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Collectionwise normality and the extension of functions on product spaces

by

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1. Introduction and preliminary results. Let S be a nonempty subset of a topological space X. The subset S is said to be P-*embedded* in X if every continuous pseudometric on S extends to a continuous pseudometric on X. The subset S is C-*embedded* (respectively C^*-*embedded*) in X if every continuous (respectively bounded continuous) real valeud function on S extends to a continuous (respectively bounded continuous) real valued function on X. It is clear that every C-embedded subset is a C^*-embedded subset; moreover every P-embedded subset is a C-embedded one (see Theorem 2.4 of [7]). The concept of P-embedding characterizes collectionwise normal spaces in the same way as C-embedding (and also C^*-embedding) characterize a normal spaces. Specifically a topological space X is collectionwise normal if and only if every closed subset of X is P-embedded in X (see [12]).

Since a pseudometric on a space X is a function on the product set $X \times X$, it is of interest to relate the extension of pseudometrics to the extension of functions on $X \times X$ (without the triangle inequality). In [1] we showed that a subset S is P-embedded in X if and only if every continuous function from S into a bounded, closed convex subset of a Banach space extends to a continuous function on X with values in the convex subset. Using results developed in [1] to demonstrate this result, we also showed in [1] that if L is any Fréchet space, then every uniformly continuous function from S into L can be extended to a continuous function on X. Also uniform continuity of the extended function is shown not to be attainable.

We now turn our attention to relating P-embedding to the extension of functions from product sets. By utilizing results from [1] we show that a subspace S is P-embedded in the Tichonov space X if and only if for all locally compact hemicompact Hausdorff spaces A, the product set $S \times A$ is C^*-embedded in the product space $X \times A$ if and only if the product set $S\times\beta S$ is C^*-embedded in the product space $X\times\beta S$, where βS is the Stone–Čech compactification of S. As a corollary we have that the subspace S is P-embedded in the Tichonov space X if and only if for all compact Hausdorff spaces A, $S\times A$ is C^*-embedded in $X\times A$. For Tichonov spaces X this implies that X is P-embedded in vX if and only if $v(X\times\beta X)=vX\times\beta X$. Moreover for spaces X of non-measurable cardinality, this also implies that $v(X\times A)=vX\times A$ for all compact Hausdorff spaces A. This is similar to a result in [4].

In section 3 the results of section 2 are generalized to give results concerning P^γ-embedding and γ-collectionwise normality. See [1] for definitions of these terms. From these results we are able to show that if S is a C-embedded subset of a topological space X then $S\times M$ is C-embedded in $X\times M$ for all compact metric spaces M. In [10] Morita considers γ-paracompact normal spaces and such spaces are always γ-collectionwise normal. We obtain characterizations of γ-collectionwise normal spaces. As a corollary to results of Morita in [10], we show that if X is a γ-paracompact normal space and if A is a compact Hausdorff space with a base for its natural uniformity of cardinality at most γ, then $X\times A$ is a γ-paracompact normal space. This theorem is then related to our results on P^γ-embedding and γ-collectionwise normal spaces.

In Theorem 4.1 we improve a characterization on collectionwise normal spaces given by H. Tamano in [13] and close with some open questions concerning P-embedding.

The major tool of this paper will be Theorem 1.2 below (which appeared as Theorem 2.3 of [1]). We state the relevant portion of this result here for convenience. However first we need the following definitions.

Definition 1.1. Let X be a topological space and let γ be an infinite cardinal number. A function f on X is said to be a (L, M)-*valued function* on X if f maps X into a complete, convex, metrizable subset M of a locally convex topological vector space L. A function f on X is said to be a (L,γ, M)-*valued function* on X if it is a (L, M)-valued function and if the image of X under f is a γ-separable subset of M (that is, there is a dense subset A of $f(X)$ and the cardinality of A is not greater than γ). A *Fréchet space* is a complete, metrizable locally convex topological vector space.

Let us recall that the set of all bounded, real valued continuous functions on X is a Banach space under the sup norm, that is $\|f\| = \sup_{x\in X}|f(x)|$. This Banach space will be denoted by $C^*(X)$.

Theorem 1.2. *Let S be a nonempty subspace of X and let γ be an infinite cardinal number. The following statements are equivalent*:

(1) *The subspace S is P^γ-embedded in X.*

(2) *Every continuous (L,γ, M)-valued function on S extends to a continuous (L, M)-valued function on X.*

(3) *Every continuous (L,γ, M)-valued function on S extends to a continuous function from X to L.*

(4) *Every continuous function from S to a Fréchet space, such that the image of S is γ-separable, extends to a continuous function on X.*

(5) *Every continuous function from S into $C^*(S)$, such that the image of S is γ-separable, extends to a continuous function on X.*

Furthermore, the above conditions are also equivalent to the conditions obtained from (2) *through* (5) *by requiring the image of S to be a bounded subset of the locally convex space in question.*

Since a subspace S of a topological space X is P-embedded in X if and only if it is P^γ-embedded in X for all infinite cardinal numbers γ (Theorem 2.8 of [12]), it is clear that we obtain characterizations of P-embedding from Theorem 1.2 by removing all mention of cardinality.

2. P-embedding and the extension of functions on product spaces. Now we develop the material needed to characterize P-embedding in terms of product spaces. If A and B are topological spaces, $C(A, B)$ will denote the set of all continuous functions from A to B equipped with the compact-open topology. A subbase for this topology is the collection of all sets $(K, W)=\{f\in C(A,B)\colon f(K)\subset W\}$ where K is a compact subset of A and W is an open subset of B. On page 80 of [11], the following proposition is shown.

Proposition 2.1. *If A is a Hausdorff topological space and if B is a locally convex topological vector space, then $C(A, B)$ is also a locally convex topological vector space.*

In addition we need to know when $C(A, B)$ is a Fréchet space. Consequently a *hemicompact space* is defined as one that is a countable union of compact subsets $(K_i)_{i\in N}$ such that every compact subset of the space is contained in some K_i. Compact spaces, Euclidean spaces R^n, and countable direct sums of compact spaces are examples of hemicompact spaces. In [3] Richard Arens showed that the concept of hemicompact spaces is useful in showing when $C(A, B)$ is a Fréchet space.

Proposition 2.2. *If A is a hemicompact Hausdorff space and if B is a Fréchet space, then $C(A, B)$ is a Fréchet space.*

We note that if A is a compact Hausdorff space and if B is a Banach space, then by the above $C(A, B)$ is a Fréchet space. More than this, it is a Banach space under the norm $\|f\| = \sup_{a\in A}\|f(a)\|_B$ (see [3]). It is easily verified that in this case the topology defined by this norm is equivalent to the compact-open topology.

The following lemma is due to Ralph Fox (see [6]).

LEMMA 2.3. *Let X, A, and B be topological spaces and let f be a continuous function from $X\times A$ into B. The function φ defined from X into $C(A, B)$ by $(\varphi(x))(a) = f(x, a)$ for all x in X and a in A is continuous. Conversely, if A is regular and locally compact, and if φ is a continuous map from X into $C(A, B)$, then the map f from $X\times A$ into B defined by $f(x, a) = (\varphi(x))(a)$ for all (x, a) in $X\times A$ is continuous.*

Using Theorem 1.2 as stated in section 1, we can now prove the following product space characterizations of P-embedding.

THEOREM 2.4. *Let S be a subspace of a completely regular T_1 space X. The following are equivalent.*

(1) *The subspace S is P-embedded in X.*

(2) *For all locally compact, hemicompact Hausdorff spaces A, the product set $S\times A$ is P-embedded in the product space $X\times A$.*

(3) *For all locally compact, hemicompact Hausdorff spaces A, the product set $S\times A$ is C-embedded in the product space $X\times A$.*

(4) *For all locally compact, hemicompact Hausdorff spaces A, the product set $S\times A$ is C^*-embedded in the product space $X\times A$.*

(5) *The product set $S\times\beta S$ is P-embedded in the product space $X\times\beta S$.*

(6) *The product set $S\times\beta S$ is C-embedded in the product space $X\times\beta S$.*

(7) *The product set $S\times\beta S$ is C^*-embedded in the product space $X\times\beta S$.*

Proof. To show that (1) implies (2), let A be a locally compact, hemicompact Hausdorff space. By the equivalence of (1) and (4) in Theorem 1.2 and the remark after Theorem 1.2 relating to P-embedding, it is sufficient to prove that if f is a continuous function from the product set $S\times A$ into a Fréchet space B, then f extends to a continuous function on $X\times A$. Let f be a continuous function from $S\times A$ into a Fréchet space B, and define a map φ from S into $C(A, B)$ by $(\varphi(x))(a) = f(x, a)$ for all a in A and all x in S. Then φ is continuous by Lemma 2.3 and $C(A, B)$ is a Fréchet space by Proposition 2.1. By assumption S is P-embedded in X. Hence by Theorem 1.2 the map φ extends to a continuous function φ^* from X into $C(A, B)$. Define a map f^* from $X\times A$ into B by $f^*(x, a) = (\varphi^*(x))(a)$ for all (x, a) in $X\times A$. The function f^* is an extension of f and is continuous by Lemma 2.3.

The implications (2) implies (3) implies (4) implies (7), and (2) implies (5) implies (6) implies (7) are clear. It remains to show that (7) implies (1). By Theorem 1.2 it is sufficient to prove that every bounded continuous function from S into $C^*(S)$ extends to a continuous function on X. Let φ be such a function. The Banach space $C^*(S)$ is isomorphic to $C^*(\beta S)$, by the mapping $f\to f^\beta$ which assigns to every bounded real valued continuous function f on S its unique extension f^β to βS. Hence we may think of φ as mapping S into $C^*(\beta S)$.

Define f from $S\times\beta S$ into R by $f(x, z) = (\varphi(x))(z)$ for all (x, z) in $S\times\beta S$. The map f is continuous by Lemma 2.3. Since φ is bounded, there is a constant K such that $\|\varphi(x)\|\leqslant K$ for all x in S. Hence $\sup_{z\in\beta S}|\varphi(x)(z)| = \sup_{z\in\beta S}|f(x, z)|\leqslant K$ for all x in S. Therefore, f is a bounded continuous function on $S\times\beta S$. By hypothesis f extends to a continuous real valued function f^* on $X\times\beta S$. Defining a function φ^* from X into $C^*(\beta S)$ by $(\varphi^*(x))(z) = f^*(x, z)$ for all x in X and z in βS, we see that φ^* is continuous by Lemma 2.3 and is the desired extension of the map φ.

As a result of this theorem extending pseudometrics from a subspace S to the space X is the same as extending bounded continuous real valued functions from $S\times A$ to $X\times A$ where A is a compact Hausdorff space.

COROLLARY 2.5. *Let S be a subspace of a completely regular T_1 space X. Then S is P-embedded in X if and only if $S\times A$ is C^*-embedded in $X\times A$ for all compact Hausdorff spaces A.*

As mentioned in the introduction, Theorem 2.4 gives new characterizations of Tichonov spaces which are collectionwise normal. Recall that a space is collectionwise normal if and only if every closed subset is P-embedded. We state here two of the characterizations for these spaces which result from 2.4.

COROLLARY 2.6. *Let X be a completely regular T_1 space. The following are equivalent.*

(1) *The space X is collectionwise normal.*

(2) *For all locally compact, hemicompact Hausdorff spaces A and for all closed subsets F of X, the product set $F\times A$ is C^*-embedded in $X\times A$.*

(3) *For all closed subsets F of X, the product set $F\times\beta F$ is C^*-embedded in $X\times\beta F$.*

The other characterizations can be obtained in a likewise fashion. We also obtain a result on the Hewitt realcompactification of a product that is similar to a result of W. W. Comfort (see Theorem 1.2 of [4]).

COROLLARY 2.7. *If X is a completely regular T_1 space, then X is P-embedded in νX if and only if $\nu(X\times\beta X) = \nu X\times\beta X$. Moreover, if X has nonmeasurable cardinality, then $\nu(X\times A) = \nu X\times A$ for all compact Hausdorff spaces A.*

Proof. If X is P-embedded in νX, then by Theorem 2.4 the space $X\times\beta X$ is C-embedded in $\nu X\times\beta X$. The product of a compact Hausdorff space and a realcompact space is realcompact. Therefore, $\nu X\times\beta X$ is a realcompact space in which $X\times\beta X$ is dense and C-embedded. By the uniqueness of the Hewitt realcompactification, it follows that $\nu(X\times\beta X) = \nu X\times\beta X$. Conversely, if $\nu(X\times\beta X) = \nu X\times\beta X$, then $X\times\beta X$ is C-embedded in $\upsilon X\times\beta X$. Therefore, by Theorem 2.4 it follows that X is P-embedded in υX.

To prove the second statement we note that in [12] it was shown that if S is a dense C-embedded subset of a completely regular T_1 space X and if the cardinality of S is nonmeasurable, then S is P-embedded in X. Consequently if X has nonmeasurable cardinality then X is P-embedded in υX. Thus the statement follows from (1) implies (2) of Theorem 2.4. This completes the proof.

A nonempty subset S of a topological space X is *Z-embedded* in X if for every zero set Z of S there is a zero set Z' of X such that $Z'\cap S = Z$. Every C^*-embedded subset is Z-embedded but the converse is not true. This concept was studied in [2]. In Theorem 2.4, statement (7) cannot be improved by stating "the product set $S\times\beta S$ is Z-embedded in the product space $X\times\beta S$". In fact an equivalence to (1) cannot be obtained even if S is required to be closed. The following examples are instructive.

Let R be the real line and let S be the open interval $(0,1)$. Since S is a cozero set of R and since $S\times\beta S$ is a cozero set of $R\times\beta S$, then S is Z-embedded in R and $S\times\beta S$ is Z-embedded in $R\times\beta S$ (see [2]). However S is not P-embedded in R since it is not C-embedded.

Now let X be the Tichonov plank, that is $X = [0,\Omega]\times[0,\omega]\setminus(\Omega,\omega)$ and let F be the closed subset $\{(\Omega,\alpha)\colon \alpha<\omega\}$. Since F is a Lindelöf subset of X and since $F\times\beta F$ is a Lindelöf subset of $X\times\beta F$ it follows that $F\times\beta F$ is Z-embedded in $X\times\beta F$. However, F is not P-embedded in X since it is not C^*-embedded in X.

3. P^γ-embedding and product spaces. This section attempts to generalize the results of the previous section to the case of P^γ-embedding. It will be seen that a generalization of Corollary 2.5 is possible, with suitable restrictions on the spaces A referred to in the corollary.

LEMMA 3.1. *Let A be any topological space and let γ be an infinite cardinal number. Let S be a subspace of a topological space X. If $S\times A$ is P^γ-embedded in $X\times A$, then S is P^γ-embedded in X. If $S\times A$ is P-embedded in $X\times A$, then S is P-embedded in X.*

Proof. The second statement follows from the first by choosing γ to be the cardinality of S. For if $S\times A$ is P-embedded in $X\times A$, by Theorem 2.8 of [12], it is P^γ-embedded in $X\times A$. Therefore by the first statement, S is P^γ-embedded in X. But since every continuous pseudometric on S is γ-separable, this means that S is P-embedded in X.

To prove the first statement let d be a γ-separable continuous pseudometric on S. By Theorem 2.1 of [12] it is sufficient to show that d extends to a continuous pseudometric on X. Define a pseudometric e on $S\times A$ by

$$e\big((x,a),(x',a')\big) = d(x,x')$$

for all (x,a) and (x',a') in $S\times A$. Then e is a continuous pseudometric on $S\times A$ since $S_e\big((x,a),\varepsilon\big) = S_d(x,\varepsilon)\times A$ for all $\varepsilon>0$. From this relationship it is also clear that the γ-separability of d implies the γ-separability of e. Therefore by assumption e extends to a continuous pseudometric e^* on $X\times A$. Fix b in A and define a function d^* on $X\times X$ by $d^*(x,x') = e^*\big((x,b),(x',b)\big)$, for all x, x' in X. Then d^* is a pseudometric on X which extends d. To see that d^* is continuous, let $y_0\in S_{d^*}(x_0,\varepsilon)$. Then $(y_0,b)\in S_{e^*}\big((x_0,b),\varepsilon\big)$. Hence there are U and V, neighborhoods of y_0 and b respectively such that $U\times V\subset S_{e^*}\big((x_0,b),\varepsilon\big)$. Therefore $y_0\in U \subset S_{d^*}(x_0,\varepsilon)$. This completes the proof of the lemma.

The following result is probably known, but we cannot find a proof of it in the literature. The proof will be included here for completeness. Recall that a compact Hausdorff space possesses a unique admissible uniformity, that is generated by all continuous pseudometrics on the space. In the case of a compact Hausdorff space, this uniformity will be referred to as its *natural uniformity*.

LEMMA 3.2. *Let X be a compact Hausdorff space and let γ be an infinite cardinal number. Then X has a base for its natural uniformity of cardinality at most γ if and only if X has a base for its topology of cardinality at most γ.*

Proof. Suppose that $\mathfrak{D}$ forms a base of cardinality at most γ for the natural uniformity of X. For each d in $\mathfrak{D}$ and each n in N, consider the cover of X by d-spheres about the points of X with radii $1/n$. Let $\big(S_d(x_i,1/n)\big)_{i=1,\ldots,m(d)}$ be a finite subcover. The collection of the spheres in these finite subcovers, for all d in $\mathfrak{D}$ and n in N, has cardinality at most γ and forms a base for the topology of X. To see this let $x_0\in U$, an open subset of X. The natural uniformity is admissible. Therefore there exists d in $\mathfrak{D}$ and n in N such that $x_0\in S_d(x_0,1/n)\subset U$. Let x_0 be in $S_d(x_i,1/2n)$, an element of the finite subcover associated with d and the natural number $2n$. It follows that $S_d(x_i,1/2n)$ is contained in U.

Conversely, suppose that $\mathfrak{U}$ is an open base for the topology of X of cardinality at most γ. We may assume that a finite union of elements in $\mathfrak{U}$ is also in $\mathfrak{U}$, since these unions will not increase the cardinality of $\mathfrak{U}$. Suppose that $F\subset G$, where F is a closed set and G is an open set in X. By an easy argument involving the compactness and normality of X, it can be shown that there are U and V in $\mathfrak{U}$ such that $F\subset U \subset \operatorname{cl} U\subset V\subset G$. Consider a pair (U,V), where $\operatorname{cl} U\subset V$ and U, V are in $\mathfrak{U}$. Since every compact Hausdorff space is normal, there exists a continuous function f on X with values in $[0,1]$ such that $f(x)=0$ for all x in $\operatorname{cl} U$, and $f(y)=1$ for all y in $X\setminus V$. Pick a function f with these properties for each pair (U,V) such that $\operatorname{cl} U\subset V$ and U, V are in $\mathfrak{U}$. The collection of finite supremum's of the ψ_f is a base for the natural uniformity on X. Note that this collection has the proper cardinality.

To show this we will show that for every continuous pseudometric d on X and $\varepsilon > 0$, there exists $f_1, \ldots, f_n$ in this collection and $\delta > 0$ such that $\psi_{f_1} \vee \ldots \vee \psi_{f_n}(x, y) \leqslant \delta$ implies that $d(x, y) \leqslant \varepsilon$ for all x, y in X.

Let d be a continuous pseudometric on X and let $0 < \varepsilon < 1$. Let $\big(S_d(x_i, \tfrac{1}{2}\varepsilon)\big)_{i=1,\ldots,n}$ be a finite subcover of the covering of X by d-spheres of radii $\tfrac{1}{2}\varepsilon$. Since this cover is normal, there is an open cover $(W_i)_{i=1,\ldots,n}$ of X such that $\mathrm{cl} W_i \subset S_d(x_i, \tfrac{1}{2}\varepsilon)$ for $i = 1, \ldots, n$. By the remark above, for each $1 \leqslant i \leqslant n$ we may choose U_i and V_i in $\mathfrak{U}$ such that $U_i \subset \mathrm{cl}\, U_i \subset V_i \subset S_d(x_i, \tfrac{1}{2}\varepsilon)$ and such that $(U_i)_{i=1,\ldots,n}$ covers X. Then for $1 \leqslant i \leqslant n$, (U_i, V_i) are in the pairs mentioned above, and therefore we have the corresponding functions f_i for $i = 1, \ldots, n$. Assume that $\psi_{f_i}(x, y) \leqslant \tfrac{1}{2}\varepsilon$ for $1 \leqslant i \leqslant n$. This means that $|f_i(x) - f_i(y)| \leqslant \tfrac{1}{2}\varepsilon$ for $1 \leqslant i \leqslant n$. Since $(U_i)_{i=1,\ldots,n}$ is an open cover, x is in some U_j. If y is not in $S_d(x_j, \tfrac{1}{2}\varepsilon)$, then y is in $X \setminus V_j$. Hence $f_j(x) = 0$ and $f_j(y) = 1$, which cannot be. Therefore $y \in S_d(x_j, \tfrac{1}{2}\varepsilon)$, which implies that $d(x, y) \leqslant \varepsilon$.

We can now state the generalization of Corollary 2.5 for P^γ-embedding.

THEOREM 3.3. *Let S be a subspace of a topological space X, and let A be a compact Hausdorff space with a base for its natural uniformity of cardinality at most γ. If S is P^γ-embedded in X, then $S \times A$ is P^γ-embedded in $X \times A$.*

Proof. Let S be P^γ-embedded in X, and let A be a compact Hausdorff space with a base for its natural uniformity of cardinality at most γ. Let f be a continuous function from $S \times A$ into a Banach space B such that $f(S \times A)$ is a γ-separable subset of B. By the equivalence of (1), (4), and (5) in Theorem 1.2, it is sufficient to show that f extends to a continuous function on $X \times A$.

Let $\mathfrak{B}$ be a dense subset of $f(S \times A)$ of cardinality at most γ, and let $\mathfrak{D}$ be a base of cardinality at most γ of continuous pseudometrics for the natural uniformity of A. Define a function g from S into $C(A, B)$ by $\big(g(x)\big)(a) = f(x, a)$ for all x in S and a in A. As was mentioned before Lemma 2.3, $C(A, B)$ with the compact open topology is a Banach space with norm $\|f\| = \sup_{a \in A} \|f(a)\|_B$. Hence Lemma 2.3 shows that g is continuous. Now if $g(S)$ is a γ-separable subset of $C(A, B)$, then by the equivalence of (1) and (4) in Theorem 1.2, the mapping g will extend to a continuous map g^* from X into $C(A, B)$. Define then a function f^* from $X \times A$ into B by $f^*(x, a) = g^*(x)(a)$ for all (x, a) in $X \times A$. The mapping f^* is an extension of f and Lemma 2.3 shows that it is continuous. The remainder of the proof then will be devoted to showing that $g(S)$ is a γ-separable subset of $C(A, B)$.

For each d in $\mathfrak{D}$ and each n in N let $\big(S_d(a_i, 1/n)\big)_{i=1,\ldots,m}$ be a finite subcover of the cover of A by d-spheres of radii $1/n$. Every open cover of a compact Hausdorff space is a normal cover (see [12]). Therefore,

this cover has a partition of unity $(h_i)_{i=1,\ldots,m}$ subordinate to it such that $A \setminus Z(h_i) \subset S_d(a_i, 1/n)$ for $1 \leqslant i \leqslant m$.

For each subset $\beta = (b_i)_{i=1,\ldots,m}$ of m elements of $\mathfrak{B}$ consider the function which maps A into B_m and is defined by

$$\Big(\sum_{i=1}^{m} h_i b_i\Big)(a) = \sum_{i=1}^{m} h_i(a)\, b_i \qquad (a \in A).$$

Now for each d in $\mathfrak{D}$ and each n in N form the function

$$f_{d,n,\beta} = \sum_{i=1}^{m} h_i b_i .$$

Each function $h_i b_i$ is continuous. To see this, let ψ denote the continuous function which maps r in R to rb in β, where b is a fixed element of β. Then $h_i b = \psi \circ h_i$, and consequently it is continuous. Therefore, each $f_{d,n,\beta}$ is continuous. The totality of these functions is a collection whose cardinality does not exceed γ. We now show that every function in $g(S)$ is uniformly approximated by functions in this collection. Since any γ-separable metric space is hereditarily γ-separable, this will prove that $g(S)$ is itself γ-separable.

Let k be an element of $g(S)$ and let $\varepsilon > 0$. The function k maps A into B, and is uniformly continuous. Therefore there is a d in $\mathfrak{D}$ and an n in N such that $d(a, a') \leqslant 1/n$ implies that $\|k(a) - k(a')\| \leqslant \tfrac{1}{2}\varepsilon$. Let $(h_i)_{i=1,\ldots,m}$ be the partition of unity subordinate to the cover $\big(S_d(a_i, 1/n)\big)_{i=1,\ldots,m}$. For each $1 \leqslant i \leqslant m$, the image $k(a_i)$ is an element of $f(S \times A)$. Hence there is a b_i in $\mathfrak{B}$ such that $\|k(a_i) - b_i\| \leqslant \tfrac{1}{2}\varepsilon$. We will show that

$$\Big\|k - \sum_{i=1}^{m} h_i b_i\Big\| \leqslant \varepsilon ,$$

$$\Big\|k - \sum h_i b_i\Big\| = \sup_{a \in A} \Big\|k(a) - \sum h_i(a)\, b_i\Big\|.$$

Let a be a fixed element of A. Then

$$\Big\|k(a) - \sum h_i(a)\, b_i\Big\| \leqslant \Big\|k(a) - \sum h_i(a)\, k(a_i)\Big\| + \Big\|\sum h_i(a)\, k(a_i) - \sum h_i(a)\, b_1\Big\| .$$

Since $k(a) = \sum h_i(a)\, k(a)$, the first term of this sum becomes:

$$\begin{aligned} \Big\|\sum h_i(a)\, k(a) - \sum h_i(a)\, k(a_i)\Big\| &= \Big\|\sum h_i(a)\big(k(a) - k(a_i)\big)\Big\| \\ &\leqslant \sum \big\|h_i(a)\big(k(a) - k(a_i)\big)\big\| \\ &= \sum h_i(a)\, \|k(a) - k(a_i)\| . \end{aligned}$$

If $h_i(a) \neq 0$, then $a \in A \setminus Z(h_i)$. Hence $d(a, a_i) < 1/n$, so $\|k(a) - k(a_i)\| \leqslant \frac{1}{2}\varepsilon$. Therefore

$$\sum h_i(a)\|k(a) - k(a_i)\| \leqslant \sum h_i(a)\tfrac{1}{2}\varepsilon = \tfrac{1}{2}\varepsilon .$$

Similarly, the second term of the sum can be shown to be $\leqslant \frac{1}{2}\varepsilon$. This completes the proof.

The following corollary combines Theorem 3.3 and Lemma 3.1.

COROLLARY 3.4. *Let A be a compact Hausdorff space with a base for its natural uniformity of cardinality at most γ. A subspace S is P^γ-embedded in X if and only if $S \times A$ is P^γ-embedded in $X \times A$.*

Since $P^{\aleph_0}$-embedding is equivalent to C-embedding, we obtain the following from Lemma 3.2 and Theorem 3.3.

COROLLARY 3.5. *If S is C-embedded in X, then $S \times M$ is C-embedded in $X \times M$ for all compact metric spaces M.*

From Corollary 3.4, the following characterization of γ-collectionwise normality is obtained.

COROLLARY 3.6. *Let X be a topological space, let γ be an infinite cardinal number, and let A be a compact Hausdorff space with a base for its natural uniformity of cardinality at most γ. The following statements are equivalent:*

(1) *The space X is γ-collectionwise normal.*

(2) *For all closed subsets F of X, the product $F \times A$ is P^γ-embedded in $X \times A$.*

It is known that every γ-paracompact normal space is γ-collectionwise normal. K. Morita proved the following theorems (see Theorems 2.1 and 2.2 of [10]): (1) *If X is a γ-paracompact space and if A is a compact space, then $X \times A$ is γ-paracompact*; and (2) *If X is a γ-paracompact normal space and if A is a compact normal space with an open base of power at most γ, then $X \times A$ is normal.* Combining these two theorems and recalling Lemma 3.2, we have the following result.

THEOREM 3.7. *If X is a γ-paracompact normal space and if A is a compact Hausdorff space with a base for its natural uniformity of power at most γ, then $X \times A$ is a γ-paracompact normal space.*

This result is parallel to that of Corollary 3.6 in the following sense. If X is a γ-paracompact normal space and if A is a compact Hausdorff space with a base for its natural uniformity of power at most γ, then $X \times A$ is a γ-paracompact normal space by 3.7. Hence $X \times A$ is γ-collectionwise normal, therefore every closed subset of the form $F \times A$, where F is closed in X, is P^γ-embedded in $X \times A$. But the latter can also be obtained by first noting that X is γ-collectionwise normal, and then using Corollary 3.6 to argue that if F is closed in X, then $F \times A$ is P^γ-embedded in $X \times A$.

The next section deals with some special results for collectionwise normal spaces.

4. Product space characterizations of collectionwise normality. Recall that every paracompact normal space is collectionwise normal. H. Tamano proved that if BX is any Hausdorff compactification of a completely regular T_1 space X, then X is paracompact if and only if $X \times BX$ is normal (see [13]). Since $X \times BX$ is paracompact if X is paracompact, this can be restated as: A Tichonov space X is paracompact if and only if $X \times BX$ is normal if and only if $X \times BX$ is paracompact. The next theorem is a parallel result for collectionwise normality. If A is a compact Hausdorff space in which the space X is C^*-embedded, this result will show that the collectionwise normality of X is equivalent to the following conditions: (1) For all closed subsets F of X, the product $F \times A$ is C^*-embedded in $X \times A$, and (2) For all closed subsets F of X, the product $F \times A$ is P-embedded in $X \times A$. We know that every closed subset of a normal space is C^*-embedded and every closed subset of a paracompact Hausdorff space is P-embedded. Hence it is clear that these conditions are weaker than the normality or paracompactness of $X \times A$.

THEOREM 4.1. *Let A be a compact Hausdorff space in which X is C^*-embedded. The following statements are equivalent:*

(1) *The space X is collectionwise normal.*

(2) *For all closed subsets F of X, the product set $F \times A$ is P-embedded in the product space $X \times A$.*

(3) *For all closed subsets F of X, the product set $F \times A$ is C^*-embedded in the product space $X \times A$.*

Proof. The implication (1) implies (2) follows from (1) implies (2) of Theorem 2.4 and the fact that every closed subset of a collectionwise normal space is P-embedded in it. The implication (2) implies (3) is immediate. Hence it suffices to show that (3) implies (1).

It is easy to show that (3) implies that every closed subset of X is C^*-embedded in X; hence X is normal. Let $(F_\alpha)_{\alpha \in I}$ be a discrete family of closed subsets of X. For each α in I, the sets F_α and $H_\beta = \bigcup_{\beta \neq \alpha} F_\beta$ are disjoint closed subsets of X. For each α in I, by the normality of X, there exists a continuous function f_α on X with values in $[0, 1]$ such that $f_\alpha(x) = 0$ for all $x \in F_\alpha$, and $f_\alpha(y) = 1$ for all $y \in H_\alpha$. Since X is C^*-embedded in A, let f_α^* be a continuous mapping of A into $[0, 1]$ such that f_α^* restricted to X is f_α. Let F be the union of F_α for α in I. Then F is a closed subset of X.

Define a real-valued function f on $F \times A$ by $f(x, a) = f_\alpha^*(a)$ for the unique α in I such that $f_\alpha(x) = 0$. The function f is clearly bounded and is continuous. To see this, let (x_0, a_0) be in $F \times A$, let $\varepsilon > 0$, and suppose that x_0 is in F_a. There is a neighborhood U of a_0 such that if a is in U, then $|f_a^*(a)-f_a^*(a_0)| < \varepsilon$. If (x, a) is in $F_a\times U$, then $|f(x, a)-f(x_0, a_0)| < \varepsilon$. Therefore by assumption f extends to a continuous real-valued function f^* on $X\times A$.

Define g from X into $C(A)$ by $\big(g(x)\big)(a) = f^*(x, a)$ for all a in A and x in X. By Lemma 2.3 the function g is continuous. Hence the pseudo-metric d^* defined on X by $d^*(x, y) = \|g(x)-g(y)\|$ for all x, y in X is continuous. Let $G_a = \bigcup_{x\in F_a} S_{d^*}(x, \frac{1}{4})$ for all a in I. Then G_a is an open subset of X containing F_a. It remains to show that $(G_a)_{a\in I}$ is a pairwise disjoint family.

Suppose t is in G_a and G_β, where $a\neq\beta$. There exist x in F_a and y in F_β such that $d^*(t, x) < \frac{1}{4}$ and $d^*(t, y) < \frac{1}{4}$. Hence $\|g(t)-g(x)\| < \frac{1}{4}$ which means that $\sup_{a\in A} |f^*(t, a)-f^*(x, a)| < \frac{1}{4}$. Similarly, $\sup_{a\in A} |f^*(t, a)-f^*(y, a)| < \frac{1}{4}$. Let $a = x$. Then

$$|f^*(t, x)-f^*(x, x)| = |f^*(t, x)-f(x, x)| = |f^*(t, x)| < \tfrac{1}{4}$$

and

$$|f^*(t, x)-f^*(y, x)| = |f^*(t, x)-f(y, x)| = |f^*(t, x)-1| < \tfrac{1}{4},$$

which is a contradiction. This completes our proof.

Now as a corollary we have Tamano's original result. (See [14].)

COROLLARY 4.2. *A completely regular T_1 space X is collectionwise normal if and only if $F\times\beta X$ is C^*-embedded in $X\times\beta X$ for all closed subsets F of X.*

5. Concluding remarks. We have just obtained Tamano's original result on collectionwise normality as a corollary to Theorem 4.1. We have pointed out that our conditions are weaker than requiring the paracompactness or normality of $X\times A$ for A any compact Hausdorff space. It is an open question whether the above condition characterizes P-embedding. That is, is it true that if S is a subspace of X, then S is P-embedded in X if and only if $S\times\beta X$ is C^*-embedded in $X\times\beta X$?

In section 3 we discussed two results of Morita from [10]. In this paper his main result is the following: A topological space X is γ-paracompact and normal if and only if $X\times I^\gamma$ is normal, where I denotes the unit interval. We do not know if there is a parallel result for γ-collectionwise normality. It might be conjectured that X is γ-collectionwise normal if and only if $F\times I^\gamma$ is C^*-embedded in $X\times I^\gamma$ for all closed subsets F of X. This would be parallel to the result that X is collectionwise normal if and only if for all closed subsets F of X, the set $F\times\beta F$ is C^*-embedded in $X\times\beta F$.

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