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On the existence of maps having graphs connected and dense

by

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The paper contains a proof of the existence of maps $f\colon X\to Y$ with connected and dense graphs in $X\times Y$, where X and Y are connected spaces satisfying some additional conditions. We also state Theorems 2 and 3, which are generalizations of Theorems 3 and 4 proved by D. Phillips in [2]. The proofs of these theorems are reproduced from [2].

Let us fix some notation and symbols; $\pi\colon X\times Y\to X$ means the projection, Fr_A and Int_A means boundary and interior operations in the space A, and $w(A)$, $\mathrm{card}\,A$ means respectively the weight of A and the cardinality of A.

LEMMA 1. *Let X, Y be connected spaces. If $0\neq G\subset X\times Y$ is open, then*

(a) $\mathrm{Int}_X\pi(\mathrm{Fr}_{X\times Y}G)\neq\emptyset$, *or*

(b) *there exists an $x\in X$ such that $\pi^{-1}(x)\subset \mathrm{Fr}_{X\times Y}G$, or*

(c) *G is dense in $X\times Y$.*

Proof. (I) Let us assume that there exists a point $(x,y)\in X\times Y$ such that $(x,y)\notin\bar{G}$ and $x\in\pi(G)$. Then there exists an open set $U_x\subset\pi(G)$ such that $x\in U_x$ and (a) $U_x\subset\pi(\mathrm{Fr}_{X\times Y}G)$. Indeed, there are open sets $U_x\subset\pi(G)$ and $U_y\subset Y$ such that $(x,y)\in U_x\times U_y\subset(X\times Y)-\bar{G}$. We show that $U_x\subset\pi(\mathrm{Fr}_{X\times Y}G)$. Suppose that there exists an $x'\in U_x-\pi(\mathrm{Fr}_{X\times Y}G)$. A subspace $\{x'\}\times Y\subset X\times Y$ is homeomorphic with Y. We have

$$\emptyset\neq G\cap(\{x'\}\times Y)\neq\{x'\}\times Y$$

and

$$\mathrm{Fr}_{\{x'\}\times Y}\big(G\cap(\{x'\}\times Y)\big)\subset(\mathrm{Fr}_{X\times Y}G)\cap(\{x'\}\times Y)=\emptyset,$$

and this contradicts the fact that $\{x'\}\times Y$ is connected.

(II) Let us assume that the condition (I) is not satisfied. We have $\mathrm{Fr}_X\pi(G)\neq\emptyset$ or $\mathrm{Fr}_X\pi(G)=\emptyset$.

(b) If $x\in\mathrm{Fr}_X\pi(G)$ then $\pi^{-1}(x)\subset\mathrm{Fr}_{X\times Y}G$. Indeed, suppose that there exist an $y\in Y$ and such an open neighbourhood $U_x\times U_y$ of point

(x, y) that $U_x \times U_y \subset X \times Y - \bar{G}$. Then there exists an $x' \in \pi(G) \cap U_x$. Hence we have $(x', y) \notin \bar{G}$ and $x' \in \pi(G)$, which is a contradiction of assumption (II).

(c) If $\mathrm{Fr}_X \pi(G) = \emptyset$, then G is dense in $X \times Y$. Indeed, in this case we have $\pi(G) = X$. Thus for every point $(x, y) \in X \times Y$ we have $x \in \pi(G)$ and hence, according to assumption (II), it follows that $(x, y) \in \bar{G}$.

LEMMA 2. *Let Z be a connected space and $D \subset Z$ a dense subspace of Z. If for every open set $G \not\supset D$, $\emptyset \neq G$, we have $D \cap \mathrm{Fr}_Z G \neq \emptyset$, then D is a connected subspace of Z.*

Proof. Let $H = D \cap G$ where G is open in Z and $\emptyset \neq G \not\supset D$. $\mathrm{Fr}_D H = \tilde{H} - H = D \cap \bar{G} - D \cap G = D \cap (\bar{G} - G) = D \cap \mathrm{Fr}_Z G \neq \emptyset$, where $\tilde{H}$ means the closure of H in D. Thus H is not a closed-open set in D, and hence D is connected.

DEFINITION. A set $C \subset X$ is called *τ-dense in X* if, for every open and non-empty set $G \subset X$, $\mathrm{card}\, C \cap G \geqslant \tau$.

THEOREM 1. *Let X, Y be connected spaces such that $w(Y) \leqslant w(X)$, $w(X) \geqslant \chi_0$ and, for every non-empty open set $U \subset X$, $\mathrm{card}\, U = 2^{w(X)}$.*

Then for every $2^{w(X)}$-dense set B in X there exists a map $f: B \to Y$ such that each extension $f^*: X \to Y$ of f has a connected and dense graph in $X \times Y$.

Proof. The family of all open sets in $X \times Y$ is of cardinality not greater than $2^{w(X)}$. Let T_0 be a family of open sets in $Y \times Y$ satisfying the condition (a) of Lemma 1 or being of form $U \times V$. From the assumption of X and from Lemma 1 it follows that if $G \in T_0$, then $\mathrm{card}\, \mathrm{Fr}_{X \times Y} G = 2^{w(X)}$. We may assume that T_0 is well ordered; $T_0 = \{G_\xi\colon \xi < \gamma]$, where γ is an initial number of power $\leqslant 2^{w(X)}$.

We define by transfinite induction a sequence of pairs of points $\{(P_0^1, P_0^2), \ldots, (P_\xi^1, P_\xi^2), \ldots\}_{\xi<\gamma}$ such that;

1° $P_\xi^1 \in G_\xi$, $P_\xi^2 \in \bar{G}_\xi - G_\xi$ and $\pi(P_\xi^i) \in B$,

2° if $\xi \neq \xi'$ or $i \neq j$ then $\pi(P_\xi^i) \neq \pi(P_{\xi'}^j)$ for $\xi, \xi' < \gamma$, $i, j = 1, 2$.

Let P, Q be points such that $P \in G_0$, $Q \in \bar{G}_0 - G_0$, $\pi(P) \neq \pi(Q)$ and $\pi(P), \pi(Q) \in B$. We put $P_0^1 = P$, $P_0^2 = Q$. Let us suppose that the points P_ξ^1, P_ξ^2 are defined for $\xi < \alpha$, and that they satisfy the conditions 1° and 2°. The cardinality of $C_\alpha = \{\pi(P_\xi^i)\colon \xi < \alpha,\ i = 1, 2\}$ is less than $2^{w(X)}$. Hence there exist points P_α^1, P_α^2 such that $\pi(P_\alpha^1) \neq \pi(P_\alpha^2)$, $P_\alpha^1 \in G_\alpha$, $P_\alpha^2 \in \bar{G}_\alpha - G_\alpha$ and $\pi(P_\alpha^i) \in (X - C_\alpha) \cap B$, $i = 1, 2$. Let $G(f) = \{P_\xi^i\colon \xi < \gamma,\ i = 1, 2\}$ and let $B_0 = \pi[G(f)]$. The virtue of 2°, the set $G(f)$ defines a unique map $f: B_0 \to Y$ whose graph is $G(f)$.

Let $D \subset X \times Y$ be a set such that $G(f) \subset D$ and $\pi(D) = X$. We shall prove that D is dense and connected in $X \times Y$. The density of D follows from the fact that open sets of form $U \times V$, belong to T_0. Hence $G(f)$ and D are dense in $X \times Y$.

To prove that D is connected in $X \times Y$ it suffices to show (see Lemma 2) that for every open set $G \subset X \times Y$, $G \not\supset D$, $G \neq \emptyset$, we have $D \cap \mathrm{Fr}_{X \times Y} G \neq \emptyset$.

(a) If $\mathrm{Int}_X \pi(\mathrm{Fr}_{X \times Y} G) \neq \emptyset$, then $D \cap \mathrm{Fr}_{X \times Y} G = D \cap (\bar{G} - G) \supset G(f) \cap (\bar{G} - G) \neq \emptyset$, since $G \in T_0$.

(b) If there exists a point $x \in X$ such that $\pi^{-1}(x) \subset \mathrm{Fr}_{X \times Y} G$, then $D \cap \mathrm{Fr}_{X \times Y} G \supset D \cap \pi^{-1}(x) \neq \emptyset$.

(c) If $G \not\supset D$ is dense in $X \times Y$, then $D \cap \mathrm{Fr}_{X \times Y} G = D \cap [(X \times Y) - G] = D - G \neq \emptyset$.

Thus, according to Lemma 2, it is proved that D is connected. Now, it is obvious that if $g: X \to Y$ is such that $g|B_0 = f$, then the graph of g is dense and connected in $X \times Y$.

COROLLARY. *If X, Y satisfies the assumptions of the Theorem 1 and $X - B$ is $2^{w(X)}$-dense in X, then for every map $g: X \to Y$ there exists a map $g^*: X \to Y$ having a connected and dense graph and such that $g^*|B = g|B$.*

Proof. It follows from Theorem 1 that there is a map $f: X - B \to Y$ such that each extension $g^*: X \to Y$ has a dense and connected graph. We define

$$g^*(x) = \begin{cases} f(x) & \text{if} \quad x \in X - B, \\ g(x) & \text{if} \quad x \in B. \end{cases}$$

LEMMA 3. *Let X be a space such that $w(X) \geqslant \chi_0$ and, for every non-empty open set G, $\mathrm{card}\, G \geqslant \tau$, where $\tau \geqslant w(X)$.*

Then X is a union of τ mutually disjoint sets, each of them being τ-dense in X.

Proof. (I) Let $\tau > w(X)$ and let R of $\mathrm{card}\, R = w(X)$ be a base for the topology on X. Let β be an initial number of the power τ. We shall define sets $A_\xi = \{x_G^\xi\colon G \in R\}$, $\xi < \beta$ such that

1° $x_G^\xi \in G$ for every $G \in R$ and $\xi < \beta$,

2° if $\xi \neq \xi'$, $\xi, \xi' < \beta$, then $A_\xi \cap A_{\xi'} = \emptyset$.

For every $G \in R$ we assume x_G^0 to be an element of G. Let us assume that the points $x_G^{\xi'}$ are defined for every $G \in R$ and $\xi' < \xi$. Notice that $\mathrm{card}\{x_G^{\xi'}\colon G \in R,\ \xi' < \xi\} < \tau$. This makes it possible to define the set $A_\xi \subset X - \bigcup_{\xi' < \xi} A_{\xi'}$ whose elements satisfy 1°.

Since $\tau \cdot \tau = \tau$, there exists a one-to-one map $\varphi: \{(\gamma, \gamma')\colon \gamma, \gamma' < \beta\} \xrightarrow{\text{onto}} \{\xi\colon \xi < \beta\}$. Let $B_\gamma = \bigcup_{\gamma' < \beta} A_{\varphi(\gamma, \gamma')}$. From 1° and 2° it follows that $\mathrm{card}(B_\gamma \cap G) \geqslant \tau$ and $B_\gamma \cap B_{\gamma'} = \emptyset$ for every $\gamma \neq \gamma'$ and a non-empty open set G. The sets $C_0 = X - \bigcup_{\gamma > 0} B_\gamma$ and $C_\gamma = B_\gamma$ if $\gamma > 0$, are mutually disjoint and each of them is τ-dense in X.

(II) Let $\tau = w(X)$ and let $\{G_\xi: \xi < \alpha\}$ be a base for the topology, where α is an initial number of the power τ. We shall define points x_ξ^γ, $\xi, \gamma < \alpha$, such that

1° $x_\xi^\gamma \in G_\xi$ for every $\gamma, \xi < \alpha$,

2° if $\gamma \neq \gamma'$ or $\xi \neq \xi'$, then $x_\xi^\gamma \neq x_{\xi'}^{\gamma'}$.

Let us assume that the points $x_{\xi'}^{\gamma'}$, $\gamma', \xi' < \xi$, are defined. Since $\operatorname{card}\{x_{\xi'}^{\gamma'}: \gamma', \xi' < \xi\} < \tau$, it is possible to define points satisfying 1° and 2°. We put $C_0 = X - \{x_\xi^\gamma: \xi < \alpha, \gamma > 0\}$ and $C_\gamma = \{x_\xi^\gamma: \xi < \alpha\}$ if $\gamma > 0$.

THEOREM 2. *If Y is a group and X, Y satisfy the assumptions of Theorem 1, then every map $f: X \to Y$ is a sum of two maps $h, k: X \to Y$, $f(x) = h(x) + k(x)$, where h and k have connected and dense graphs in $X \times Y$.*

Proof. It follows from Lemma 3 that there exist mutually disjoint sets B_1, B_2, each $2^{w(X)}$-dense in X and such that $B_1 \cup B_2 = X$. Let $h_1: X \to Y$, $k_2: X \to Y$ be extensions of maps having properties as in Theorem 1, respectively for sets B_1 and B_2. We define

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in B_1, \\ f(x) - k_2(x) & \text{if } x \in B_2, \end{cases} \qquad k(x) = \begin{cases} f(x) - h_1(x) & \text{if } x \in B_1, \\ k_2(x) & \text{if } x \in B_2. \end{cases}$$

THEOREM 3. *If X, Y satisfy the assumptions of Theorem 1, then every map $f: X \to Y$ is the point-wise limit of a sequence $f_1, f_2, \ldots$ where every map $f_n: X \to Y$, $n = 1, 2, \ldots$ has a connected and dense graph. More precisely, for every $x \in X$ there exists an n_0 such that if $n \geq n_0$, then $f_n(x) = f(x)$.*

Proof. Let $B_1, B_2, \ldots$ be a sequence of mutually disjoint sets each of them being $2^{w(X)}$-dense in X and $X = \bigcup_{i=1}^{\infty} B_i$. Let $A_n = \bigcup_{i=1}^{n} B_i$. The sets A_n and $X - A_n$ are $2^{w(X)}$-dense in X. From the Corollary it follows that for every n there exists a map $f_n: X \to Y$ such that the graph of f_n is dense and connected in $X \times Y$ and $f_n | A_n = f | A_n$.

LEMMA 4. *If X is a vector space satisfying the assumptions of Theorem 1, then there exists a $2^{w(X)}$-dense vector base for X.*

Proof. A set A is called linearly independent if $r_1 x_1 + \ldots + r_n x_n = 0$ iff $r_1 = \ldots = r_n = 0$, where r_i is rational and $x_i \in A$ for $i = 1, 2, \ldots, n$, $n = 1, 2, \ldots$ Let us write $S(A) = \{r_1 x_1 + \ldots + r_n x_n: x_i \in A,\ r_i \text{ is rational}, i = 1, \ldots, n,\ n = 1, 2, \ldots\}$.

Let $\{G_\xi: \xi < \alpha\}$ be a base for the topology on X, where α is an ordinal number of power $w(X)$. Let β be an initial number of power $2^{w(X)}$. We shall define by induction points x_ξ^η, $\xi < \alpha$, $\eta < \beta$, such that:

1° $x_\xi^\eta \in G_\xi$ for every $\xi < \alpha$, $\eta < \beta$,

2° $B_0 = \{x_\xi^\eta: \xi < \alpha, \eta < \beta\}$ is linearly independent.

Let $0 \neq x_0^0 \in G_0$. Let us assume that the points $x_{\xi'}^0$, are defined. Since $\operatorname{card} S(x_{\xi'}^0: \xi' < \xi) < 2^{w(X)}$, we may put $x_\xi^0 \in G_\xi - S(x_{\xi'}^0: \xi' < \xi)$.

Now, let us assume that the points $x_\xi^{\eta'}$, $\eta' < \eta$, $\xi < \alpha$ are defined. Let us notice that $\operatorname{card} S(x_\xi^{\eta'}: \eta' < \eta, \xi < \alpha) < 2^{w(X)}$. This makes it possible to define points x_ξ^η, $\xi < \alpha$ such that $x_\xi^\eta \in G_\xi - S(x_{\xi'}^{\eta'}: \eta' < \eta$ and $\xi' < \alpha$ or $\eta' = \eta$ and $\xi' < \xi)$.

From the Kuratowski–Zorn Lemma it follows that there exists a vector base B containing B_0. The base B is $2^{w(X)}$-dense in X because B_0 is $2^{w(X)}$-dense in X.

THEOREM 4. *If X, Y are vector spaces satisfying the assumptions of Theorem 1, then there exists a map $F: X \to Y$ satisfying Cauchy's equation $F(x+y) = F(x) + F(y)$ and having a dense and connected graph.*

Proof. Let B be a $2^{w(X)}$-dense vector base in X, and let $f: B \to Y$ be a map such that every extension $f^*: X \to Y$ of f has a dense and connected graph. We define $F: X \to Y$; $F(x) = r_1 f(x_1) + \ldots + r_n f(x_n)$, where $x = r_1 x_1 + \ldots + r_n x_n$ and $x_i \in B$, r_i is rational for $i = 1, \ldots, n$, $n = 1, 2, \ldots$

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