On the existence of maps having graphs connected and dense

by

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The paper contains a proof of the existence of maps $f: X \to Y$ with connected and dense graphs in $X \times Y$, where $X$ and $Y$ are connected spaces satisfying some additional conditions. We also state Theorems 2 and 3, which are generalizations of Theorems 3 and 4 proved by D. Phillips in [3]. The proofs of these theorems are reproduced from [3].

Let us fix some notation and symbols; $\pi: X \times Y \to X$ means the projection, $Fr_{X}$ and $Int_A$ means boundary and interior operations in the space $A$, and $w(A)$, $\text{card}A$ means respectively the weight of $A$ and the cardinality of $A$.

**Lemma 1.** Let $X$, $Y$ be connected spaces. If $0 \neq G \subset X \times Y$ is open, then

(a) $\text{Int}_{\pi}(Fr_{X} \times \pi G) \neq \emptyset$, or

(b) there exists an $x \in X$ such that $\pi^{-1}(x) \subset Fr_{X} \times \pi G$, or

(c) $G$ is dense in $X \times Y$.

**Proof.** (I) Let us assume that there exists a point $(x, y) \in X \times Y$ such that $(x, y) \in G$ and $x \in \pi(G)$. Then there exists an open set $U_x \subset \pi(G)$ such that $x \in U_x$ and $(x) = \pi(U_x \cap \{x\})$. Indeed, there are open sets $U_x \subset \pi(G)$ and $U_y \subset X$ such that $(x, y) \in U_x \times U_y \subset (X \times Y) - G$. We show that $U_x \subset \pi(Fr_{X} \times \pi G)$. Suppose that there exists an $x' \in U_x - \pi(Fr_{X} \times \pi G)$. A subspace $(x') \times Y \subset X \times Y$ is homeomorphic with $Y$. We have

$G \neq \emptyset \cap ((x') \times Y) \neq (x') \times Y$

and

$Fr_{X} \pi(G) \subset Fr_{X} \pi(G) \cap ((x') \times Y) = G$,

and this contradicts the fact that $(x') \times Y$ is connected.

(II) Let us assume that the condition (I) is not satisfied. We have $Fr_{X} \pi(G) \neq \emptyset$ or $Fr_{X} \pi(G) = \emptyset$.

(b) If $x \in Fr_{X} \pi(G)$ then $\pi^{-1}(x) \subset Fr_{X} \times \pi G$. Indeed, suppose that there exist an $y \in Y$ and such an open neighbourhood $U_x \times U_y$ of point
(x, y) that \( U_x \times U_y \subseteq X \times Y - \emptyset \). Then there exists an \( x' \in \pi(G) \cap U_x \). Hence we have \( (x', y) \notin \mathcal{G} \) and \( x' \in \pi(G) \), which is a contradiction of assumption (II).

(c) If \( \text{Fr}_x \pi(G) = \emptyset \), then \( G \) is dense in \( X \times Y \). Indeed, in this case we have \( \pi(G) = X \). Thus for every point \( (x, y) \in X \times Y \) we have \( x \in \pi(G) \) and hence, according to assumption (II), it follows that \( (x, y) \in \mathcal{G} \).

**Lemma 2.** Let \( Z \) be a connected space and \( D \subseteq Z \) a dense subspace of \( Z \). If for every open set \( G \subseteq D \), \( \emptyset \neq G \), we have \( D \cap \text{Fr}_x \mathcal{G} \neq \emptyset \), then \( D \) is a connected subspace of \( Z \).

**Proof.** Let \( H = D \cap \mathcal{G} \) where \( \mathcal{G} \) is open in \( Z \) and \( \emptyset \neq \mathcal{G} \subseteq D \). Let \( \text{Fr}_x H = \overline{H} - H = D \cap \overline{\mathcal{G}} - D = D \cap \overline{\mathcal{G}} - D = D \cap \text{Fr}_x \mathcal{G} \neq \emptyset \), where \( \overline{H} \) means the closure of \( H \) in \( D \). Thus \( H \) is not a closed-open set in \( D \), and hence \( D \) is connected.

**Definition.** A set \( G \subseteq X \) is called \( \tau \)-dense in \( X \) if, for every open and non-empty set \( G \subseteq X \), \( \text{card} \, C \cap \mathcal{G} \geq \tau \).

**Theorem 1.** Let \( X, Y \) be connected spaces such that \( \text{w}(X) \leq \text{w}(X) \), \( \text{w}(X) \geq \lambda \) and, for every non-empty open set \( U \subseteq X \), \( \text{card} \, U = 2^{\text{w}(X)} \).

Then for every \( 2^{\text{w}(X)} \)-dense set \( B \subseteq X \) there exists a map \( f : B \to Y \) such that each extension \( \mathcal{G} : X \to Y \) of \( f \) has a connected and dense graph in \( X \times Y \).

**Proof.** The family of all open sets in \( X \times Y \) is of cardinality not greater than \( 2^{\text{w}(X)} \). Let \( T_b \) be a family of open sets in \( X \times Y \) satisfying the condition (a) of Lemma 1 or being of form \( U \times X \). From the assumption of \( X \) and from Lemma 1 it follows that if \( \mathcal{G} \subseteq T_b \), then \( \text{Fr}_x \mathcal{G} = 2^{\text{w}(X)} \). We may assume that \( T_b \) is well ordered; \( T_b = \{ \mathcal{G}_i : \xi < \gamma \} \), where \( \gamma \) is an initial number of power \( \leq 2^{\text{w}(X)} \).

We define by transfinite induction a sequence of pairs of points \( (P_i, P'_i)_{\xi < \gamma} \) such that:

1° \( P_\xi \in G_\xi \), \( P'_\xi \in G_\xi - G_\xi \), \( \pi(P'_\xi) \neq \emptyset \).

2° If \( \xi \neq \xi' \) or \( i \neq j \), then \( \pi(P'_i) \neq \pi(P'_j) \) for \( \xi, \xi' < \gamma \), \( i, j = 1, 2 \).

Let \( P, Q \) be points such that \( P \notin G_\xi \), \( Q \notin G_\xi \), \( \pi(P) \neq \pi(Q) \). We put \( P_0 = P \), \( P'_0 = Q \). Let us now suppose that the points \( P'_1 \), \( P'_2 \) are defined for \( \xi < \alpha \), and that they satisfy the conditions 1° and 2°. The cardinality of \( C_\xi = \{ \pi(P'_i) : \xi < \alpha, i = 1, 2 \} \) is less than \( 2^{\text{w}(X)} \). Hence there exists points \( P'_2 \), \( P''_2 \) such that \( \pi(P'_2) \neq \pi(P''_2) \), \( P'_2 \notin G_\xi \), \( P''_2 \notin G_\xi \), \( \pi(P''_2) \in (X - C_\xi) \cap B_\xi \), \( \xi = 1, 2 \).

Let \( G(f) = \{ P'_2 : \xi < \alpha, i = 1, 2 \} \) and let \( B_\xi = \pi(G(f)) \). The virtue of 2°, the set \( G(f) \) defines a unique map \( f : B_{\xi} \to Y \) whose graph is \( G(f) \).

Let \( C \subseteq X \times Y \) be a set such that \( G(f) \subseteq C \) and \( \pi(D) = X \). We shall prove that \( D \) is dense and connected in \( X \times Y \). The density of \( D \) follows from the fact that open sets of form \( U \times Y \), belong to \( T_b \). Hence \( G(f) \) and \( D \) are dense in \( X \times Y \).

To prove that \( D \) is connected in \( X \times Y \) it suffices to show (see Lemma 2) that for every open set \( G \subseteq X \times Y \), \( G \cap D \neq \emptyset \), \( G \neq \emptyset \), we have \( D \cap \text{Fr}_x \mathcal{G} \cap \mathcal{G} \neq \emptyset \).

(a) If \( \text{Int} \, \pi(\text{Fr}_x \mathcal{G}) \neq \emptyset \), then \( D \cap \mathcal{G} = D \cap (\overline{\mathcal{G}} - \emptyset) \).

(b) If \( \text{Fr}_x \mathcal{G} \subseteq G \)

(c) If \( \mathcal{G} \neq \emptyset \) and \( G \) is dense in \( X \times Y \), then \( D \cap \text{Fr}_x \mathcal{G} = D \cap (\overline{\mathcal{G}} - \emptyset) \).

Thus, according to Lemma 2, it is proved that \( D \) is connected. Now, it is obvious that if \( g : X \to Y \) is such that \( g[B] = f \), then the graph of \( g \) is dense and connected in \( X \times Y \).

**Corollary.** If \( X, Y \) satisfies the assumptions of the Theorem 1 and \( X - B \) is \( 2^{\text{w}(X)} \)-dense in \( X \), then for every map \( g : X \to Y \) there exists a map \( g^* : X \to Y \) having a connected and dense graph and such that \( g^*[B] = g[B] \).

**Proof.** It follows from Theorem 1 that there is a map \( f : X - B \to Y \) such that each extension \( g^* : X \to Y \) has a dense and connected graph. We define

\[
 g^*(x) = \begin{cases} f(x) & \text{if } x \notin X - B, \\ g(x) & \text{if } x \in B. \end{cases}
\]

**Lemma 3.** Let \( X \) be a space such that \( \text{w}(X) \geq \lambda \) and, for every non-empty open set \( G, \text{card} \, G \geq \tau \), where \( \tau \geq \text{w}(X) \).

Then \( X \) is a union of \( \tau \) mutually disjoint sets, each of them being \( \tau \)-dense in \( X \).

**Proof.** (1) Let \( \tau > \text{w}(X) \) and let \( R \subseteq \text{card} \, E = \text{w}(X) \) be a base for the topology on \( X \). Let \( \beta \) be an initial number of the power \( \tau \). We shall define sets \( A_i = \{ x_i : G \in R, \xi < \beta \} \) such that

1° \( x_i \in G \) for every \( G \in R \) and \( \xi < \beta \).

2° If \( \xi \neq \xi' \), \( \xi, \xi' < \beta \), then \( A_{\xi} \cap A_{\xi'} = \emptyset \).

For every \( G \in R \) we assume \( x_0 \) to be an element of \( G \). Let us assume that the points \( x_0 \) are defined for every \( G \in R \) and \( \xi < \beta \). Notice that \( \text{card} \, A_0 = \{ x_0 : G \in R \} \) and \( \text{card} \, A_{\xi} = \text{card} \, R \).

Since \( \tau > \text{w}(X) \), there exists a one-to-one map \( \varphi : ((\gamma, \gamma') : (\gamma, \gamma' < \beta) \mapsto \{ (\xi) : \xi < \beta) \). Let \( B_\xi = \text{card} \, A_{\xi < \beta} \).

Since \( \varphi < \beta \), \( B_\xi < \beta \). Let \( B_{\xi} = \text{card} \, A_{\xi < \beta} \).

From 1° and 2° it follows that \( \text{card}(B_0 \cap R) = \tau \) and \( B_\xi \cap B_{\xi'} = \emptyset \) for every \( \xi \neq \xi' \) and a non-empty open set \( G \). The sets \( C_{\xi} = X - \bigcup_{\gamma < \beta} C_{\xi} \), and \( C_{\xi} = B_{\xi} \) if \( \xi > 0 \), are mutually disjoint and each of them is \( \tau \)-dense in \( X \).
Now, let us assume that the points $x_i', y_i' < \eta$, $\xi < \alpha$ are defined. Let us notice that $\operatorname{card}(x_i': \eta < \eta', \xi < \alpha) < 2^{\omega \cdot \alpha \cdot \eta'}$. This makes it possible to define points $x_i': \xi < \alpha$ such that $x_i' \in G_{\xi' \eta'} - S(x_i': \eta < \eta'$ and $\xi < \alpha$ or $\eta' = \eta$ and $\xi' < \xi$).

From the Kuratowski–Zorn Lemma it follows that there exists a vector base $B$ containing $B_0$. The base $B$ is $2^{\omega \cdot \alpha \cdot \eta'}$-dense in $X$ because $B_0$ is $2^{\omega \cdot \alpha \cdot \eta'}$-dense in $X$.

**Theorem 4.** If $X$, $Y$ are vector spaces satisfying the assumptions of Theorem 1, then there exists a map $f': X \to Y$ satisfying Cauchy's equation $F(x+y) = F(x) + F(y)$ and having a dense and connected graph.

**Proof.** Let $B$ be a $2^{\omega \cdot \alpha \cdot \eta'}$-dense vector base in $X$, and let $f: B \to Y$ be a map such that every extension $f': X \to Y$ of $f$ has a dense and connected graph. We define $f': X \to Y; f'(x) = r_1 f(x_1) + \ldots + r_n f(x_n)$, where $x = x_1 + \ldots + x_n$ and $x_i \in B, r_i$ is rational for $i = 1, \ldots, n, n = 1, 2, \ldots$.

References


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