{C}$, see [6] or [10].

There are known various conditions upon T which are sufficient for $T \in \mathfrak{B}$, resp. $T \in \mathfrak{C}$, and also conditions upon T which are necessary for $T \in \mathfrak{B}$, resp. $T \in \mathfrak{C}$. For example, if T has a positive inner Lebesgue measure, then $T \in \mathfrak{B}$, i.e. each additive function upper-bounded on T is continuous. This is the famous theorem of A. Ostrowski [18] generalized by Á. Császár [2] and S. Marcus [15]. Other results can be found in [8], [17], [14], [11], [12], [16], [4], [5], [6], [10], [9]. However, none of those results gives a condition which is both sufficient and necessary for $T \in \mathfrak{B}$, resp. $T \in \mathfrak{C}$. And the following is the problem of M. Kuczma [13]: give a characterization of the members of the classes $\mathfrak{B}$ and $\mathfrak{C}$. The present paper is devoted to this problem.

The set R_n can be interpreted as a vector space over the field Q of rational numbers, and each additive function $f\colon R_n \to R$ (R denotes the set R_1) as a morphism from the vector space R_n to the vector space R, since each such additive function is Q-linear, i.e. $f(\alpha x+\beta y) = \alpha f(x)+$ $+\beta f(y)$, for $\alpha, \beta \in Q$, $x, y \in R_n$, see [1]. Hence it is natural to apply some methods and results from vector analysis to the functional equation (1). Let E be a vector space over Q. Then each basis of E is referred to as a Hamel basis. A subset C of a vector space E is Q-convex if

$$(1-\alpha)C+\alpha C \subset C$$

whenever $\alpha \in Q$, $0 \leqslant \alpha \leqslant 1$. Let A be a subset of E and let x be a point from A. Then A is Q-radial at the point x if for each $y \in E$ there is a positive real c_y such that $x+\alpha y \in A$ whenever $|\alpha| < c_y$, $\alpha \in Q$.

Marcin E. Kuczma [10] has considered the subsets $T \subset R_n$ which are both Q-convex and Q-radial at some point. He has shown that such a set T belongs to $\mathfrak{B}$ if and only if T contains an open sphere. In the present paper there is proved a similar result with the class $\mathfrak{B}$ replaced by $\check{C}$ (see Theorem 3 below) and this result is a key to the characterization of the class $\mathcal{C}$ (see Theorem 4 below). However, a similar characterization of the class $\mathfrak{B}$ is impossible (see Remark 1 after Theorem 4). In the paper we use some ideas from the above-mentioned paper of Marcin E. Kuczma.

Finally, here are some remarks concerning the notation. In the sequel, the usual set-theoretic operations are denoted by $\bigcup$, $\bigcap$, $\setminus$ (difference), $\times$. The symbols $+, -, \cdot, \sum$, denote the algebraic operations; they may be applied also to sets, e.g. $A+B$ denotes the set of all elements of the form $a+b$, $a \in A$, $b \in B$. Rational numbers are always denoted by Greek letters.

Theorem 1, which follows presently is an analogon of the famous Hahn–Banach theorem on linear functionals, applied to vector spaces over Q. See also [10]. First we prove the following

LEMMA. *Let E be a vector space over Q, and let X and Y be subspaces of E such that $X \subset Y$ and X has codimension 1 in Y; let C be a Q-convex subset of E which is Q-radial at 0 and symmetric with respect to 0 (i.e. $C = -C$); finally, let $f\colon X \to R$ be an additive function such that $|f(x)| \leqslant 1$ for $x \in X \cap C$. Then there exists an additive function $g\colon Y \to R$ which is an extension of f and is such that $|g(x)| \leqslant 1$ for $x \in Y \cap C$.*

Proof. According to the supposition we may write

$$Y = X + Qy$$

where y is a point in $Y \setminus X$. Consider the following sets $U, V \subset X \times Q$:

$$U = \{(x, \xi)\colon x \in X,\ \xi > 0,\ (x-y)/\xi \in C\},$$

$$V = \{(x, \xi)\colon x \in X,\ \xi > 0,\ (x+y)/\xi \in C\}.$$

Since C is Q-radial at 0, for each $x \in X$ there is a number $\xi > 0$ such that $(x \pm y)/\xi \in C$. Hence U and V are non-empty sets.

Write

$$u = \sup\{f(x)-\xi\colon (x, \xi) \in U\},$$

$$u' = \inf\{f(x)+\xi\colon (x, \xi) \in U\},$$

$$v = \inf\{\xi - f(x)\colon (x, \xi) \in V\},$$

$$v' = \sup\{-\xi - f(x)\colon (x, \xi) \in V\}.$$

We are going to show that

$$u' \geqslant u, \quad v \geqslant v', \quad v \geqslant u, \quad \text{and} \quad u' \geqslant v'. \tag{2}$$

Suppose, on the contrary, that $u' < u$. Then there exist $(x_1, \xi_1) \in U$ and $(x_2, \xi_2) \in U$ such that $f(x_1)+\xi_1 < f(x_2)-\xi_2$; hence, by the Q-linearity of f,

$$f\left(\frac{x_2-x_1}{\xi_1+\xi_2}\right) > 1. \tag{3}$$

On the other hand, $(x_1-y)/\xi_1 \in C$, $(x_2-y)/\xi_2 \in C$. Since C is Q-convex and symmetric with respect to 0, we have

$$x = \frac{x_2-x_1}{\xi_1+\xi_2} = \frac{\xi_2}{\xi_1+\xi_2}\cdot\frac{x_2-y}{\xi_2} - \frac{\xi_1}{\xi_1+\xi_2}\cdot\frac{x_1-y}{\xi_1} \in C.$$

Thus $x \in X \cap C$, which implies $|f(x)| \leqslant 1$, contrary to (3).

Similarly, let $v' > v$. Then there exist $(z_1, \eta_1) \in V$, $(z_2, \eta_2) \in V$ such that $-\eta_1 - f(z_1) > \eta_2 - f(z_2)$, and hence

$$f(z) > 1, \tag{4}$$

where $z = (z_2-z_1)/(\eta_1+\eta_2)$. On the other hand, $(z_1+y)/\eta_1 \in C$, $(z_2+y)/\eta_2 \in C$; hence from the Q-convexity and symmetry of C, similarly to the preceding case, it follows that $z \in C$. Thus $|f(z)| \leqslant 1$, contrary to (4).

If $v < u$, then there exist $(w_1, \xi_1) \in V$, $(w_2, \xi_2) \in U$ such that $\xi_1 - f(w_1) < f(w_2) - \xi_2$, or equivalently,

$$f(w) > 1, \tag{5}$$

where $w = (w_1+w_2)/(\xi_1+\xi_2)$. On the other hand, $(w_1+y)/\xi_1 \in C$, $(w_2-y)/\xi_2 \in C$; hence, by the Q-convexity of C, $w \in C$, and consequently $f(w) \leqslant 1$, contrary to (5).

Finally, assume that $u' < v'$. Then there are $(t_1, \tau_1) \in U$, $(t_2, \tau_2) \in V$ such that $f(t_1)+\tau_1 < -f(t_2)-\tau_2$, so

$$f(t) < -1, \tag{6}$$

where $t = (t_1+t_2)/(\tau_1+\tau_2)$. On the other hand, $(t_1-y)/\tau_1 \in C$, $(t_2+y)/\tau_2 \in C$; hence, by the Q-convexity of C, $t \in C$, and consequently $f(t) \geqslant -1$, contrary to (6).

Thus (2) holds. Hence there exists a real number c such that $v \geqslant c \geqslant v'$ and $u' \geqslant c \geqslant u$. Define a function g by

$$g(x) = f(x) \quad \text{for } x \in X, \qquad g(y) = c,$$

and extend it by Q-linearity onto the whole of Y. It remains to verify that $|g(z)| \leqslant 1$, for each $z \in Y \cap C$.

Let $x+ay \in Y \cap C$, where $x \in X$; we may assume that $a \neq 0$. If $a > 0$, then $\left(\frac{x}{a}, \frac{1}{a}\right) \in V$; hence

$$\frac{1}{a}\big(1-f(x)\big) \geqslant v \geqslant c \quad \text{and} \quad \frac{1}{a}\big(-1-f(x)\big) \leqslant v' \leqslant c\,,$$

and so

$$-1 \leqslant g(x+ay) = f(x)+ac \leqslant 1\,.$$

If $a < 0$, then $\left(-\frac{x}{a}, -\frac{1}{a}\right) \in U$, whence

$$\frac{1}{a}\big(1-f(x)\big) \leqslant u \leqslant c \quad \text{and} \quad \frac{1}{a}\big(-1-f(x)\big) \geqslant u' \geqslant c\,,$$

and so again

$$-1 \leqslant g(x+ay) \leqslant 1\,.$$

Now we are able to formulate the following

THEOREM 1. *Let E be a vector space over Q, X a subspace of E, and C a Q-convex subset of E, Q-radial at 0 and symmetric with respect to 0. If $f\colon X \to R$ is an additive function which is bounded on the set $X \cap C$, then there exists an additive function $F\colon E \to R$ which is an extension of f and which is bounded on C.*

Proof. The proof of the theorem is based on the lemma proved above and on Zorn's lemma. It is much the same as the proof of the Hahn-Banach theorem and so we omit it. The reader is referred to [3], [19] or [10].

In Theorems 2 and 3, which follow, there are constructed certain discontinuous additive functions. Theorem 2 is devoted to functions defined on R and Theorem 3 is a generalization of Theorem 2 to functions defined on R_n.

THEOREM 2. *Let C be a Q-convex subset of the real line, Q-radial at 0 and symmetric with respect to 0. Then either C is an interval or there exists a discontinuous additive function $F\colon R \to R$ bounded (bilaterally) on C.*

Proof. Assume that C is not an interval. The set C is dense in itself, since if $z \in C$ then also each $az \in C$, for $a \in Q$, $|a| \leqslant 1$. From this and from the Q-convexity of C it follows that C contains no non-trivial interval. Hence there exist reals x and y such that

(7) $$x \notin C\,, \quad y \in C\,, \quad 0 < |x| < \tfrac{1}{3}|y|\,.$$

The proof of Theorem 2 is based on Theorem 1. We construct a discontinuous additive function g on the set $Qx+Qy$, bounded on the set $(Qx+Qy) \cap C$, and then we extend g to the desired function F.

For each integer $k \geqslant 2$ define a set $A_k \subset Q\times Q$ by

$$A_k = \{(a, \beta);\ ax+\beta y \in C,\ a, \beta \in Q,\ a > k\}\,.$$

It is easy to verify that if $(a, \beta) \in A_k$, then $\beta \neq 0$; otherwise there would exist some $a > k > 1$ such that $ax \in C$ and hence, by the Q-convexity of C, $\frac{1}{a}(ax) = x \in C$, contrary to (7). Thus we can define the sets $B_k \subset Q$ as follows:

$$B_k = \left\{\frac{a}{\beta};\ (a, \beta) \in A_k\right\}.$$

If there exists a k such that $A_k = \emptyset$, define an additive function g as follows: $g(ax+\beta y) = a$. By the symmetry of C, if $(a'x+\beta'y) \in C$, then $|a'| \leqslant k$. Hence g is bounded on $C \cap (Qx+Qy)$. Now Theorem 1 guarantees the existence of an additive function F, bounded on C and such that F is an extension of g. Clearly F is discontinuous since, for each $a \in Q$, $F(ay) = g(ay) = 0$.

Thus we may assume that $A_k \neq \emptyset$, for each k. We show that all numbers from B_k have the same sign. Indeed, if there are positive rationals $a_1, a_2, \beta_1, \beta_2$, such that $a_1x+\beta_1y \in C$, $a_2x-\beta_2y \in C$, $a_1 > k \geqslant 2$, $a_2 > k \geqslant 2$, and, say, $\beta_1 \geqslant \beta_2$ (in the case of $\beta_1 \leqslant \beta_2$ the proof is similar), then by the Q-convexity and symmetry of C we have

$$\frac{1}{2}\left(\frac{\beta_2}{\beta_1}(a_1x+\beta_1y)+(a_2x-\beta_2y)\right) = \frac{1}{2}\left(\frac{\beta_2 a_1}{\beta_1}+a_2\right)x \in C\,,$$

whence $\frac{\beta_2 a_1}{\beta_1}+a_2 < 2$; on the other hand, $a_2 > 2$, $\frac{\beta_2 a_1}{\beta_1} > 0$, and so $\frac{\beta_2 a_1}{\beta_1}+a_2 > 2$ — a contradiction.

In the sequel we may assume without loss of generality that each set B_k contains positive numbers (otherwise it suffices to replace y by $-y$ in (7)).

We show that, for each $(a, \beta) \in A_k$,

(8) $$\frac{a}{\beta} < 2\,.$$

Assume, on the contrary, that $\frac{a}{\beta} \geqslant 2$ for some $(a, \beta) \in A_k$. If $\beta \leqslant 1$, then $\beta y \in C$ and hence, by the Q-convexity and symmetry of C, $\frac{1}{2}\big((ax+\beta y)-\beta y\big) = \left(\frac{a}{2}\right)x \in C$; but $a > n \geqslant 2$, whence $\frac{2}{a}\left(\frac{a}{2}x\right) = x \in C$, contrary to (7). If

$\beta > 1$, then $\frac{1}{2}\left(\frac{1}{\beta}(\alpha x+\beta y)-y\right) = \left(\frac{1}{2}\frac{\alpha}{\beta}\right)x \in C$, which again implies $x \in C$ — a contradiction.

Since, for each $k \geqslant 2$, $B_k \supset B_{k+1}$, there exists a $\lim_{k\to\infty}(\sup B_k) = c$. In view of (8), $0 \leqslant c \leqslant 2$. Define an additive function g on the set $Qx+Qy$ as follows: $g(\alpha x+\beta y) = \alpha-\beta c$. We show that g is bounded on $(Qx+Qy)\cap C$. Let $(\alpha x+\beta y) \in C$. Since C is symmetric, it suffices to consider the case of $\beta > 0$. Let $\varepsilon > 0$. For each integer $k \geqslant 2$ choose a pair $(\alpha_k, \beta_k) \in A_k$ such that $\left|\frac{\alpha_k}{\beta_k}-c\right| < \varepsilon$. Since $\lim_{k\to\infty}\alpha_k = +\infty$, from (8) we have $\lim_{k\to\infty}\beta_k = +\infty$. Let m be an integer such that $\beta_m > \beta$. By the Q-convexity and symmetry of C we have $\frac{\beta}{\beta_m}(\alpha_m x+\beta_m y) = \left(\frac{\beta\alpha_m}{\beta_m}x+\beta y\right) = \omega x+\beta y \in C$ and

$$-\varepsilon+\frac{\omega}{\beta} < c < \varepsilon+\frac{\omega}{\beta}. \tag{9}$$

By the Q-convexity and symmetry of C we have $\frac{1}{2}\big((\alpha x+\beta y)-(\omega x+\beta y)\big) = \frac{\alpha-\omega}{2}x \in C$, whence

$$-2 < \alpha-\omega < 2. \tag{10}$$

From (9) and (10) it follows that

$$g(\alpha x+\beta y) = \alpha-\beta c < \alpha+\beta\varepsilon-\omega < 2+\beta\varepsilon,$$

and similarly

$$g(\alpha x+\beta y) > \alpha-\beta\varepsilon-\omega > -2-\beta\varepsilon.$$

Since ε is arbitrary, we get

$$-2 \leqslant g(\alpha x+\beta y) \leqslant 2.$$

Thus g is bounded on the set $(Qx+Qy)\cap C$.

Finally, we show that g is discontinuous on $Qx+Qy$. Let $\eta > 0$. Choose a positive rational number $\delta < 2$ such that $|\delta-c| < \eta$. For each β such that $|\beta| < 1/\eta$ we have $|g(\beta\delta x+\beta y)| = |\beta\delta-\beta c| = |\beta||\delta-c| < 1/\eta\cdot\eta = 1$. On the other hand, since $0 < \delta < 2$, we get from (7) $|\delta x+y| > |y|/3$. Thus the set of those z from $Qx+Qy$ for which $|g(z)| < 1$ is dense in the interval $(-|y|/3\eta, |y|/3\eta)$. Since η is arbitrary, we conclude that there is a subset A of $Qx+Qy$ which is dense in R and is such that $|g(z)| < 1$ for $z \in A$. But $g(x) = 1$, whence g is a non-zero additive function. From this it follows that g is discontinuous on $Qx+Qy$.

Now Theorem 1 applied to the function g guarantees the existence of a desired extension F of g, q.e.d.

THEOREM 3. *Let C be a Q-convex subset of the n-dimensional Euclidean space R_n, Q-radial at 0, and symmetric with respect to 0. Then either C contains a sphere centred at 0 or there exists a discontinuous additive function F: $R_n \to R$ bounded on C.*

Proof. Let $e_1, e_2, \ldots, e_n$ be the usual orthonormal basis for R_n. If for each i, $1 \leqslant i \leqslant n$, $C_i = R\cdot e_i \cap C$ is an interval (which is, by the symmetry of C, centred at 0) let d be the length of the smallest interval C_i. Let $(x_1, x_2, \ldots, x_n)$, where $x_i \in R$, be a point from R_n such that $|x_i| < d/n$, for each i. Then $nx_i\cdot e_i \in C_i \subset C$ and hence, by the Q-convexity of C,

$$(x_1, x_2, \ldots, x_n) = \frac{1}{n}(nx_1\cdot e_1+nx_2\cdot e_2+\ldots+nx_n\cdot e_n) \in C,$$

and consequently C contains a sphere.

Thus we may assume that, for some i, C_i is not an interval. In this case, by Theorem 2, there exists a discontinuous additive function f: $R\cdot e_i \to R$, bounded on $R\cdot e_i \cap C = C_i$, which can be extended, by Theorem 1, to a discontinuous additive function F: $R_n \to R$ bounded on C, q.e.d.

In the next sections we shall use the following notation: If T is a subset of R_n, let $Q(T)$ denote the Q-convex hull of T.

Now we are able to prove the main result.

THEOREM 4. *Let T be a subset of the n-dimensional Euclidean space R_n. Then each additive function f: $R_n \to R$ bounded on T is continuous if and only if the Q-convex hull of $T-T$ contains a sphere.*

In other words, $T \in \mathfrak{C}$ if and only if $Q(T-T)$ contains a ball.

Proof. Assume that $Q(T-T)$ contains a certain sphere. Let f be bounded on T, i.e. let $|f(x)| \leqslant M$ for $x \in T$. Then f is also bounded on $T-T$ with the bounding constant $2M$. If $z \in Q(T-T)$, then $z = a_1x_1+$ $+\ldots+a_mx_m$, where $x_i \in T-T$, $a_i \in Q$, $a_i > 0$, $a_1+\ldots+a_m = 1$. Hence

$$|f(z)| = |a_1f(x_1)+\ldots+a_mf(x_m)| \leqslant a_1|f(x_1)|+\ldots+a_m|f(x_m)| \leqslant (a_1+\ldots+a_m)2M = 2M;$$

thus f is bounded on the set $Q(T-T)$ of positive inner Lebesgue measure and consequently f is continuous (see [15]).

Now assume that $Q(T-T)$ contains no sphere. Clearly $Q(T-T)$ is Q-convex and symmetric with respect to 0. If $Q(T-T)$ is also Q-radial at 0, then there exists, by Theorem 3, a discontinuous additive function f: $R_n \to R$, bounded on $Q(T-T)$. Let a be a fixed point from T. Since $T-a \subset Q(T-T)$, we conclude that f is bounded on $T-a$ and consequently f is also bounded on T.

Hence it remains to consider the case where $Q(T-T)$ is not Q-radial at 0. In this case $Q(T-T)$ cannot contain a Hamel basis. Indeed, if H is a Hamel basis contained in $Q(T-T)$, then each element $x \in R_n$ can be written in the form $x \in \sum a_i b_i$ (finite sum) where $b_i \in H \cup -H \subset Q(T-T)$, $a_i > 0$. From the Q-convexity of $Q(T-T)$ it follows that $x/\sum a_i \in Q(T-T)$ and consequently $ax \in Q(T-T)$ for each positive $a \leqslant \sum a_i$. Thus $Q(T-T)$ is Q-radial at 0 — a contradiction. Hence $Q(T-T)$ does not contain any Hamel basis and so the vector subspace (over Q) spanned by $Q(T-T)$ cannot be the whole R_n. To finish our proof we use the following result of R. Ger and Marek Kuczma [6]: If $A \in \mathfrak{C}$, then the vector subspace spanned by A is R_n. Hence $Q(T-T) \notin \mathfrak{C}$ and, similarly to the preceding case, we conclude that also $T \notin \mathfrak{C}$, q.e.d.

Remark 1. In connection with Theorem 4 one may expect that the sets from the class $\mathfrak{B}$ can be characterized as follows: A set T is in $\mathfrak{B}$ if and only if the Q-convex hull of T contains a sphere, or at least has the positive inner Lebesgue measure. However, this hypothesis is false as is shown on an example by Marcin E. Kuczma [10].

Remark 2. A set $A \subset R_n$ is called to be *midpoint convex* if for each $x, y \in A$, $\frac{1}{2}(x+y) \in A$. R. Ger and Marek Kuczma [6] have proved the following result: Let $T \subset R_n$, and let $J(T)$ denote the midpoint convex hull of T. If the set $J(T)-J(T)$ has a positive inner Lebesgue measure, then $T \in \mathfrak{C}$. The authors have conjectured that this condition is not necessary for $T \in \mathfrak{C}$. Their conjecture is true, as can be shown on a rather complicated example. In fact, there exists a midpoint convex symmetric set $T \in \mathfrak{C}$, which has the zero inner measure. Consequently, Q-convexity in Theorem 4 cannot be replaced by midpoint convexity.

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