On boundedness and discontinuity of additive functions

by

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Let \( f \) be a real-valued function defined on the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \); \( f \) is said to be additive if it satisfies the functional equation

\[
    f(x + y) = f(x) + f(y)
\]

for each \( x, y \in \mathbb{R}^n \). R. Ger and Marek Kuczma [6] have introduced the following classes of sets: A set \( T \subset \mathbb{R}^n \) belongs to the class \( \mathcal{S} \) if and only if each additive function upper-bounded on \( T \) is continuous. A set \( T \subset \mathbb{R}^n \) belongs to the class \( \mathcal{C} \) if and only if each additive function bounded (bilateral) on \( T \) is continuous. It is known that \( \mathcal{S} \not= \mathcal{C} \), see [6] or [10].

There are known various conditions upon \( T \) which are sufficient for \( T \in \mathcal{S} \), resp. \( T \in \mathcal{C} \), and also conditions upon \( T \) which are necessary for \( T \in \mathcal{S} \), resp. \( T \in \mathcal{C} \). For example, if \( T \) has a positive inner Lebesgue measure, then \( T \in \mathcal{S} \), i.e. each additive function upper-bounded on \( T \) is continuous. This is the famous theorem of A. Ostrowski [18] generalized by Á. Császár [2] and S. Marcus [15]. Other results can be found in [8], [17], [14], [11], [12], [16], [4], [5], [9], [10], [9]. However, none of those results give a condition which is both sufficient and necessary for \( T \in \mathcal{S} \), resp. \( T \in \mathcal{C} \). And the following is the problem of M. Kuczma [13]: give a characterization of the members of the classes \( \mathcal{S} \) and \( \mathcal{C} \). The present paper is devoted to this problem.

The set \( \mathbb{R}^n \) can be interpreted as a vector space over the field \( \mathbb{Q} \) of rational numbers, and each additive function \( f: \mathbb{R}^n \to \mathbb{R} \) (\( \mathbb{R} \) denotes the set \( \mathbb{R} \)) as a morphism from the vector space \( \mathbb{R}^n \) to the vector space \( \mathbb{R} \), since each such additive function is \( \mathbb{Q} \)-linear, i.e. \( f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \) for \( \alpha, \beta \in \mathbb{Q}, \ x, y \in \mathbb{R}^n \), see [12]. Hence it is natural to apply some methods and results from vector analysis to the functional equation (1).

Let \( E \) be a vector space over \( \mathbb{Q} \). Then each basis of \( E \) is referred to as a Hamel basis. A subset \( C \) of a vector space \( E \) is \( \mathbb{Q} \)-convex if

\[
    (1 - \alpha) C + \alpha C \subset C
\]

whenever \( \alpha \in \mathbb{Q}, \ 0 \leq \alpha \leq 1 \). Let \( A \) be a subset of \( E \) and let \( x \) be a point

\[ 17 = \text{Fundamenta Mathematicae T. LXXVI} \]
from $A$. Then $A$ is $Q$-radial at the point $x$ if for each $y \in E$ there is a positive real $c_y$ such that $x + ay \in A$ whenever $|a| < c_y$, $a \in Q$.

Marcin E. Kuczma [10] has considered the subsets $T \subset R_n$ which are both $Q$-convex and $Q$-radial at some point. He has shown that such a set $T$ belongs to $S$ if and only if $T$ contains an open sphere. In the present paper there is proved a similar result with the class $S$ replaced by $C$ (see Theorem 3 below) and this result is a key to the characterization of the class $C$ (see Theorem 4 below). However, a similar characterization of the class $S$ is impossible (see Remark 1 after Theorem 4). In the paper we use some ideas from the above-mentioned paper of Marcin E. Kuczma.

Finally, here are some remarks concerning the notation. In the sequel, the usual set-theoretic operations are denoted by $\cup$, $\cap$, $\setminus$ (difference), $\times$. The symbols $\vee$, $\wedge$, $\veevee$, $\wedge\wedge$ denote the algebraic operations; they may be applied also to sets, e.g. $A + B$ denotes the set of all elements of the form $a + b$, $a \in A$, $b \in B$. Rational numbers are always denoted by Greek letters.

Theorem 1, which follows presently is an analogon of the famous Hahn--Banach theorem on linear functionals, applied to vector spaces over $Q$. See also [10]. First we prove the following

**LEMMA.** Let $E$ be a vector space over $Q$, and let $X$ and $Y$ be subspaces of $E$ such that $X \subset Y$ and $X$ has codimension 1 in $Y$; let $C$ be a $Q$-convex subset of $E$ which is $Q$-radial at 0 and symmetric with respect to 0 (i.e. $C = -C$); finally, let $f : X + E$ be an additive function such that $|f(x)| < 1$ for $x \in X \cap C$. Then there exists an additive function $g : X \to E$ which is an extension of $f$ and is such that $|g(x)| < 1$ for $x \in X \cap C$.

**Proof.** According to the supposition we may write

$$X = X + Qy$$

where $y$ is a point in $Y \setminus X$. Consider the following sets $U, V \subset X \times Q$:

$$U = \{(x, \xi) : x \in X, \xi > 0, (x-y)\xi \in C\},$$

$$V = \{(x, \xi) : x \in X, \xi > 0, (x+y)\xi \in C\}.$$

Since $C$ is $Q$-radial at 0, for each $x \in X$ there is a number $\xi > 0$ such that $(x+\xi)\xi \in C$. Hence $U$ and $V$ are non-empty sets.

Write

$$u' = \sup\{f(x) - \xi : (x, \xi) \in U\},$$

$$u' = \inf\{f(x) + \xi : (x, \xi) \in U\},$$

$$v = \inf\{-f(x) + \xi : (x, \xi) \in V\},$$

$$v' = \sup\{-f(x) : (x, \xi) \in V\}.$$

We are going to show that

(2) $u' > u, v > v', v \geq u$, and $u' > v'$.

Suppose, on the contrary, that $u' < u$. Then there exist $(x_1, \xi_1) \in U$ and $(x_2, \xi_2) \in U$ such that $f(x_1) + \xi_1 < f(x_2) - \xi_2$; hence, by the $Q$-linearity of $f$,

$$\frac{\xi_1}{\xi_1 + \xi_2} < 1.$$

On the other hand, $(x_1 - y)/\xi_1 \in C$, $(x_2 - y)/\xi_2 \in C$. Since $C$ is $Q$-convex and symmetric with respect to 0, we have

$$z = \frac{\xi_1 - \xi_2}{\xi_1 + \xi_2} = \frac{\xi_1}{\xi_1 + \xi_2} = \frac{\xi_2}{\xi_1 + \xi_2} = \frac{\xi_1 - \xi_2}{\xi_1} \in C.$$

Thus $z \in X \cap C$, which implies $|f(z)| < 1$, contrary to (3).

Similarly, let $v' > v$. Then there exist $(x_1, \eta_1) \in V$, $(x_2, \eta_2) \in V$ such that $-\eta_1 - f(x_2) > \eta_2 - f(x_1)$, and hence

(4) $f(z) > 1$, where $z = (x_2 - x_1)/(\eta_1 + \eta_2)$. On the other hand, $(x_1 + y)/\eta_1 \in C$, $(x_2 + y)/\eta_2 \in C$; hence, by the $Q$-convexity and symmetry of $C$, similarly to the preceding case, it follows that $x \in C$. Thus $|f(x)| < 1$, contrary to (4).

If $v < u$, then there exist $(x_1, \xi_1) \in V$, $(x_2, \xi_2) \in U$ such that $\xi_2 - f(x_2) < f(x_1) - \xi_1$, or equivalently,

(5) $f(w) > 1$, where $w = (x_1 + x_2)/(\xi_1 + \xi_2)$. On the other hand, $(x_1 + y)/\xi_1 \in C$, $(x_2 + y)/\xi_2 \in C$; hence, by the $Q$-convexity of $C$, $w \in C$, and consequently $f(w) < 1$, contrary to (5).

Finally, assume that $u' < v'$. Then there are $(t_1, \tau_1) \in U$, $(t_2, \tau_2) \in V$ such that $f(t_1) + \tau_1 < f(t_2) - \tau_2$, so

(6) $f(t) < -1$, where $t = (t_1 + t_2)/(\tau_1 + \tau_2)$. On the other hand, $(t_1 - y)/\tau_1 \in C$, $(t_2 - y)/\tau_2 \in C$; hence, by the $Q$-convexity of $C$, $t \in C$, and consequently $f(t) \geq -1$, contrary to (6).

Thus (2) holds. Hence there exists a real number $c$ such that $v > c \geq u'$ and $u' > c \geq u$. Define a function $g$ by

$$g(x) = f(x) \quad \text{for } x \in X, \quad g(y) = c,$$

and extend it by $Q$-linearity onto the whole of $X$. It remains to verify that $|g(x)| 

\leq 1$, for each $x \in X \cap C$.
Let \( x + ay \in Y \cap C \), where \( x \in X \); we may assume that \( a \neq 0 \). If \( a > 0 \), then \( \frac{|x|}{|a|} \in V \); hence

\[
\frac{1}{a} (1 - f(x)) > v \Rightarrow v > c \quad \text{and} \quad \frac{1}{a} (1 - f(x)) \leq v' \leq c,
\]

and so

\[
-1 \leq g(x + ay) = f(x) + ac \leq 1.
\]

If \( a < 0 \), then \( \left( \frac{x}{a}, \frac{1}{a} \right) \in U \), whence

\[
\frac{1}{a} (1 - f(x)) \leq u \leq c \quad \text{and} \quad \frac{1}{a} (1 - f(x)) \geq u' \geq c,
\]

and so again

\[
-1 \leq g(x + ay) \leq 1.
\]

Now we are able to formulate the following

**Theorem 1.** Let \( E \) be a vector space over \( Q \), \( X \) a subspace of \( E \), and \( C \) a \( Q \)-convex subset of \( E \), \( Q \)-radial at 0 and symmetric with respect to 0. If \( f \colon X \to R \) is an additive function which is bounded on the set \( X \cap C \), then there exists an additive function \( F \colon E \to R \) which is an extension of \( f \) and which is bounded on \( C \).

**Proof.** The proof of the theorem is based on the lemma proved above and on Zorn’s lemma. It is much the same as the proof of the Hahn-Banach theorem and so we omit it. The reader is referred to [3], [9], [19] or [29].

In Theorems 2 and 3, which follow, there are constructed certain discontinuous additive functions. Theorem 2 is devoted to functions defined on \( E \) and Theorem 3 is a generalization of Theorem 2 to functions defined on \( R_a \).

**Theorem 2.** Let \( C \) be a \( Q \)-convex subset of the real line, \( Q \)-radial at 0 and symmetric with respect to 0. Then either \( C \) is an interval or there exists a discontinuous additive function \( F \colon R \to R \) bounded (bilaterally) on \( C \).

**Proof.** Assume that \( C \) is not an interval. The set \( C \) is dense in itself, since if \( x \in C \) then also each \( ax + ay \in C \), for \( a \in Q \), \( |a| < 1 \). From this and from the \( Q \)-convexity of \( C \) it follows that \( C \) contains no non-trivial interval. Hence there exist reals \( x \) and \( y \) such that:

\[
(7) \quad x \in C, \quad y \in C, \quad 0 < |x| < \frac{1}{2}|y|.
\]

The proof of Theorem 2 is based on Theorem 1. We construct a discontinuous additive function \( g \) on the set \( Qx + Qy \), bounded on the set \( (Qx + Qy) \cap C \), and then we extend \( g \) to the desired function \( F \).

For each integer \( k \geq 2 \) define a set \( A_k \subseteq Q \times Q \) by

\[
A_k = \{(x, y) : ax + by \in C, a, b \in Q, a > b\}.
\]

It is easy to verify that if \( (a, b) \in A_k \), then \( b > 0 \); otherwise there would exist some \( c \geq k > 1 \) such that \( ax \in C \) and hence, by the \( Q \)-convexity of \( C \), \( \frac{1}{a} (ax) = x \in C \), contrary to (7). Thus we can define the sets \( B_k \subseteq Q \) as follows:

\[
B_k = \left\{a \in Q : (a, b) \in A_k\right\}.
\]

If there exists a \( k \) such that \( A_k = \emptyset \), define an additive function \( g \) as follows: \( g(ax + by) = a \). By the symmetry of \( C \), if \( (ax + by) \in C \), then \( |a| < k \). Hence \( g \) is bounded on \( C \cap (Qx + Qy) \). Now Theorem 1 guarantees the existence of an additive function \( F \), bounded on \( C \) and such that \( F \) is an extension of \( g \). Clearly \( F \) is discontinuous since, for each \( a \in Q \), \( F(ax) = g(ax) = 0 \).

Thus we may assume that \( A_k = \emptyset \), for each \( k \). We show that all numbers from \( B_k \) have the same sign. Indeed, if there are positive rationals \( a_1, a_2, b_1, b_2 \) such that \( a_1 x + b_1 y \in C \), \( a_2 x - b_2 y \in C \), \( a_1 > k \geq 2 \), \( a_2 > k \geq 2 \), and, say, \( b_2 > b_1 \) (in the case of \( b_1 = b_2 \), the proof is similar), then by the \( Q \)-convexity and symmetry of \( C \) we have

\[
\frac{1}{2} \left( \begin{array}{c} b_1 \\ a_1 \\ a_2 \end{array} \right) (a_1 x + b_1 y) + \left( \begin{array}{c} b_2 \\ a_2 \end{array} \right) (a_2 x - b_2 y) = \frac{1}{2} \left( \begin{array}{c} b_1 a_1 + a_2 \\ b_1 \\ b_1 \end{array} \right) x + a_2 y \in C,
\]

whence \( b_2 a_1 + a_2 < 2 \); on the other hand, \( a_2 > 2 \), \( b_2 a_1 > 0 \), and so

\[
\frac{b_2 a_1}{b_1} + a_2 < 2 \quad \text{a contradiction.}
\]

In the sequel we may assume without loss of generality that each set \( B_k \) contains positive numbers (otherwise it suffices to replace \( y \) by \( -y \) in (7)).

We show that, for each \( (a, b) \in A_k \),

\[
(8) \quad \frac{a}{b} < 2.
\]

Assume, on the contrary, that \( \frac{a}{b} \geq 2 \) for some \( (a, b) \in A_k \). If \( b \leq 1 \), then \( \beta y \in C \) and hence, by the \( Q \)-convexity and symmetry of \( C \),

\[
\frac{a}{b} (ax + by) = a \in C,
\]

contrary to (7). If
\[ \beta > 1, \text{ then } \frac{1}{\beta} (ax + \beta y) - y = \left( \frac{a}{\beta} \right) x \in C, \text{ which again implies } x \in C. \]

A contradiction.

Since, for each \( k \geq 2 \), \( B_k \supset B_{k+1} \), there exists a \( \lim_{k \to \infty} (\sup B_k) = c. \)

In view of (8), \( 0 < c \leq 2 \). Define an additive function \( g \) on the set \( Q_x + Q_y \) as follows: \( g(ax + \beta y) = a - \beta c. \) We show that \( g \) is bounded on \( (Q_x + Q_y) \cap C \).

Let \( (ax + \beta y) \in C \). Since \( C \) is symmetric, it suffices to consider the case of \( \beta > 0 \). Let \( \epsilon > 0 \). For each integer \( k \geq 2 \) choose a pair \( (a_k, \beta_k) \in A_k \)

such that \( \left| \frac{a_k}{\beta_k} - c \right| < \epsilon \). Since \( \lim a_k = +\infty \), from (8) we have \( \lim \beta_k = +\infty \).

Let us be an integer such that \( \beta_m > \beta \). By the \( Q \)-convexity and symmetry of \( C \) we have

\[ \frac{1}{\beta_m} (a_m ax + \beta_m y) = \left( \frac{a_m}{\beta_m} \right) x + \beta y \in C \text{ and} \]

\[ -\epsilon < \frac{a_m}{\beta_m} < c < \frac{a_m}{\beta_m} + \frac{\epsilon}{\beta} \]

By the \( Q \)-convexity and symmetry of \( C \) we have

\[ \frac{1}{2} (ax + \beta y) - (ax + \beta y) = \frac{a - \omega}{2} x \in C, \text{ whence} \]

\[ -2 < a - \omega < 2. \]

From (9) and (10) it follows that

\[ g(ax + \beta y) = a - \beta c < a + \beta e - \omega < 2 + \beta e, \]

and similarly

\[ g(ax + \beta y) > a - \beta e - \omega > -2 - \beta e. \]

Since \( \omega \) is arbitrary, we get

\[ -2 \leq g(ax + \beta y) \leq 2. \]

Thus \( g \) is bounded on \( Q_x + Q_y \) \( \cap C \).

Finally, we show that \( g \) is discontinuous on \( Q_x + Q_y \). Let \( \eta > 0 \). Choose a positive rational number \( \delta < 2 \) such that \( |\beta - c| < \eta \). For each \( \beta \)

such that \( |\beta - 1/\eta| \) we have

\[ |g(\beta ax + \beta y) - g(ax + \beta y)| = |\beta - 1/\eta| = |\beta - c| < 1/\eta \leq 1. \]

On the other hand, since \( 0 < \delta < 2 \), we get from (7) \( |\delta x + y| > |y|/3 \). Thus the set of those \( z \) from \( Q_x + Q_y \) for which \( |g(z)| < 1 \) is dense in the interval \( (-|y|/3\eta, |y|/3\eta) \). Since \( \eta \) is arbitrary, we conclude that there is a subset \( A \) of \( Q_x + Q_y \) which is dense in \( R \) and is such that \( |g(z)| < 1 \) for \( z \in A \).

But \( g(z) = 1 \), whence \( g \) is a non-zero additive function. From this it follows that \( g \) is discontinuous on \( Q_x + Q_y \).

Now Theorem 1 applied to the function \( g \) guarantees the existence of a desired extension \( F \) of \( g \), q.e.d.

**Theorem 3.** Let \( C \) be a \( Q \)-convex subset of the \( n \)-dimensional Euclidean space \( R_n \), \( Q \)-radial at 0, and symmetric with respect to 0. Then either \( C \) contains a sphere centred at 0 or there exists a discontinuous additive function \( F: R_n \to R \) bounded on \( C \).

Proof. Let \( e_1, \ldots, e_n \) be the usual orthonormal basis for \( R_n \). If for each \( i, 1 \leq i \leq n, \ C_i = R \cdot e_i \cap C \) is an interval (which is, by the symmetry of \( C \), centred at 0) let \( d \) be the length of the smallest interval \( C_i \). Let \( \{x_1, x_2, \ldots, x_n\} \), where \( x_i \in R_i \) be a point from \( R_n \) such that \( |x_i| < d/2 \), for each \( i \). Then \( n e_1, e_2, \ldots, e_n \in C \subset C \) and hence, by the \( Q \)-convexity of \( C \),

\[ (x_1, x_2, \ldots, x_n) = \frac{1}{n} (n a_1 e_1 + n a_2 e_2 + \ldots + n a_n e_n) \in C, \]

and consequently \( C \) contains a sphere.

Thus we may assume that, for some \( i, C_i \) is not an interval. In this case, by Theorem 2, there exists a discontinuous additive function \( f: R \cdot e_i \to R \), bounded on \( R \cdot e_i \cap C \subset C \), which can be extended, by Theorem 1, to a discontinuous additive function \( F: R_n \to R \) bounded on \( C \), q.e.d.

In the next sections we shall use the following notation: If \( T \) is a subset of \( R_n \), let \( Q(T) \) denote the \( Q \)-convex hull of \( T \).

Now we are able to prove the main result.

**Theorem 4.** Let \( T \) be a subset of the \( n \)-dimensional Euclidean space \( R_n \).

Then each additive function \( f: R_n \to R \) bounded on \( T \) is continuous if and only if the \( Q \)-convex hull of \( T - T \) contains a sphere.

In other words, \( T \subset C \) if and only if \( Q(T - T) \) contains a ball.

Proof. Assume that \( Q(T - T) \) contains a certain sphere. Let \( f \) be bounded on \( T \), i.e., let \( |f(x)| \leq M \) for \( x \in T \). Then \( f \) is also bounded on \( T - T \) with the bounding constant \( 2M \). If \( x \in Q(T - T) \), then \( x = a_1 e_1 + \ldots + a_n e_n \), where \( e_i \in T - T, a_i \in Q, a_i > 0, a_1 + \ldots + a_n = 1. \)

\[ |f(x)| = |a_1 f(x_1) + \ldots + a_n f(x_n)| \leq a_1 |f(x_1)| + \ldots + a_n |f(x_n)| \]

\[ \leq (a_1 + \ldots + a_n) 2M = 2M; \]

thus \( f \) is bounded on the set \( Q(T - T) \) of positive inner Lebesgue measure and consequently \( f \) is continuous (see [15]).

Now assume that \( Q(T - T) \) contains no sphere. Clearly \( Q(T - T) \) is \( Q \)-convex and symmetric with respect to 0. If \( Q(T - T) \) is also \( Q \)-radial at 0, then there exists, by Theorem 3, a discontinuous additive function \( f: R_n \to R \), bounded on \( Q(T - T) \). Let \( a \) be a fixed point from \( T \). Since \( T - a \subset Q(T - T) \), we conclude that \( f \) is bounded on \( T - a \) and consequently \( f \) is also bounded on \( T \).

Hence it remains to consider the case where \( Q(T - T) \) is not \( Q \)-radial at 0. In this case \( Q(T - T) \) cannot contain a Hamel basis. Indeed, if \( R \) is
a Hamel basis contained in \( Q(T - T) \), then each element \( x \in R_a \) can be written in the form \( x = \sum a_i h_i \) (finite sum) where \( h_i \in H \) and \( -H \subset C \subset Q(T - T) \).

From the \( Q \)-convexity of \( Q(T - T) \) it follows that \( a \sum a_i \in Q(T - T) \) and consequently \( a \in Q(T - T) \) for each positive \( a \leq \sum a_i \). Thus \( Q(T - T) \) is \( Q \)-radial at \( 0 \) — a contradiction. Hence \( Q(T - T) \) does not contain any Hamel basis and so the vector subspace (over \( Q \)) spanned by \( Q(T - T) \) cannot be the whole \( R_a \). To finish our proof we use the following result of E. Ger and Marek Kuczma [6]: If \( A \in C \), then the vector subspace spanned by \( A \) is \( R_a \). Hence \( Q(T - T) \in C \) and, similarly to the preceding case, we conclude that also \( T \not\in C \), q.e.d.

Remark 1. In connection with Theorem 4 one may expect that the sets from the class \( S \) can be characterized as follows: A set \( T \) is in \( S \) if and only if the \( Q \)-convex hull of \( T \) contains a sphere, or at least has the positive inner Lebesgue measure. However, this hypothesis is false as is shown on an example by Marcin E. Kuczma [10].

Remark 2. A set \( A \subset R_a \) is called to be midpoint convex if for each \( x, y \in A \), \( \frac{1}{2}(x + y) \in A \). E. Ger and Marek Kuczma [6] have proved the following result: Let \( T \subset R_a \), and let \( J(T) \) denote the midpoint convex hull of \( T \). If the set \( J(T) - J(T) \) has a positive inner Lebesgue measure, then \( T \in C \). The authors have conjectured that this condition is not necessary for \( T \in C \). Their conjecture is true, as can be shown on a rather complicated example. In fact, there exists a midpoint convex symmetric set \( T \in C \), which has the zero inner measure. Consequently, \( Q \)-convexity in Theorem 4 cannot be replaced by midpoint convexity.

References