c-continuous fundamental groups

by

Karl R. Gentry and **Hughes B. Hoyle*, III** (Greensboro, N. C.)

1. Introduction. The idea of c-continuous functions was first introduced in [1]. The purpose of this paper is to define a new type of fundamental group using c-continuous homotopy for the equivalence relation instead of the usual homotopy.

Section 2 consists of a necessary preliminary result.

In Section 3, we give our definitions and prove our main theorems most of which parallel the usual theorems about fundamental groups.

In Section 4, examples are given which show that our type of fundamental group is non-trivial and different from the usual fundamental group.

Throughout this paper the symbol I will be used for the closed unit interval and the symbol $f \underset{y_0}{\sim} g$ will mean that f is homotopic to g modulo y_0. The reader is referred to [1] for definitions not covered in this paper.

2. Preliminary result.

DEFINITION 1. [1] Let X and Y be topological spaces, let $f\colon X \to Y$ be a function, and let $p \in X$. Then f is said to be *c-continuous* at p provided if U is an open subset of Y containing $f(p)$ such that $Y-U$ is compact, then there is an open subset V of X containing p such that $f(V) \subset U$. The function f is said to be *c-continuous* (on X) provided f is c-continuous at each point of X.

THEOREM 1. *If $f\colon X \to Y$ is c-continuous and $g\colon Y \to Z$ is a homeomorphism from Y onto Z, then $gf\colon X \to Z$ is c-continuous.*

Proof. Let U be an open subset of Z such that $Z-U$ is compact. Then since g is a homeomorphism, $Y-g^{-1}(U) = g^{-1}(Z-U)$ is compact and $g^{-1}(U)$ is open. By [1, Th. 1, p. 1], since f is c-continuous, $f^{-1}(g^{-1}(U))$ is open. Thus $(gf)^{-1}(U)$ is open and by [1, Th. 1, p. 1], gf is c-continuous.

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3. Definitions and main results.

DEFINITION 2. Let Y be a topological space and let $y_0 \in Y$. Then $C(Y, y_0)$ is the set of all continuous functions $f\colon I \to Y$ such that $f(0) = y_0 = f(1)$. We say that f is *c-continuous homotopic to g modulo* y_0, denoted by $f \overset{c}{\underset{y_0}{\sim}} g$, provided there is a c-continuous function $F\colon I\times I \to Y$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$, and $F(0, t) = y_0 = F(1, t)$ for all $x \in I$, $t \in I$.

THEOREM 2. *The relation* $\overset{c}{\underset{y_0}{\sim}}$ *is an equivalence relation on* $C(Y, y_0)$.

Proof. Let $f \in C(Y, y_0)$. Since $f \underset{y_0}{\sim} f$ and every continuous function is c-continuous, then $f \overset{c}{\underset{y_0}{\sim}} f$.

Let $f, g \in C(Y, y_0)$ and suppose $f \overset{c}{\underset{y_0}{\sim}} g$. Then there is a c-continuous function $F\colon I\times I \to Y$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$, and $F(0, t) = y_0 = F(1, t)$ for all $x \in I$, $t \in I$. Define $K\colon I\times I \to I\times I$ by $K(x, t) = (x, 1-t)$ and define $G = FK$. Since F is c-continuous and K is continuous by [1, Th. 3, p. 4], G is c-continuous. Now $G(x, 0) = F\big(K(x, 0)\big) = F(x, 1) = g(x)$, $G(x, 1) = F\big(K(x, 1)\big) = F(x, 0) = f(x)$, and $G(0, t) = F(0, 1-t) = y_0 = F(1, 1-t) = G(1, t)$ for all $x \in I$, $t \in I$. Hence, $g \overset{c}{\underset{y_0}{\sim}} f$.

Let $f, g, h \in C(Y, y_0)$ and suppose that $f \overset{c}{\underset{y_0}{\sim}} g$ and $g \overset{c}{\underset{y_0}{\sim}} h$. Then there are c-continuous functions $F, G\colon I\times I \to Y$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$, and $F(0, t) = y_0 = F(1, t)$ for all $x \in I$, $t \in I$ and $G(x, 0) = g(x)$, $G(x, 1) = h(x)$, and $G(0, t) = y_0 = G(1, t)$ for all $x \in I$, $t \in I$. Define $H\colon I\times I \to Y$ by

$$H(x, t) = \begin{cases} F(x, 2t) & \text{if} \quad x \in I,\ 0 \leqslant t \leqslant \tfrac{1}{2}\,. \\ G(x, 2t-1) & \text{if} \quad x \in I,\ \tfrac{1}{2} \leqslant t \leqslant 1\,. \end{cases}$$

Since F and G are c-continuous, by [1, Th. 3, p. 4], $H|_{I\times[0,1/2]}$ and $H|_{I\times[1/2,1]}$ are c-continuous. Therefore, by [1, Th. 4, p. 4], H is c-continuous. Now $H(x, 0) = F(x, 0) = f(x)$ and $H(x, 1) = G(x, 1) = h(x)$ for all $x \in I$. Also

$$H(0, t) = \begin{cases} F(0, 2t) & \text{if} \quad 0 \leqslant t \leqslant \tfrac{1}{2} \\ G(0, 2t-1) & \text{if} \quad \tfrac{1}{2} \leqslant t \leqslant 1 \end{cases} = y_0 \quad \text{for all } t \in I\,,$$

and

$$H(1, t) = \begin{cases} F(1, 2t) & \text{if} \quad 0 \leqslant t \leqslant \tfrac{1}{2} \\ G(1, 2t-1) & \text{if} \quad \tfrac{1}{2} \leqslant t \leqslant 1 \end{cases} = y_0 \quad \text{for all } t \in I\,.$$

Therefore, $f \overset{c}{\underset{y_0}{\sim}} h$ and hence $\overset{c}{\underset{y_0}{\sim}}$ is an equivalence relation on $C(Y, y_0)$.

DEFINITION 3. Let $f, g \in C(Y, y_0)$. Then $f * g$ is the function in $C(Y, y_0)$ defined by

$$(f * g)(x) = \begin{cases} f(2x) & \text{if} \quad 0 \leqslant x \leqslant \tfrac{1}{2}\,, \\ g(2x-1) & \text{if} \quad \tfrac{1}{2} \leqslant x \leqslant 1\,. \end{cases}$$

The equivalence relation $\overset{c}{\underset{y_0}{\sim}}$ breaks $C(Y, y_0)$ into disjoint equivalence classes. Let $C_1(Y, y_0)$ denote this set of equivalence classes. If $[f], [g] \in C_1(Y, y_0)$, then we define $[f]\cdot[g]$ to be $[f * g]$.

LEMMA 3.1. *If* $[f], [g] \in C_1(Y, y_0)$, *then* $[f]\cdot[g]$ *is well-defined.*

Proof. Let $f_1, f_2 \in [f]$ and $g_1, g_2 \in [g]$. Since $f_1, f_2 \in [f]$, there is a c-continuous function $F\colon I\times I \to Y$ such that $F(x, 0) = f_1(x)$, $F(x, 1) = f_2(x)$, and $F(0, t) = y_0 = F(1, t)$ for all $x \in I$, $t \in I$. Since $g_1, g_2 \in [g]$, there is a c-continuous function $G\colon I\times I \to Y$ such that $G(x, 0) = g_1(x)$, $G(x, 1) = g_2(x)$, and $G(0, t) = y_0 = G(1, t)$ for all $x \in I$, $t \in I$. Define a function $H\colon I\times I \to Y$ by

$$H(x, t) = \begin{cases} F(2x, t) & \text{if} \quad 0 \leqslant x \leqslant \tfrac{1}{2},\ t \in I\,, \\ G(2x-1, t) & \text{if} \quad \tfrac{1}{2} \leqslant x \leqslant 1,\ t \in I\,. \end{cases}$$

Since F and G are c-continuous functions, by [1, Th. 3, p. 4], $H|_{[0,1/2]\times I}$ and $H|_{[1/2,1]\times I}$ are c-continuous functions. Therefore, by [1, Th. 4, p. 4], H is a c-continuous function. Now

$$H(x, 0) = \begin{cases} F(2x, 0) \\ G(2x-1, 0) \end{cases} = \begin{cases} f_1(2x) \\ g_1(2x-1) \end{cases} = (f_1 * g_1)(x) \quad \text{for all } x \in I\,,$$

and

$$H(x, 1) = \begin{cases} F(2x, 1) \\ G(2x-1, 1) \end{cases} = \begin{cases} f_2(2x) \\ g_2(2x-1) \end{cases} = (f_2 * g_2)(x) \quad \text{for all } x \in I\,,$$

and

$$H(0, t) = F(0, t) = y_0 = G(1, t) = H(1, t) \quad \text{for all } t \in I\,.$$

Thus $f_1 * g_1 \overset{c}{\underset{y_0}{\sim}} f_2 * g_2$ and hence $[f]\cdot[g]$ is well-defined.

DEFINITION 4. The *identity element* of $C_1(Y, y_0)$, denoted by $[e]$, is the equivalence class which contains the function $e\colon I \to Y$ defined by $e(x) = y_0$ for all $x \in I$.

LEMMA 3.2. *If* $[f] \in C_1(Y, y_0)$, *then* $[f]\cdot[e] = [f]$.

Proof. Since $f * e \underset{y_0}{\sim} f$, then $f * e \overset{c}{\underset{y_0}{\sim}} f$. Thus $[f]\cdot[e] = [f]$.

DEFINITION 5. If $[f] \in C_1(Y, y_0)$, then $[f]^{-1}$ is the element of $C_1(Y, y_0)$ containing the function $g\colon I \to Y$ defined by $g(t) = f(1-t)$ for all $t \in I$.

LEMMA 3.3. *If* $[f] \in C_1(Y, y_0)$, *then* $[f]\cdot[f]^{-1} = [e]$.

Proof. Let $g(t) = f(1-t)$ for all $t \in I$. Then $[f]\cdot[f]^{-1} = [f]\cdot[g] = [f * g] = [e]$.

THEOREM 3. *The ordered pair* $(C_1(Y, y_0), \cdot)$ *is a group.*

Proof. It remains only to show that $\cdot$ is associative. But this follows immediately since $(f*g)*h \underset{y_0}{\sim} f*(g*h)$ implies $(f*g)*h \overset{c}{\underset{y_0}{\sim}} f*(g*h)$.

From now on, the symbol $C_1(Y, y_0)$ will denote the set $C_1(Y, y_0)$ together with the operation $\cdot$ and $C_1(Y, y_0)$ will be called the *c-continuous fundamental group of* Y *with respect to* y_0.

THEOREM 4. *Let* $y_0 \in Y$. *Then there is an epimorphism* λ: $\pi_1(Y, y_0) \to C_1(Y, y_0)$.

Proof. Let $[f] \in \pi_1(Y, y_0)$. Define $\lambda([f])$ to be the equivalence class in $C_1(Y, y_0)$ which contains f.

Let $[f] \in \pi_1(Y, y_0)$ and let $f, g \in [f]$. Then $f \underset{y_0}{\sim} g$ and thus $f \overset{c}{\underset{y_0}{\sim}} g$. Thus, λ is well-defined.

Let $M \in C_1(Y, y_0)$ and let $f \in M$. Then $[f] \in \pi_1(Y, y_0)$ and $\lambda([f]) = M$. Therefore, λ is onto.

Let $[f], [g] \in \pi_1(Y, y_0)$. Then $\lambda([f]\cdot[g]) = \lambda([f*g]) = [f*g] = [f]\cdot[g] = \lambda([f])\cdot\lambda([g])$. Hence, λ is an epimorphism.

THEOREM 5. *Let* $y_1 \in Y_1$, *let* $y_2 \in Y_2$, *and let* H: $Y_1 \to Y_2$ *be a homeomorphism from* Y_1 *onto* Y_2 *with* $H(y_1) = y_2$. *Then* $C_1(Y_1, y_1)$ *is isomorphic to* $C_1(Y_2, y_2)$.

Proof. Let $f \in C(Y_1, y_1)$. Then f: $I \to Y_1$ and since H: $Y_1 \to Y_2$, Hf: $I \to Y_2$. Since Hf is continuous and $Hf(0) = H(f(0)) = H(y_1) = y_2 = H(y_1) = H(f(1)) = Hf(1)$, $Hf \in C(Y_2, y_2)$. Define λ: $C_1(Y_1, y_1) \to C_1(Y_2, y_2)$ by $\lambda([f]) = [Hf]$.

Let $[f] \in C_1(Y_1, y_1)$ and let $f, g \in [f]$. Then $f \overset{c}{\underset{y_1}{\sim}} g$. Thus, there is a c-continuous function F: $I \times I \to Y_1$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$, and $F(0, t) = y_1 = F(1, t)$ for all $x \in I$, $t \in I$. Then HF: $I \times I \to Y_2$ and by Theorem 1, HF is a c-continuous function. Now $HF(x, 0) = H(F(x, 0)) = H(f(x)) = Hf(x)$ for all $x \in I$, $HF(x, 1) = H(F(x, 1)) = H(g(x)) = Hg(x)$ for all $x \in I$, and $HF(0, t) = H(F(0, t)) = H(y_1) = y_2 = H(y_1) = H(F(1, t)) = HF(1, t)$ for all $t \in I$. Therefore, $Hf \overset{c}{\underset{y_2}{\sim}} Hg$ and $[Hf] = [Hg]$. Hence, λ is well-defined.

Let $[f] \in C_1(Y_2, y_2)$. Then $f \in C(Y_2, y_2)$. Since f is continuous, and $H^{-1}f(0) = H^{-1}(f(0)) = H^{-1}(y_2) = y_1 = H^{-1}(y_2) = H^{-1}(f(1)) = H^{-1}f(1)$, $H^{-1}f \in C(Y_1, y_1)$. Therefore, $[H^{-1}f] \in C_1(Y_1, y_1)$ and since $\lambda([H^{-1}f]) = [HH^{-1}f] = [f]$, λ is onto.

Let $[f], [g] \in C_1(Y_1, y_1)$ such that $\lambda([f]) = \lambda([g])$. Then $[Hf] = [Hg]$ and therefore, $Hf \overset{c}{\underset{y_2}{\sim}} Hg$. Thus there is a c-continuous function K: $I \times I \to Y_2$ such that $K(x, 0) = Hf(x)$, $K(x, 1) = Hg(x)$, and $K(0, t) = y_2 = K(1, t)$ for all $x \in I$, $t \in I$. Thus by Theorem 1, $H^{-1}K$ is a c-continuous function. Now

$$
\begin{aligned}
H^{-1}K(x, 0) &= H^{-1}(K(x, 0)) = H^{-1}(Hf(x)) = f(x) && \text{for all } x \in I,\\
H^{-1}K(x, 1) &= H^{-1}(K(x, 1)) = H^{-1}(Hg(x)) = g(x) && \text{for all } x \in I,\\
H^{-1}K(0, t) &= H^{-1}(K(0, t)) = H^{-1}(y_2) = y_1 && \text{for all } t \in I,\\
H^{-1}K(1, t) &= H^{-1}(K(1, t)) = H^{-1}(y_2) = y_1 && \text{for all } t \in I.
\end{aligned}
$$

Therefore $f \overset{c}{\underset{y_1}{\sim}} g$ and thus $[f] = [g]$. Hence, λ is one-to-one.

Let $[f], [g] \in C_1(Y_1, y_1)$. From the definition of $f * g$, it is obvious that $H(f*g) = Hf * Hg$. Then

$$
\begin{aligned}
\lambda([f]\cdot[g]) &= \lambda([f*g]) = [H(f*g)] = [Hf * Hg] = [Hf]\cdot[Hg]\\
&= \lambda([f])\cdot\lambda([g]).
\end{aligned}
$$

Hence, λ is an isomorphism.

THEOREM 6. *If* Y *is pathwise connected and* $y_0, y_1 \in Y$, *then* $C_1(Y, y_0)$ *is isomorphic to* $C_1(Y, y_1)$.

Proof. Since Y is pathwise connected, there is a continuous function p: $I \to Y$ such that $p(0) = y_0$ and $p(1) = y_1$. Define $\bar{p}$: $I \to Y$ by $\bar{p}(x) = p(1-x)$ for all $x \in I$. Then $\bar{p}$ is continuous. Let e_0 and e_1 be the functions defined by $e_0(x) = y_0$ and $e_1(x) = y_1$ for all $x \in I$. It is well known that $p * \bar{p} \underset{y_0}{\sim} e_0$ and $\bar{p} * p \underset{y_1}{\sim} e_1$. Therefore, $[p * \bar{p}] = [e_0]$ and $[\bar{p} * p] = [e_1]$. Define λ: $C_1(Y, y_0) \to C_1(Y, y_1)$ by $\lambda([f]) = [\bar{p} * (f * p)]$. Since $*$ satisfies the associative law up to homotopy, for convenience in the future the parenthesis in $[\bar{p} * (f * p)]$ will be omitted.

Let $f, g \in C(Y, y_0)$ such that $f \overset{c}{\underset{y_0}{\sim}} g$. Then there is a c-continuous function F: $I \times I \to Y$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$, $F(0, t) = y_0 = F(1, t)$ for all $x \in I$, $t \in I$. Now

$$
(\bar{p} * (f * p))(x) = \begin{cases} \bar{p}(2x) & \text{if } 0 \leqslant x \leqslant \tfrac{1}{2},\\ f(4x-2) & \text{if } \tfrac{1}{2} \leqslant x \leqslant \tfrac{3}{4},\\ p(4x-3) & \text{if } \tfrac{3}{4} \leqslant x \leqslant 1, \end{cases}
$$

and

$$
(\bar{p} * (g * p))(x) = \begin{cases} \bar{p}(2x) & \text{if } 0 \leqslant x \leqslant \tfrac{1}{2},\\ g(4x-2) & \text{if } \tfrac{1}{2} \leqslant x \leqslant \tfrac{3}{4},\\ p(4x-3) & \text{if } \tfrac{3}{4} \leqslant x \leqslant 1. \end{cases}
$$

Define $G\colon I\times I\to Y$ by

$$G(x,t)=\begin{cases}\overline{p}(2x) & \text{if}\quad 0\leqslant x\leqslant \tfrac{1}{2},\ t\in I\,,\\ F(4x-2,t) & \text{if}\quad \tfrac{1}{2}\leqslant x\leqslant \tfrac{3}{4},\ t\in I\,,\\ p(4x-3) & \text{if}\quad \tfrac{3}{4}\leqslant x\leqslant 1,\ t\in I\,.\end{cases}$$

It is easy to check that G is well-defined. Since $G|_{[0,1/2]\times I}$ is continuous and $G|_{[1/2,3/4]\times I}$ is c-continuous (by [1, Th. 3, p. 4]), $G|_{[0,3/4]\times I}$ is c-continuous by [1, Th. 4, p. 4]. Since $G|_{[3/4,1]\times I}$ is continuous, by [1, Th. 4, p. 4], G is c-continuous. Now

$$G(x,0)=\begin{cases}\overline{p}(2x)\\ F(4x-2,0)\\ p(4x-3)\end{cases}=\begin{cases}\overline{p}(2x)\\ f(4x-2)=\big(\overline{p}*(f*p)\big)(x)\\ p(4x-3)\end{cases}\quad\text{for all } x\in I,$$

and

$$G(x,1)=\begin{cases}\overline{p}(2x)\\ F(4x-2,1)\\ p(4x-3)\end{cases}=\begin{cases}\overline{p}(2x)\\ g(4x-2)=\big(\overline{p}*(g*p)\big)(x)\\ p(4x-3)\end{cases}\quad\text{for all } x\in I,$$

and

$$G(0,t)=\overline{p}(0)=p(1)=y_1=p(1)=G(1,t)\quad\text{for all } t\in I\,.$$

Therefore, $\overline{p}*(f*p)\underset{y_1}{\overset{c}{\sim}}\overline{p}*(g*p)$. Hence, if $f\underset{y_0}{\overset{c}{\sim}}g$, then $\overline{p}*(f*p)\underset{y_1}{\overset{c}{\sim}}\overline{p}*(g*p)$. Similarly, if $h\underset{y_1}{\overset{c}{\sim}}k$, then $p*(h*\overline{p})\underset{y_0}{\overset{c}{\sim}}p*(k*\overline{p})$.

Let $[f]\in C_1(Y,y_0)$. Then $f\in C(Y,y_0)$ and $\overline{p}*(f*p)$ is continuous. Now $\big(\overline{p}*(f*p)\big)(0)=\overline{p}(0)=p(1)=y_1$ and $\big(\overline{p}*(f*p)\big)(1)=(f*p)(1)=p(1)=y_1$. Therefore, $\overline{p}*(f*p)\in C(Y,y_1)$. Thus, $\lambda([f])\in C_1(Y,y_1)$. Hence, λ is into.

Let $[f]\in C_1(Y,y_0)$ and let $f,g\in[f]$. Then $f\underset{y_0}{\overset{c}{\sim}}g$. Thus, $\overline{p}*(f*p)\underset{y_1}{\overset{c}{\sim}}\overline{p}*(g*p)$. Therefore, λ is well-defined.

Let $[f],[g]\in C_1(Y,y_0)$ such that $\lambda([f])=\lambda([g])$. Then $[\overline{p}*f*p]=[\overline{p}*g*p]$ and therefore $\overline{p}*(f*p)\underset{y_1}{\overset{c}{\sim}}\overline{p}*(g*p)$. But this means that $p*\overline{p}*f*p*\overline{p}\underset{y_0}{\overset{c}{\sim}}p*\overline{p}*g*\overline{p}*p$ and thus $e_0*f*e_0\underset{y_0}{\overset{c}{\sim}}e_0*g*e_0$. Therefore, $f\underset{y_0}{\overset{c}{\sim}}g$. Hence, $[f]=[g]$ and λ is one-to-one.

Let $[f]\in C_1(Y,y_1)$. Then $[p*f*\overline{p}]\in C_1(Y,y_0)$ and $\lambda([p*f*\overline{p}])=[\overline{p}*p*f*\overline{p}*p]=[e_1*f*e_1]=[f]$. Hence λ is onto.

Let $[f],[g]\in C_1(Y,y_0)$. Then

$$\lambda([f]\cdot[g])=\lambda([f*g])=[\overline{p}*f*g*p]=[\overline{p}*f*e_0*g*p]$$
$$=[\overline{p}*f*p*\overline{p}*g*p]=[\overline{p}*f*p]\cdot[\overline{p}*g*p]=\lambda([f])\cdot\lambda([g])\,.$$

Hence, λ is an isomorphism.

From now on in view of the previous theorem, if Y is pathwise connected, then $C_1(Y)$ will denote the c-continuous fundamental group of Y with respect to any point of Y.

THEOREM 7. *If Y is a compact space and $y_0\in Y$, then $\pi_1(Y,y_0)=C_1(Y,y_0)$.*

Proof. This follows immediately from the fact that if the range space is compact then a function is continuous if and only if it is c-continuous.

4. Examples.

THEOREM 8. *Let $Y=\text{Plane}-\{(0,0)\}$, let T be the usual induced topology for Y and let $y_0=(1,0)$. If $f(x)=y_0$ for all $x\in I$ and $g\colon I\to Y$ is a loop at y_0, then $f\underset{y_0}{\overset{c}{\sim}}g$.*

Proof. Let D be the closed unit disc and $H\colon I\times I\to D$ be a homeomorphism from $I\times I$ onto D. Then it is clear that H maps the boundary of $I\times I$ onto the boundary of D and the interior of $I\times I$ onto the interior of D. Define $F\colon[0,1)\to[1,\infty)$ by $F(x)=1/(1-x)$ for all $x\in[0,1)$ and define $G\colon\{\text{interior } D\}\to(1,\infty)$ by $G(x,y)=F\big((x^2+y^2)^{1/2}\big)$. Then F is a homeomorphism and G is continuous and onto. Define $K\colon I\times I\to Y$ by

$$K(x,y)=\begin{cases}GH(x,y) & \text{if}\quad 0<x<1,\ 0<y<1\,,\\ y_0 & \text{if}\quad x=0,\ \text{or}\ x=1,\ \text{or}\ y=1\,,\\ g(x) & \text{if}\quad y=0\,.\end{cases}$$

Now $K(0,t)=y_0=K(1,t)$, $K(x,0)=g(x)$, and $K(x,1)=f(x)$ for all $x\in I$, $t\in I$. It is clear that K is continuous on the interior of $I\times I$ and that if B is the boundary of $I\times I$, then $K|_B$ is continuous. So it remains to show that K is c-continuous on B. Let $p\in B$ and let U be an open subset of Y with compact compliment containing $K(p)$. Since $K|_B$ is continuous, there exists an open subset M of B containing p such that $K(M)\subset U$. Since U has compact compliment, there is a number $r>1$ such that $(r,\infty)\subset U$. Let $R=\{(x,y)\mid 1>(x^2+y^2)^{1/2}>1-(1/r)\}$. Then R is open and $G(R)\subset U$. Now $H^{-1}(R)$ is an open subset of $I\times I$. Let $N=H^{-1}(R)\cup M$. If $x\in H^{-1}(R)$, then $H(x)\in R$ and $K(x)=GH(x)\in G(R)\subset U$. Thus $K(N)\subset U$. Let $q\in M$ and suppose q is a limit point of $(I\times I)-\big(B\cup H^{-1}(R)\big)$. Then since H is a homeomorphism $H(q)$ is a limit point in D of $\{(x,y)\mid (x^2+y^2)^{1/2}\leqslant 1-(1/r)\}$. But q is in B, so $H(q)$ is in the boundary of D which is impossible. Thus q is not a limit point of $(I\times I)-\big(B\cup H^{-1}(R)\big)$. Also q is not a limit point of $B-M$. Therefore q is an interior point of $H^{-1}(R)\cup M$. Thus, since $H^{-1}(R)$ is open and each point of M is an interior point of $H^{-1}(R)\cup M$, $N=H^{-1}(R)\cup M$ is open. Hence, K is c-continuous and $f\underset{y_0}{\overset{c}{\sim}}g$.

COROLLARY 8.1. *Let $Y = \text{Plane} - \{(0,0)\}$ and let T be the usual induced topology for Y. Then $C_1(Y)$ is the trivial group.*

Proof. This follows immediately from Theorem 8.

The following example shows that two spaces may have the same fundamental group and yet have different c-continuous fundamental groups.

EXAMPLE 1. Let X be the unit circle with the usual topology and let $Y = \text{Plane} - \{(0,0)\}$ with the usual induced topology. Then $\pi_1(X)$ is isomorphic to $\pi_1(Y)$ but $C_1(X)$ and $C_1(Y)$ are not isomorphic.

Proof. By Theorem 7, $\pi_1(X)$ is isomorphic to $C_1(X)$. Thus both are isomorphic to the integers under addition. However $\pi_1(Y)$ is isomorphic to the integers under addition while by Corollary 8.1, $C_1(Y)$ is the trivial group.

The following example shows that there are non-compact spaces with non-trivial c-continuous fundamental groups.

EXAMPLE 2. Let $A = \{(x,y)|\ x^2+y^2 = 1\}$, let $B = \bigcup_{n=2}^{\infty}\{(x,y)|\ x = n,\ 0 \leqslant y \leqslant n\}$, let $Y = A \cup B$, and let T be the usual induced plane topology for Y. Then Y is not compact and if $y_0 = (1,0)$, $C_1(Y, y_0)$ is not trivial.

Proof. Let $f\colon I \to Y$ be defined by $f(x) = y_0$ for all $x \in I$. Let $g\colon I \to Y$ be defined by $g(x) = (\cos 2\pi x, \sin 2\pi x)$ for all $x \in I$. Suppose $f \overset{c}{\underset{y_0}{\sim}} g$. Then there exists a c-continuous function $F\colon I\times I \to Y$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$, and $F(0,t) = y_0 = F(1,t)$ for all $x \in I$, $t \in I$. If the range of F is bounded, then the range of F is contained in some compact subset of Y and by [1], Th. 5, p. 5, F is continuous and hence, $f \underset{y_0}{\sim} g$. But f is not homotopic to g modulo y_0, and thus the range of F is unbounded. Therefore the range of F must intersect an infinite number of components of Y. Let $\mathcal{M}$ be the collection of components of Y which intersect the range of F. Then $I\times I = \bigcup\{F^{-1}(M)|\ M \in \mathcal{M}\}$. If $M \in \mathcal{M}$, then by [1, Th. 1, p. 1], since F is c-continuous and M is compact, then $F^{-1}(M)$ is closed. It is clear that if $M, N \in \mathcal{M}$, then $F^{-1}(M) \cap F^{-1}(N) = \emptyset$. Thus $I\times I$ is the union of a countably infinite collection of disjoint closed sets which is impossible. Thus f is not c-continuous homotopic to g and hence $C_1(Y, y_0)$ is not trivial.

The authors have not yet been able to show that there is a non-compact, pathwise connected space which has a non-trivial c-continuous fundamental group. A candidate for such a space might be as follows: Let Y be the subset of the plane under the usual topology defined by

$$Y = \{(x,y)|\ x^2+y^2 = 1\} \cup \Big(\bigcup_{n=1}^{\infty}\{(x,y)|\ y = n(x-2)+n,\ 1 \leqslant x \leqslant 2\}\Big).$$

Another interesting problem is that of almost continuous fundamental groups and connectivity fundamental groups. In [2], the necessary preliminary theorems are proved about almost continuous functions and connectivity maps to insure that these fundamental groups can be defined in a similar manner to c-continuous fundamental groups. These groups also turn out to have the usual properties of fundamental groups. The problem is to decide whether or not these groups are different in some cases from the usual fundamental groups.

References

[1] K. R. Gentry and H. B. Hoyle, III, *C-continuous functions*, Yokohama Mathematical Journal 18 (1970), pp. 71–76.

[2] H. B. Hoyle, III, *Connectivity maps and almost continuous functions*, Duke Mathematical Journal 37 (1970), pp. 671–680.

THE UNIVERSITY OF NORTH CAROLINA AT GREENSBORO

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