A note on E-compact spaces [1]

by

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1. Introduction. This paper is divided into two parts. In part I we give a semi-internal characterization of E-compact spaces whereas in part 2 we study one-point E-completely regular extensions, with the purpose of establishing necessary and sufficient conditions for an E-completely regular space X to have a one-point E-compactification. Let us recall that $X \in \boldsymbol{C}(E)$ is E-compact (written $X \in K(E)$) if $X_{\mathrm{cl}} \subset E^m$ [2]. S. Mrówka [9] has proved that X is E-compact iff, for every E-completely regular proper extension εX of X, the class $C(X, E)$ is non-extendable over εX.

2. E-compactness. In this section we study E-compactness and some related concepts. We show that for $X \in \boldsymbol{C}(E)$ the following conditions are equivalent:

THEOREM 2.1 [3].

1) *X is E-compact.*

2) *For every net x_a in X, x_a converges if $f(x_a)$ converges for every $f \in C(X, E)$.*

Proof. 1) $\Rightarrow$ 2) Let us index the family $C(X,E)$ by a set Ξ and let $m = \operatorname{card}\Xi$. Since $X \in K(E)$, the parametric mapping h (induced by $C(X,E)$) where $h\colon X \to E^m$, is a closed homeomorphism. So let x_a be a net in X such that $f_\xi(x_a) \to x_\xi$ for every $\xi \in \Xi$. Since $\pi_\xi\big(h(x)\big) = f_\xi(x)$, we obtain $h(x_a) \to x \in E^m$, where $\pi_\xi(x) = x_\xi$. Recall that $h\{X\}$ is closed in E^m and hence $x \in h\{X\}$. Therefore, since h is a homeomorphism, x_a converges in X.

2) $\Rightarrow$ 1) Let $h\colon X \to E^m$ be the evaluation mapping induced by $C(X, E)$. Choose $x \in \overline{h\{X\}}^{E^m}$ and let x_a be a net in $h\{X\}$ such that $x_a \to x$. Select

[1] This paper is a part of a doctoral dissertation prepared at the Pennsylvania State University under the supervision of professor Stanisław Mrówka.

[2] $X_{\mathrm{cl}} \subset E^m$ means that X is embeddable as a closed subset of E^m.

[3] In the remainder of this paper E is assumed to be Hausdorff unless otherwise specified.

a net x_a in X with $h(x_a) = x_a$. Since $\pi_\xi(h(x_a)) = f_\xi(x_a)$ for every $\xi \in \Xi$ and $h(x_a) \to x$, we obtain $f_\xi(x_a) \to \pi_\xi(x)$ for $\xi \in \Xi$. Therefore by condition 2) $x_a \to x$ for some $x \in X$, and since E^m is Hausdorff, we infer that $h(x) = x$ and thus that $h\{X\}$ is closed in E^m.

We discuss some applications of the above theorem to the E-transformation of a Hausdorff space X.

Let X be a Hausdorff space and (X^*, φ) its E-transformation (¹). We exhibit some of the influences of $C(X, E)$ on the E-transformation X^* of X.

Corollary 2.1. *The following conditions are equivalent.*

1) $X^* \in K(X)$.

2) *For every net x_a in X and $f \in C(X, E)$, $f(x_a)$ converges iff $\varphi(x_a)$ converges.*

Proof. 1) ⇒ 2) Let (X^*, φ) be the E-transformation of X, and x_a a net in X. Given any $f \in C(X, E)$, we can select an $h \in C(X^*, E)$ such that $f = h \circ \varphi$.

Therefore by Theorem 2.1 $h(\varphi(x_a))$ converges iff $\varphi(x_a)$ converges and the latter happens iff $f(x_a)$ converges.

2) ⇒ 1) Let x_a^* be a net in X^* such that $h(x_a^*)$ converges for every $h \in C(X^*, E)$. Let x_a be a net in X such that $\varphi(x_a) = x_a^*$ for every a. Since (X^*, φ) is the E-transformation of X and $h(x_a^*)$ converges for every $h \in C(X^*, E)$, we infer that $f(x_a)$ converges for every $f \in C(X, E)$ and hence that $\varphi(x_a)$ converges. Therefore, by Theorem 2.1, $X^* \in K(E)$.

We remark that the above corollary is not sharp enough since it does not depend only upon $C(X, E)$. However, the following corollary does depend only upon $C(X, E)$.

Corollary 2.2. *The following conditions are equivalent.*

1) *X has an E-compact modification X^*.*

2) *For a net x_a in X, $f(x_a)$ converges for every $f \in C(X, E)$ iff there exists $x_0 \in X$ such that $f(x_a) \to f(x_0)$ for every $f \in C(X, E)$.*

Proof. Let (X^*, φ) be the E-transformation of X. Then the result is a direct consequence of Corollary 2.1. It remains only to choose $x_0 \in X$ such that $\varphi(x_a) \to \varphi(x_0)$.

We next consider the following situation. Let X be a Hausdorff space such that its E-modification X^* is E-compact, and select any Hausdorff proper extension εX of X. We proceed to give conditions in terms of (X^*, φ) for the space X to be E-embedded in εX.

(¹) Let X be a topological space; the pair (X^*, φ) is the E-transformation of X if (X^*, φ) satisfies the following conditions: (i) X^* is E-completely regular ($X^* \in C(E)$). (ii) φ: $X \to X^*$ is continuous and onto. (iii) For every $f \in C(X, E)$ there exists an $h \in C(X^*, E)$ such that $f = h \circ \varphi$.

Corollary 2.3. *X is E-embedded in εX iff φ is continuously extendable over εX.*

Proof. ⇒ Assume that φ: $X \to X^*$ can be continuously extended to a function φ^*: $\varepsilon X \to X^*$. Let $f \in C(X, E)$, and for $x \in \varepsilon X/X$ define $f^*(x) = h(\varphi^*(x))$, where $h \in C(X^*, E)$ is such that $f = h \circ \varphi$. Then f^* is continuous and $f \subset f^*$.

⇐ Let X be E-embedded in εX. We seek a function φ^*: $\varepsilon X \to X^*$ such that φ^* is continuous and $\varphi \subset \varphi^*$. To this end let $x \in \varepsilon X/X$ and let x_a be a net in X such that $x_a \to x$. For every $f \in C(X, E)$ let f^* be its unique extension to εX. Observe that $f^*(x_a) \to f^*(x)$ and since $f(x_a) = f^*(x_a)$, $f(x_a)$ converges for every $f \in C(X, E)$. Therefore, by Lemma 2.2, there exists an $x_0 \in X$ such that $f(x_a) \to f(x_0)$ for every $f \in C(X, E)$, and thus $f^*(x) = f(x_0)$. Let $\varphi^*(x) = \varphi(x_0)$. Since (X^*, φ) is the E-transformation of X, φ^* is well defined; hence it remains only to show that φ^* is continuous. Let $h \in C(X^*, E)$ be defined by $f = h \circ \varphi$. It is clear from our definition of φ^* that $h \circ \varphi^* = f^*$. Therefore, by Theorem 1.2 of [6], φ^* is continuous.

We consider next the case where the E-modification X^* of X is not assumed to be E-compact. In this case we have the following

Corollary 2.4. *Let (X^*, φ) be the E-trasformation of X and εX a proper Hausdorff extension of X. Then the following conditions are equivalent.*

1) *The mapping φ: $X \to X^*$ has a continuous extension* (¹) *φ^*: $\varepsilon X \to \beta_E X^*$.*

2) *X is E-embedded in εX.*

Proof. 1) ⇒ 2) Let f: $X \to E$ and choose $h \in C(X^*, E)$ such that $f = h \circ \varphi$. Since h admits a continuous extension h^*: $\beta_E X^* \to E$, define $f^* = h^* \circ \varphi^*$.

2) ⇒ 1) Let $x \in \varepsilon X/X$ and choose a net x_a in X such that $x_a \to x$. For every $f \in C(X, E)$ and $h \in C(X^*, E)$ let f^* and h^* be the continuous extensions to εX and to $\beta_E X^*$ respectively. Notice that $f^*(x_a) \to f^*(x)$ for every $f^* \in C(\varepsilon X, E)$ and thus that $h(\varphi(x_a)) = h^*(\varphi(x_a))$ converges for every $h^* \in C(\beta_E X^*, E)$. Therefore, by Theorem 2.1, there exists an $\hat{x} \in \beta_E X^*$ such that $\varphi(x_a) \to \hat{x}$. We show next that φ^* is well defined for every $x \in \varepsilon X/X$. To see this, choose two nets x_a and x_a in X such that x_a and $x_{a'}$ both converge to x. Thus we obtain $\hat{x}$ and $\hat{x}'$ in $\beta_E X^*$ such that $\varphi(x_a) \to \hat{x}$ and $\varphi(x_{a'}) \to \hat{x}'$. If $\hat{x} \neq \hat{x}'$, choose $h^* \in C(\beta_E X^*, E)$ such that $h^*(x) \neq h^*(x')$, and this implies that $f = f^*(\varphi(x_a))$ converges both to $h^*(\hat{x})$ and to $h^*(x')$. This contradicts the fact that E is Hausdorff and

(¹) $\beta_E X^*$ is an E-compact extension of X^* satisfying i) Every continuous function : $X^* \to E$ admits a continuous extension f^*: $\beta_E X^* \to E$. For more details on $\beta_E X^*$ see S. Mrówka [10].

thus φ^* is a function. To prove the continuity of φ^*, where $\varphi^*\colon \varepsilon X \to \beta_E X^*$, observe that every $f \in C(X, E)$ is of the form $h \circ \varphi$ and that $f^* = h^* \circ \varphi^*$ and thus, by Theorem 1.2 of [6], φ^* is continuous.

We conclude this section with a condition for E-compactness which resembles the characterization of compactness in terms of nets.

COROLLARY 2.5. *The following are equivalent*:

1) $X \in K(E)$.

2) *For a net x_a in X, x_a has a cluster point iff $f(x_a)$ has a cluster point for every $f \in C(X, E)$.*

The proof of this corollary can be obtained by a slight modification of the proof of Theorem 2.1 and with the aid of universal nets.

3. One-point E-completely regular extensions. In this part we study one-point E-completely regular extensions. We recall that, for every E-completely regular space X, there exists an E-compact extension $\beta_E X$ which is characterized by the following conditions: (1)

a) $\beta_E X$ is E-compact.

b) Every continuous mapping $g\colon X \to Y$ where Y is E-compact can be extended to $\beta_E X$.

c) $\beta_E X$ is uniquely determined by the above properties, i.e., if εX is an arbitrary extension of X that satisfies a) and b), then $\varepsilon X \underset{\text{ext}}{=} \beta_E X$ (2).

Let us choose an E-determining family F of $C(X, E)$. If we parallel the construction of $\beta_E X$, we obtain an extension of X, which we will denote by $\varepsilon_F X$, characterized by the following conditions:

THEOREM 3.1. *There is a unique (up to homeomorphism) extension $\varepsilon_F X$ of X characterized by the following conditions*:

a) *$\varepsilon_F X$ is E-compact.*

b) *Every continuous mapping $f\colon X \to E$, for $f \in F$ can be continuously extended to $\varepsilon_F X$.*

c) *Every Hausdorff extension εX satisfying condition* b) *can be continuously mapped into $\varepsilon_F X$, leaving the points of X fixed.*

Proof. Let $h\colon X \to E$ be the parametric map determined by the family F. Let $\varepsilon X = \overline{h(X)}^{E^F}$. Then $\varepsilon_F X$ satisfies conditions a) and b) by definition.

To prove part c) let X be any Hausdorff extension of X which satisfies condition b). Then, by Corollary 1.1 of [6] and a slight modification

(1) For more information on $\beta_E X$ see S. Mrówka [10].

(2) $\varepsilon_1 X \underset{\text{ext}}{=} \varepsilon_2 X$ means that $\varepsilon_1 X$ is homeomorphic to $\varepsilon_2 X$ by a homeomorphism which leaves the points of X fixed.

of Corollary 2.4, we obtain a continuous function $f\colon \varepsilon X \to \varepsilon X$ such that $f(x) = x$ for every $x \in X$.

To establish the uniqueness of the extension is now a simple matter. Let εX be a Hausdorff extension of X satisfying conditions a), b) and c). From part c) above we obtain a continuous function $h\colon \varepsilon X \to \varepsilon X$ leaving X invariant, and from our assumptions there exists a continuous function $g\colon \varepsilon_F X \to \varepsilon X$, which is the identity on X. From this it follows that $\varepsilon X \underset{\text{ext}}{=} \varepsilon_F X$.

We consider next the following situation. Let εX be an E-completely regular extension of X, and let F be the family of all restrictions of $C(\varepsilon X, E)$ to X. Then it is clear that εX can be obtained in a natural way in terms of the family F. If $\varepsilon X \in K(E)$, then εX is the smallest extension (in the sense of property c) of Lemma 2.1) determined by the family F. We point out that this is just a duplication of the Tihonov method used in this solution to the problem of finding all the compactifications of a completely regular space X.

In the remainder of this section we will be concerned with one-point E-completely regular extensions.

We start by introducing some definitions:

Let $X \in K(E)$ and let ξ be a class of subsets of X. Following P. Alexandroff [1], ξ is a *centred system* if ξ is closed under finite intersections and the empty set $\emptyset$ is not a member of ξ. The centred system ξ is said to be *Hausdorff* if, for every $x \in X$ such that $x \notin A$ for some $A \in \xi$, there exists a $B \in \xi$ with $x \notin \overline{B}$ ($\overline{B}$ is the closure of B in X).

Let $X \in C(E)$ and let ξ be a Hausdorff centred system of open sets of X with empty intersection. We define a topology for $X \cup \{\xi\}$ in the following manner:

X is an open subset of $X \cup \{\xi\}$, and the neighbourhood system for $\{\xi\}$ consists of all $A \cup \{\xi\}$ for $A \in \xi$. It is easy to see that $X \cup \{\xi\}$ is Hausdorff and that it contains X densely embedded. We should add that, whenever we consider $X \cup \{\xi\}$ as a topological space, we mean that it has the topology defined above.

Consider the following conditions on an E-completely regular space X.

(i) X is locally E-compact but not *E-compact* (1).

(ii) There exists an E-determining family $\mathcal{F} \subset C(X, E)$ such that $\mathcal{F}$, restricted to any closed E-compact neighbourhood U of x, produces a closed embedding of U.

Before we state our last condition, we consider the following preliminaries.

(1) A space $X \in C(E)$ is locally E-compact iff every $x \in X$ has a base system of closed neighbourhoods which are E-compact.

Let Δ be the class of all E-compact closed subsets A of X, such that the family $\mathcal{F}$ when restricted to A produces a closed embedding of A. We have the following observations on Δ.

1) *Δ is closed under finite unions* (this is a direct application of Lemma 2.5).

2) *The class $\xi = \{X/A: A \in \Delta\}$ is a Hausdorff centred system of open sets with vacuous intersection.*

Proof of 2). From 1) and the fact that X is not E-compact we infer that ξ is a centred system. Furthermore, ξ is Hausdorff and has vacuous intersection because of conditions (i) and (ii) above.

Our last condition on X now reads:

(iii) The filter base $f\{\xi\}$ converges for every $f \in \mathcal{F}$.

Combining our three conditions on X, we obtain the following

LEMMA 3.1. *For every $X \in C(E)$, if conditions* (i), (ii) *and* (iii) *hold, then $X \cup \{\xi\}$ is a one-point E-compactification of X.*

Proof. We show that $X \cup \{\xi\}$ is E-completely regular. We need only worry about $\{\xi\}$. So let x_a be a net in $X \cup \{\xi\}$ such that $f^*(x_a) \to f^*(\xi)$ for every $f^* \in \mathcal{F}^*$ (f^* and $\mathcal{F}^*$ as defined above). Suppose that $x_a \to \{\xi\}$: then there exists a subnet x_β of x_a and an E-compact closed subset $H \subset X$ such that the net x_β is in H and $\mathcal{F}^*\ H$ produces a closed embedding of H. This shows that $f^*(x_\beta) \to f^*(\xi)$ for all $f^* \in \mathcal{F}^*$, which is a contradiction. This demonstrates that $X \cup \{\xi\}$ is E-completely regular, and therefore that the family $\mathcal{F}^*$ is an E-determining family for $X \cup \{\xi\}$. We show that the family $\mathcal{F}^*$ produces a closed embedding. Let x_a be a net in $X \cup \{\xi\}$ such that $f^*(x_a)$ converges for every $f^* \in \mathcal{F}^*$: we will prove that this implies that x_a has a cluster point in $X \cup \{\xi\}$ and hence, by Lemma 2.5, $X \cup \{\xi\}$ is E-compact. Let us suppose that $\{\xi\}$ is not a cluster point of x_a. Then the net x_a is eventually in some $A \in \Delta$ (Δ as defined above) and thus, by condition ii) and Theorem 2.1, x_a converges in A and hence in $X \cup \{\xi\}$. If, on the other hand, $\{\xi\}$ is already a cluster point, we have nothing to prove.

THEOREM 3.2. *Let X be an E-completely regular space which is not E-compact. Then the following conditions are equivalent:*

1) *X admits a one-point E-compactification.*

2) *X is locally E-compact and there exist an E-determining family $\mathcal{F} \subset C(X, E)$ and a centred system ξ satisfying conditions* (i), (ii) *and* (iii) *of Lemma* 3.1.

Proof. 2) $\Rightarrow$ 1) This is Lemma 3.1.

1) $\Rightarrow$ 2) Let $X \cup \{p\}$ be a one-point E-compactification of X. Define the family $\mathcal{F} \subset C(X, E)$ as the restriction of $C(X \cup \{p\}, E)$ to X, and let ξ be any neighbourhood system of open sets of the point p in $X \cup \{p\}$. It is easy to see that conditions (i), (ii) and (iii) of Lemma 3.1 hold.

We now wish to give an application of Theorem 2.2 to a special kind of topological space E. The space E is Hausdorff and has the following property:

PROPERTY PCR. There exists a fixed pair of distinct points e_0 and e_1 in E such that for every closed set A and point x in E^n ($n \in N$), with $x \notin A$, there exists a continuous function $g: E^n \to E$ such that $g\{A\} = e_0$ and $g(x) = e_1$.

Let E have property (PCR) and let X be an E-completely regular space which is locally compact but not compact. We want to show that the one-point compactification $X \cup \{\infty\}$ of X is E-completely regular and thus E-compact.

COROLLARY 3.1. *Let E have property* (*PCR*) *and let $X \in C(E)$ be locally compact but not compact. Then the one-point compactification of X is E-compact.*

Furthermore, this can be obtained in terms of all the functions $f: X \to E$ which are constant outside compact subsets of X.

Proof. Let F be the family of all those continuous functions from X into E which have the constant value e_1 outside compact subsets of X. We claim that the family F so defined is an E-determining family for X. Namely let A and x be a closed set and a point of such that $x \notin A$. Since X is locally compact, there exists a compact neighbourhood U of x such that $U \cap A = \emptyset$. Let f be a continuous function defined as follows:

$$f: X \to E, \quad f[X/\operatorname{int} U] = e_0^1 \quad \text{and} \quad f(x) = e_1.$$

The existence of such an f is guaranteed because E has property (PCR) and, furthermore, $f \in F$.

Let ξ be the class of complements of all compact subsets of X. It is easy to see that ξ is a Hausdorff centred system of open sets with vacuous intersection. It remains to show that the filter base $f[\xi]$ converges for every $f \in F$. Let $G \in \xi$ and let f be an arbitrary function in F. Let H be a compact subset of X such that $f[X/H] = e_0$. Then $G' = G \cap (X/H) \in \xi$ and $f[G'] = e_0$, showing that the filter base $f[\xi]$ converges to e_0. Therefore by Lemma 3.1 $X \cup \{\xi\}$ is E-completely regular.

COROLLARY 3.2. *Let $E = R$ (R being the reals with the usual topology) and let X be locally compact but not compact. Then the one-point compactification of X can be obtained in terms of all the real-valued continuous functions which vanish outside compact subsets of X.*

(1) $\operatorname{int} U$ denotes the interior of U in X.

We next present certain special cases of one-point E-completely regular extensions. Let E be a regular space and $X \in \boldsymbol{C}(E)$ which has the following property.

(EN) For every closed subset $H \subset X$, H is E-embedded. Let P be a topological property [1] of E-completely regular spaces which satisfy condition (EN). We assume further that property P satisfies the following conditions:

1) If A is a closed subset of X and A has property P, then any closed subset $H \subset A$ also has property P.

2) If A_1 and A_2 are closed subsets of X and A_1, A_2 have property P, then $A_1 \cup A_2$ has property P.

We say that a space X has property P *locally* iff every point $x \in X$ has a system of closed neighbourhoods possessing property P. Let us denote by $(\mathrm{W_P})$ the following statement about E-completely regular spaces [2]:

$(\mathrm{W_P})$ For every E-completely regular space X, if $x_0 \in X$ is such that for every closed subset $H \subset X$, with $x_0 \notin H$, H has property P, then X has property P.

THEOREM 3.3. *Let X be an E-completely regular space satisfying condition* (EN). *Furthermore, assume that the following conditions are satisfied:*

(i) *X is locally E-compact but not E-compact and has property* P *locally but not globally.*

(ii) *Property* P *satisfies conditions* 1) *and* 2).

(iii) *Property* P *has feature* $(\mathrm{W_P})$.

Then there exists a one point E-compactification of X which has property P.

Proof. Let us consider the following objects:

$$\Delta = \{H\colon\ \exists H' \subset X \text{ such that } H \text{ and } H' \text{ are closed } E\text{-compact subsets satisfying property P and } H \subset \operatorname{int} H'\}.$$

Let the family F be defined as follows:

$$F = \{f \in C(X, E)\colon f[X \backslash H] = e_0,\ H \in \Delta \text{ and } e_0 \text{ being a fixed point of } E\}.$$

We show that F is an E-determining family and that F restricted to any $H \in \Delta$ produces a close embedding.

F is E-determining, namely let A and x be a closed subset and a point of X, respectively, such that $x \notin A$. Choose E-compact closed neighbourhoods U_1, U_2 of x which have property P and are such that $U_1 \subset \operatorname{int} U_2$ and $U_2 \cap A = \emptyset$. Let the continuous function f be defined as follows: $F\colon X \to E$, and $f[X \backslash \operatorname{int} U_1] = e_0$ and $f(x) = e_1$ where $e_1 \neq e_0$. Observe that $U_1 \in \Delta$ and thus $f \in F$.

Next let $H \in \Delta$: then there exists an H', which is a closed E-compact subset of X satisfying property P and such that $H \subset \operatorname{int} H'$. Since X is normal and every closed set is E-embedded we can find $H_1 \in \Delta$ such that $H \subset \operatorname{int} H_1 \subset H_1 \subset \operatorname{int} H'$. Now let $g\colon H \to E$ and let the continuous function be defined as follows: $f\colon X \to E$, and $f[X \backslash \operatorname{int} H_1] = e_0$. That such a function exists is guaranteed since X satisfies conditions (EN). Observe that $H_1 \in \Delta$ and thus that $f \in F$. This shows that the family F restricted to any $H \in \Delta$ produces a closed embedding.

Let $\xi = \{X \backslash H\colon H \in \Delta\}$. It remains to observe that ξ is a Hausdorff centred system with vacuous intersection and that $f(\xi)$ converges for every $f \in F$. Thus, by Theorem 3.2, $X \cup \{\xi\}$ is E-compact. Furthermore, since property P has feature $(\mathrm{W_P})$, we find that $X \cup \{\xi\}$ has property P.

We next examine a few specific topological properties.

PROPERTY $\mathrm{P_1}$. $[m, n]$ compactness [1].

This property $\mathrm{P_1}$ is easily seen to satisfy conditions 1) and 2). Furthermore it is clear that property $\mathrm{P_1}$ has feature $(\mathrm{W_{P_1}})$. Hence we have the following result.

COROLLARY 3.3. *Let $X \in C(E)$ satisfy condition* (EN). *Then if X has properties $\mathrm{P_1}$ and E-compactness locally but neither property globally, there always exists a one-point E-compact extension of X which has property $\mathrm{P_1}$.*

PROPERTY $\mathrm{P_2}$. For our next case we consider the space R of real numbers as the space E and the property $\mathrm{P_2}$ is that of R-compactness. Then our situation is as follows. Let X be a normal space which is locally R-compact but not R-compact. We want to show that there exists a one-point extension of X which is normal and R-compact. We start by proving that R-compactness for normal spaces satisfies conditions 1) and 2).

Condition 1). Let H be a closed R-compact subset of X: then it is clear that if $H_1 \subset H$ and H_1 is close, then H_1 is R-compact.

Condition 2). Let H_1 and H_2 be closed R-compact subsets of X: then $H_1 \cup H_2$ is R-embedded in X and thus by Corollary 2.5, $H_1 \cup H_2$ is R-compact.

Before we establish our assertion concerning property $\mathrm{P_2}$, we want to consider the following preliminaries.

[1] *Property* P will only be considered for E-completely regular spaces and will be assumed to be invariant under homeomorphisms.

[2] Our approach parallels that of S. Mrówka [8].

[1] A space X is said to be $[m, n]$-*compact* (m, n being infinite cardinals) if every open covering U of X with $n \leqslant \overline{U} \leqslant m$ admits a subcovering V_1 with $V_1 \leqslant n$.

Let X be an E-completely regular space such that there exists a point $x_0 \in X$ with the property that every closed subset $H \subset X$, with $x_0 \notin H$, implies that H is a normal subset of X. It was shown by S. Mrówka [8] that this implies that X is normal.

COROLLARY 3.4. *Let X be a normal space which is locally R-compact but not R-compact. Then X admits a one-point extension which is normal and R-compact.*

Proof. By an argument similar to that of the proof of Theorem 3.2, we can show that there exists a completely regular extension $X \cup \{\xi\}$ of X such that if H is closed in $X \cup \{\xi\}$ and $H \cap \{\xi\} = \emptyset$, then H is normal. Therefore by our preliminary remarks we find that $X \cup \{\xi\}$ is normal.

PROPERTY P_3. Let E be a Hausdorff space satisfying condition (PCR). Let P_3 be an arbitrary topological property of E-compact spaces (E as described above) having feature (W_{P_3}).

COROLLARY 3.5. *If $X \in C(E)$ has the properties* (EN) *and* P_3 *locally but neither globally, then there exists a one-point E-completely regular extension of X which has property* P_3.

Proof. Let Δ be the class of all the closed subsets H of X such that $H \in \Delta$ iff there exists a closed subset H' of X, with $H \subset \operatorname{int} H'$ where H' has properties (EN) and P_3.

We have the following observations on Δ:

1) Δ is non-empty.

This follows because X has properties (EN) and P_3 locally.

2) Δ is closed under finite unions.

To see this take $H_1, H_2 \in \Delta$. Then H_1 and H_2 are both E-embedded in $H_1 \cup H_2$. Thus $H_1 \cup H_2$ has property (EN) and hence is E-compact. Furthermore, since property P_3 is finitely additive on closed sets, we find that $H_1 \cup H_2$ has property P_3. Finally, it is easy to see that $H_1 \cup H_2 \in \Delta$.

3) If $H \in \Delta$ and H' is a closed subset of H, then $H' \in \Delta$.

This follows directly from the definition of Δ. Let us now consider the following objects:

$$F = \{f\colon f \in C(X, E), f[X \backslash H] = e_0, H \in \Delta \text{ and } e_0 \text{ is a fixed point of } E\},$$

and

$$\xi = \{X \backslash H\colon H \in \Delta\}.$$

By an argument similar to the one given in the proof of Corollary 3.2, we can show that $X \cup \{\xi\}$ is E-compact; furthermore, since property P_3 has feature (W_{P_3}), we find that $X \cup \{\xi\}$ has property P_3.

We conclude with the following special case.

COROLLARY 3.6. *If X is completely regular, and X is locally normal and locally R-compact but neither globally, then X admits a one-point extension which is normal and R-compact.*

The proof of this corollary follows from a direct application of Corollaries 3.4 and 3.5.

We want to point out that the same conclusions hold for N-compactness [1] under similar conditions. Furthermore, we have a stronger result for $E = N$. Let $X \in C(N)$ be locally N-compact, but not N-compact. Then there exists a one-point N-compactification of X.

To see this let

$$\Delta = \{H\colon H \text{ is an open closed } N\text{-compact subset of } X\}$$

and

$$F = \{f\colon f \in C(X, N) \text{ and } f[X \backslash H] = 0 \text{ for } H \in \Delta\}.$$

We observe that F is E-determining for X and, furthermore, that F, when restricted to any $H \in \Delta$, produces a closed embedding. Therefore, if we consider the Hausdorff centred system $\xi = \{X \backslash H\colon H \in \Delta\}$, we can show by an argument similar to that of Theorem 3.2 that $X \cup \{\xi\}$ is a one-point N-compactification of X.

We wish to add that the development of part 2) of this chapter has been motivated by the following question. Is every locally R-compact space an open subset of an R-compact space? The farthest that we have gone in this direction is the result of Corollary 3.6. On the other hand, for N-compact space the answer, as shown above, is affirmative.

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On the existence of maps having graphs connected and dense

by

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The paper contains a proof of the existence of maps $f\colon X\to Y$ with connected and dense graphs in $X\times Y$, where X and Y are connected spaces satisfying some additional conditions. We also state Theorems 2 and 3, which are generalizations of Theorems 3 and 4 proved by D. Phillips in [2]. The proofs of these theorems are reproduced from [2].

Let us fix some notation and symbols; $\pi\colon X\times Y\to X$ means the projection, Fr_A and Int_A means boundary and interior operations in the space A, and $w(A)$, $\operatorname{card} A$ means respectively the weight of A and the cardinality of A.

LEMMA 1. *Let X, Y be connected spaces. If $0\neq G\subset X\times Y$ is open, then*

(a) $\mathrm{Int}_X\pi(\mathrm{Fr}_{X\times Y}G)\neq\emptyset$, *or*

(b) *there exists an $x\in X$ such that $\pi^{-1}(x)\subset \mathrm{Fr}_{X\times Y}G$, or*

(c) *G is dense in $X\times Y$.*

Proof. (I) Let us assume that there exists a point $(x,y)\in X\times Y$ such that $(x,y)\notin\overline{G}$ and $x\in\pi(G)$. Then there exists an open set $U_x\subset\pi(G)$ such that $x\in U_x$ and (a) $U_x\subset\pi(\mathrm{Fr}_{X\times Y}G)$. Indeed, there are open sets $U_x\subset\pi(G)$ and $U_y\subset Y$ such that $(x,y)\in U_x\times U_y\subset(X\times Y)-\overline{G}$. We show that $U_x\subset\pi(\mathrm{Fr}_{X\times Y}G)$. Suppose that there exists an $x'\in U_x-\pi(\mathrm{Fr}_{X\times Y}G)$. A subspace $\{x'\}\times Y\subset X\times Y$ is homeomorphic with Y. We have

$$\emptyset\neq G\cap(\{x'\}\times Y)\neq\{x'\}\times Y$$

and

$$\mathrm{Fr}_{\{x'\}\times Y}\big(G\cap(\{x'\}\times Y)\big)\subset(\mathrm{Fr}_{X\times Y}G)\cap(\{x'\}\times Y)=\emptyset,$$

and this contradicts the fact that $\{x'\}\times Y$ is connected.

(II) Let us assume that the condition (I) is not satisfied. We have $\mathrm{Fr}_X\pi(G)\neq\emptyset$ or $\mathrm{Fr}_X\pi(G)=\emptyset$.

(b) If $x\in\mathrm{Fr}_X\pi(G)$ then $\pi^{-1}(x)\subset\mathrm{Fr}_{X\times Y}G$. Indeed, suppose that there exist an $y\in Y$ and such an open neighbourhood $U_x\times U_y$ of point