Product proximities

by

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Introduction. Let \( \{ X_i : i \in I \} \) be a family of sets and let \( X \) represent the Cartesian product of these sets. If \( \mathcal{T}_i \) is a topology on \( X_i \), then the product topology \( \mathcal{T} \) on \( X \) can be defined as the least topology which makes the projection maps continuous. If \( \mathcal{U}_i \) is a uniformity on \( X_i \), then the product uniformity \( \mathcal{U} \) on \( X \) can be defined as the least uniformity which makes the projection maps uniformly continuous. If \( \delta_i \) is a proximity relation on \( X_i \), it is natural to define the product proximity \( \delta \) on \( X \) as the least proximity relation on \( X \) which makes the projection maps \( p \)-continuous. For this definition to be a good one the topology induced by the product proximity should be identical to the product of the topologies induced by the coordinate proximities. Also the proximity relation induced by the product uniformity should be identical to the product of the proximity relations induced by the coordinate uniformities. The definition of the product proximity relation suggested above satisfies the former criterion but fails to satisfy the latter. It is possible, for example, that there exist uniformities \( \mathcal{U} \) and \( \mathcal{U}' \) on \( X \) which induce the same proximity relation \( \delta \) on \( X \) and yet the product uniformities \( \mathcal{U} \times \mathcal{U} \) and \( \mathcal{U}' \times \mathcal{U}' \) induce different proximity relations on \( X \times X \). This paper examines this phenomenon.

§ 1. Proximity classes of Product Uniformities. Teubell [4] observes that the products of two non-totally bounded uniformities \( \mathcal{U} \) and \( \mathcal{U}' \) induce a proximity relation distinct from the relation induced by the totally bounded uniformities \( \mathcal{U} \) and \( \mathcal{U}' \) from the same proximity classes as \( \mathcal{U} \) and \( \mathcal{U}' \) respectively. We generalize this result to \( m \)-bounded uniformities.

**Definition 1.1.** Let \( m \) be an infinite cardinal number. A uniformity \( \mathcal{U} \) on \( X \) is \( m \)-bounded iff for any \( U \in \mathcal{U} \), there exists a set \( A \) of cardinality \( < m \) such that \( U(A) = X \).

**Definition 1.2.** Let \( (X, \mathcal{U}) \) be a uniform space. A subset \( A \) of \( X \) is uniformly discrete iff there exists \( U \in \mathcal{U} \) such that \( (x, y) \notin U \) for any \( x, y \in A \), where \( x \neq y \).
Lemma 1.3. A uniform space \((X, \mathcal{U})\) is m-bounded iff every uniformly discrete subset of \(X\) has cardinality \(< m\).

Proof. See, for example, Reed and Thron [7], Theorem 1.2.

Lemma 1.4. Let \(\{(X_i, \mathcal{U}_i): i \in I\}\) be a family of uniform spaces and let \(X = \times \{X_i: i \in I\}\). Then the product uniformity \(\mathcal{U}_p\) on \(X\) has a basis consisting of sets of the form \(U \times X = \{(x, y): (x, y) \in U(i)\}\), where \(U(i) \subseteq X_i\), \(i \in I\) and \(x_i\) and \(y_i\) denote the \(i\)-th coordinate of \(x\) and \(y\), respectively.

In order to avoid awkward notation we shall employ the following nonorthodox but simplified notation. We let \(\langle \mathcal{U} \rangle\) denote the proximity relation induced by the uniformity \(\mathcal{U}\), \(\langle \mathcal{U}, \mathcal{V} \rangle\) denote the proximity induced by \(\mathcal{U}\) and \(\mathcal{V}\) and, in general \(\langle \mathcal{U}_i: i \in I\rangle\) denote the proximity induced by the product of the uniformities \(\{\mathcal{U}_i: i \in I\}\).

Theorem 1.5. Let \(\mathcal{U}\) and \(\mathcal{V}\) be uniformities on \(X\) and \(\mathcal{U}'\) and \(\mathcal{V}'\) be uniformities on \(X\). If \(\mathcal{U}\) or \(\mathcal{U}'\) is m-bounded and \(\mathcal{V}\) and \(\mathcal{V}'\) are not m-bounded, then \(\langle \mathcal{U}, \mathcal{V} \rangle \neq \langle \mathcal{U}', \mathcal{V}' \rangle\).

Proof. We prove this in a more general setting in Theorem 3.10.

Corollary 1.6. Let \(\mathcal{U}\) and \(\mathcal{U}'\) be uniformities on \(X\) and \(\mathcal{V}\) and \(\mathcal{V}'\) be uniformities on \(X\). If \(\mathcal{U}\) or \(\mathcal{U}'\) is m-bounded and \(\mathcal{V}\) and \(\mathcal{V}'\) are not m-bounded and \(\langle \mathcal{U} \rangle = \langle \mathcal{U}' \rangle\) and \(\langle \mathcal{V} \rangle = \langle \mathcal{V}' \rangle\), then \(\langle \mathcal{U}, \mathcal{V} \rangle \neq \langle \mathcal{U}', \mathcal{V}' \rangle\).

It is not the case that boundedness is the only key to the formation of distinct product proximities. The following theorem shows, for example, that the product of uniformities of the same kind may also induce distinct relations. First we recall the standard ordering of uniformities and proximity relations on a set \(X\). If \(\mathcal{U}\) and \(\mathcal{V}\) are uniformities on \(X\) we write \(\mathcal{U} < \mathcal{V}\) iff \(\mathcal{U} \subseteq \mathcal{V}\); if \(\mathcal{U} < \mathcal{V}\) and \(\mathcal{V} < \mathcal{U}\) are proximity relations on \(X\) we write \(\delta < \delta'\) iff \(\delta \subseteq \delta'\). Now it is easily shown (and will be shown in a more general setting in Theorem 3.1) that if \(\mathcal{U} < \mathcal{V}\), then \(\langle \mathcal{U} \rangle < \langle \mathcal{V} \rangle\). It is not necessarily true, however, that \(\mathcal{U} < \mathcal{V}\) implies \(\langle \mathcal{U} \rangle < \langle \mathcal{V} \rangle\). It is possible that \(\mathcal{U} < \mathcal{V}\) and \(\langle \mathcal{U} \rangle = \langle \mathcal{V} \rangle\). For products we have the following theorems.

Theorem 1.7. Let \(\mathcal{U}\), \(\mathcal{U}'\), and \(\mathcal{V}\) be uniformities on \(X\) such that \(\mathcal{U}\) or \(\mathcal{U}'\) is \(\mathcal{V}\)-bounded and \(\mathcal{V}\) is not \(\mathcal{U}\)-bounded. Then \(\langle \mathcal{U}, \mathcal{V} \rangle \neq \langle \mathcal{U}', \mathcal{V} \rangle\).

Proof. Without loss of generality we assume that \(\mathcal{U} < \mathcal{V}\).

Since \(\mathcal{U} < \mathcal{V}\) there exists a symmetric member \(V\) of \(\mathcal{V}\) such that \(U \subseteq V\) for any \(U \in \mathcal{U}\). Let \(X\) be a symmetric entourage such that \(X < X < X < V\). We shall show that \(\langle X, \mathcal{U} \rangle \neq \langle X, \mathcal{V} \rangle\) and \(\langle X, \mathcal{U}' \rangle \neq \langle X, \mathcal{V} \rangle\). This will imply that \(\langle \mathcal{U}, \mathcal{V} \rangle \neq \langle \mathcal{U}', \mathcal{V} \rangle\).

Let \(W = X \times X \times X\). We shall show that \(\langle X, \mathcal{V} \rangle \neq \langle X, \mathcal{V} \rangle\) and \(\langle X, \mathcal{U}' \rangle \neq \langle X, \mathcal{V} \rangle\). Now let \((x, y) \in U\) be such that \((x, y) \in V\). Such an \((x, y)\) exists because \(U \subseteq V\). Now \((x, y) \in U\) so \((x, x), (y, y) \in W\). Since \((x, x) \in X\) we have \((y, y) \in W\) because \((x, y) \notin W\) and \(W\) is symmetric so \(W < W < W < W\). Thus \(W < X < X < W\). Since \(W\) was arbitrary \((X, \mathcal{X})\) is symmetric so \(X < X < X < X\). Since \(X, y\) was an arbitrary member of \(W(X)\), we have \(W(X) \subseteq W\). We may conclude that \(W(X) < W < W < W\) and \(\langle X, \mathcal{V} \rangle \neq \langle X, \mathcal{U}' \rangle\).

Corollary 1.8. If \(\mathcal{U}\), \(\mathcal{U}'\), and \(\mathcal{V}\) are uniformities on \(X\) such that \(\mathcal{U} < \mathcal{V}\), then \(\langle \mathcal{U}, \mathcal{V} \rangle \neq \langle \mathcal{U}', \mathcal{V} \rangle\).

Corollary 1.9. If \(\mathcal{U}\), \(\mathcal{U}'\), and \(\mathcal{V}\) are m-bounded uniformities on \(X\) such that \(\mathcal{U} < \mathcal{V}\), then \(\langle \mathcal{U}, \mathcal{V} \rangle \neq \langle \mathcal{U}', \mathcal{V} \rangle\).

If \(m = \kappa\), the hypotheses of 1.9 cannot be satisfied because there is one and only one totally bounded uniformity compatible with \(\delta\). However, Reed and Thron [7] have shown that if \(\delta\) has a compatible m-bounded uniformity \(\mathcal{U}\) and \(m \neq \kappa\), then there is an infinite decreasing sequence of compatible m-bounded uniformities less than \(\mathcal{U}\).

§ 2. Product Proximities. As explained in the introduction there is a natural definition of a product proximity.

Definition 2.1. Let \(\{(X_i, \delta_i): i \in I\}\) be a family of proximity spaces and let \(X = \times \{X_i: i \in I\}\). The proximity relation \(\delta(1)\) is the proximity on \(X\) such that each projection map is \(p\)-continuous.


Definition 2.2. Let \(\{(X_i, \delta_i): i \in I\}\) be a family of proximity spaces and let \(X = \times \{X_i: i \in I\}\). The proximity relation \(\delta(2)\) on \(X\) is defined as follows. If \(A, B \subseteq X\), then \(\delta(2)\) iff for every finite cover \(C = \{C_j: j = 1, \ldots, n\}\) of \(A\) and \(D = \{D_k: k = 1, \ldots, m\}\) of \(B\), there exists \(C_i \subseteq C\) and \(D_k \subseteq D\) such that \(\bigcap_{j=1}^n C_j \subseteq D_k\) for all \(i \in I\) (where \(\bigcap\) denotes the projection map from \(X\) to \(X_i\)).

Cazmir's definition is essentially the same as Leader's, the only difference being that Cazmir considers finite decompositions rather than finite covers of sets \(A\) and \(B\).

There is a third definition which arises from a product of uniformities definable on \(\{X_i, \delta_i\}\). Each proximity relation has at least one compatible uniformity, the totally bounded uniformity whose base is generated by sets of the form \(\{A \times A: A = \{x_1, \ldots, x_n\}\}\), where \(A \subseteq B \subseteq B_1 \subseteq \cdots \subseteq B_n\) is an arbitrary finite cover of the space \(B\) is the relation associated with \(\delta\), where \(A \subseteq B\) iff \(\langle A \times B, \delta \rangle \neq \delta\).

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DEFINITION 2.3. Let \( (X_i, \delta_i) : i \in I \) be a family of proximity spaces, let \( X = \times \{ X_i : i \in I \} \) and let \( U_{\delta} \) be the totally bounded uniformity associated with \( \delta \). The proximity relation \( \delta(3) \) on \( X \) is the proximity relation induced by the product of the uniformities \( \{ U_{\delta} : i \in I \} \).

These three definitions are, in fact, equivalent. Leader [3] shows for his definitions (Theorem 9 ii) that the topology induced by the product proximity is identical to the product of the topologies induced by the coordinate proximity relations. Thus all three definitions satisfy the first criterion of a good definition for a product proximity. We show these facts in a more general setting.

DEFINITION 2.4. Let \( m \) be an infinite cardinal number. A proximity relation \( \delta \) on \( X \) is called an \( m \)-proximity relation iff \( (A \cup B, i \in I) \in \delta \) implies that \( (A, B) \in \delta \) for at least one \( i \in I \), where \( I \) is of cardinality \( <m \).

DEFINITION 2.5. Let \( \{ (X_i, \delta_i) : i \in I \} \) be a family of \( n_i \)-proximity spaces, where \( n_i \geq m \) for all \( i \in I \) and some infinite cardinal number \( m \). Let \( X = \times \{ X_i : i \in I \} \). Then the proximity relation \( \delta(m) \) on \( X \) is the least \( m \)-proximity relation on \( X \) such that each projection map is \( p \)-continuous.

DEFINITION 2.6. Let \( \{ (X_i, \delta_i) : i \in I \} \) and \( X \) be defined as in 2.5. The proximity relation \( \delta(m) \) on \( X \) is defined as follows. If \( A, B \subseteq X \) then \( (A, B) \in \delta(m) \) if for every cover \( \{ C_i : a < n_i < m \} \) of \( A \) and \( \{ D_i : b < n_i < m \} \) of \( B \) there exists \( a_0 \) and \( b_0 \) such that \( (p_{\delta}(C_{a_0}), p_{\delta}(D_{b_0})) \in \delta_i \) for all \( i \in I \).

These two definitions are obviously generalizations of \( \delta(1) \) and \( \delta(2) \) respectively. We may generalize \( \delta(n) \) because every \( n \)-proximity has a unique \( m \)-bounded, \( m \)-uniformity compatible with \( \delta \) for \( m \leq n \).

DEFINITION 2.7. A uniformity \( U \) on \( X \) is called an \( m \)-uniformity iff \( \bigcap \{ U_{\delta} : \delta \in U \} \), where \( I \) is of cardinality \( <m \).

THEOREM 2.8. If \( \delta \) is an \( n \)-proximity on \( X \), then for each \( m \leq n \) there exists a unique \( m \)-bounded, \( m \)-uniformity on \( X \) such that \( \{ U_{\delta} : \delta \in U \} = \delta \).

Proof. The proof is a straightforward generalization of the proof of Theorem 21.29 in Thron [8].

DEFINITION 2.9. Let \( \mathcal{U} \) be an \( n_i \)-uniformity on \( X_i \), where \( n_i \leq m \) and \( i \in I \), let \( X = \times \{ X_i : i \in I \} \), let \( H_{\mathcal{U}_{\delta}} \) be as in Lemma 1.4, and let \( \mathcal{F} \) consist of sets of the form \( \bigcap \{ H_{\mathcal{U}_{\delta}} : i \in I \} \), where \( I \) is of cardinality \( <m \).

The \textit{m-product} \( \mathcal{U} \) on \( X \) is defined to be \( \{ U : U \supseteq B \in \mathcal{F} \} \). \( \mathcal{F} \) is a base for \( \mathcal{U} \).

DEFINITION 2.10. Let \( \{ (X_i, \delta_i) : i \in I \} \) be a family of \( n_i \)-proximity spaces, where \( m \geq m \) and \( \{ U_{\delta} : i \in I \} \), let \( U \) be the \( m \)-bounded uniformity associated with \( \delta \) and let \( \mathcal{U} \) be the \( m \)-product uniformity of \( \{ U_{\delta} : i \in I \} \). The proximity relation \( \delta(m) \) on \( X \) is defined to be the proximity induced on \( X \) by \( \mathcal{U} \).

THEOREM 2.11. Let \( \{ (X_i, \delta_i) : i \in I \} \) be a family of \( n_i \)-proximity spaces, where \( n_i \geq m \) for all \( i \in I \), and let \( X = \times \{ X_i : i \in I \} \); then \( \delta(m)(1) = \delta(m)(2) = \delta(m)(3) \) on \( X \).

Proof. We prove this theorem by showing that \( \delta(m)(1) \subseteq \delta(m)(2) \subseteq \delta(m)(3) \).

First we show that \( \delta(m)(1) \subseteq \delta(m)(2) \). Let \( (A, B) \in \delta(m)(1) \). It is easily shown that \( (p_{\delta}(A), p_{\delta}(B)) \in \delta_i \) for all \( i \in I \). Suppose that \( G_i : a < n_i < m \) and \( (L_i : b < n_i < m) \) are covers of \( A \) and \( B \) respectively. Now \( \{ p_{\delta}(G_i) : a < n_i \} \cap \delta(m)(1) \) and \( \{ p_{\delta}(L_i) : b < n_i \} \) for all \( i \in I \). Thus \( (A, B) \in \delta(m)(2) \).

Next we show that \( \delta(m)(2) \subseteq \delta(m)(3) \). Suppose that \( (A, B) \notin \delta(m)(2) \). Letting \( \mathcal{U} \) denote the \( m \)-product of the \( m \)-bounded \( m \)-uniformities associated with the \( \delta_i \) we have \( U \in \mathcal{U} \) such that \( U \cap U = \emptyset \). Now \( U = \bigcap \{ H_{\mathcal{U}_{\delta}} : i \in I \} \), where \( J \) has cardinality \( <m \). Also \( U = \bigcap \{ O_{\mathcal{U}_{\delta}} : i \in I \} \), where \( K(i) \) has cardinality \( <m \).

Next cover \( A \) with elements of the form \( \bigcap \{ D(i, k) : k \in K(i) \} \), where \( D(i, k) = \{ x \in X : x \in A_k \} \) and \( \bigcap \{ D(i, k) : k \in K(i) \} = X_k(i) \).

Let \( M(i) = \{ k \in K(i) : D(i, k) \cap \delta(A) = \emptyset \} \) and \( N(i) = \{ k \in K(i) : D(i, k) \cap p_{\delta}(B) = \emptyset \} \).

Now cover \( A \) with elements of the form \( \bigcap \{ D(i, k) : k \in M(i) \} \) and \( B \) with elements of the form \( \bigcap \{ D(i, k) : k \in N(i) \} \). Note that there are \( <m \) elements in these covers. Now suppose there exists elements \( E \) and \( F \) from the covers of \( A \) and \( B \) respectively such that \( \delta(E, F) \) for all \( i \in I \). We complete the proof by showing that this is impossible. Letting \( p_{\delta}(E) = D(i, k) \) and \( p_{\delta}(F) = D(i, s) \) we have \( D(i, k) \cap D(i, s) = \emptyset \) for all \( i \in I \). Since \( D(i, k) \subseteq O(i, k) \) and \( D(i, s) \subseteq O(i, s) \) we have \( O(i, k) \cap O(i, s) = \emptyset \); denote a point in this intersection by \( x(i) \). Now let \( x \in X \) be such that \( p_{\delta}(x) = x(i) \) for all \( i \in I \) and \( p(x) = \emptyset \) for arbitrary \( i \in I \). Noting that \( E \cap F = \emptyset \) and \( F \cap B = \emptyset \) we let \( y \in E \cap A \) and \( z \in F \cap B \). Now \( x(i) \in O(i, k) \) so we have \( y(i), z(i) \in U(i) \) and hence \( y(i), z(i) \in U \) and \( x \in U(A) \). Similarly \( (z, x) \in U \) and \( x = x(B) \). But this contradicts the fact that \( U(A) \cap U(B) = \emptyset \) and so we may conclude that \( (A, B) \notin \delta(m)(2) \).

Finally we show that \( \delta(m)(3) \subseteq \delta(m)(1) \). Clearly \( \mathcal{U} \) is an \( m \)-uniformity so \( \delta(m)(3) \) is an \( m \)-proximity. Furthermore the projection maps \( p_{\delta} \) are uniformly continuous from \( (X, \delta) \) to \( (X_i, \delta_i) \) and hence \( p_{\delta} \) is \( m \)-proximities from \( \delta(m)(3) \) to \( \delta(m)(3) \). Now since \( \delta(m)(1) \) is the least \( m \)-proximity on \( X \) such that the projection maps are \( m \)-continuous we have \( \delta(m)(1) \subseteq \delta(m)(3) \); i.e., \( \delta(m)(3) \subseteq \delta(m)(1) \).
Letting \( m = \pi \), we have the promised equivalence of the different definitions of a product proximity.

**Corollary 2.12.** If \( \langle X_i; \delta_i \rangle: i \in I \) is a family of proximity spaces, then the product proximity relations \( \delta(1), \delta(2), \) and \( \delta(3) \) are all equivalent on \( \times X_i: i \in I \).

The topological counterpart to the \( m \)-uniformity and the \( m \)-proximity is the \( m \)-additive topology. A topology \( G \) is \( m \)-additive iff \( \bigcap \{ G_i: i \in I, G_i \in \mathcal{G} \} \in \mathcal{G} \), where \( i \) is of cardinality \( < m \). It is clear that if \( \delta \) is an \( m \)-proximity, then \( \mathcal{G}_\delta \) is \( m \)-additive and if \( \mathcal{U}_\delta \) is an \( m \)-uniformity, then \( \mathcal{U}_\delta \) is an \( m \)-proximity and \( \mathcal{U}_\delta \) is \( m \)-additive. If \( \langle X_i, \mathcal{G}_i \rangle: i \in I \) is a family of \( m \)-additive spaces for \( n \geq m \) the \( m \)-product topology on \( \times X_i: i \in I \) has as its base sets the form \( \bigcap \{ G(i): i \in I, G(i) \in \mathcal{G}_i \} \), where \( G(i) \subseteq \{ x: \pi_i \in G(i) \} \) and \( J \) has cardinality \( < m \). It is easily shown that \( \delta(3) \) and hence \( \delta(2) \) and \( \delta(1) \) satisfy the first criterion of a good definition of the \( m \)-product proximity.

**Theorem 2.13.** Let \( \langle X_i, \mathcal{G}_i \rangle: i \in I \) be a family of \( m \)-additive spaces for \( n \geq m \). The \( m \)-product of the associated \( m \)-additive spaces \( \mathcal{G}_i \) is the same as \( \mathcal{W}_\delta \).

### § 3. Weak and Strong Product Proximities

Letting \( \langle X_i, \mathcal{G}_i \rangle: i \in I \) be a family of proximity spaces, \( X = \times X_i: i \in I \) and letting \( \mathcal{U}_\delta \) be an arbitrary uniformity \( X_i \) compatible with \( \delta_i \) for all \( i \). Now we consider the proximity relation \( \langle \mathcal{U}_\delta \rangle: i \in I \) on \( X \). In § 1 we observed that \( \langle \mathcal{U}_\delta \rangle: i \in I \) and \( \langle \mathcal{V}_\delta \rangle: i \in I \) may be different for different choices of \( \mathcal{U}_\delta \) and \( \mathcal{V}_\delta \) on \( X_i \). In § 2 we concentrated on the particular relation \( \langle \mathcal{U}_\delta \rangle: i \in I \), where \( \mathcal{U}_\delta \) is the totally bounded uniformity on \( X_i \) compatible with \( \delta_i \). Recalling the ordering of proximities and the ordering of uniformities on \( X \) we prove the following convenient relationship between them. As in § 2 we generalize to arbitrary cardinalities. For notation we shall use \( \langle \mathcal{U}_\delta: i \in I \rangle \) to mean the \( m \)-proximity relation generated by the \( m \)-uniformity on \( \langle \mathcal{U}_\delta: i \in I \rangle \).

**Theorem 3.1.** If \( \mathcal{U}_\delta \) and \( \mathcal{V}_\delta \) are \( m \)-uniformities on \( X_i \) respectively, where \( \mathcal{U}_\delta \subseteq \mathcal{V}_\delta \) and \( n \geq m \) for all \( i \) with \( \mathcal{V}_\delta: i \in I \) and \( \mathcal{U}_\delta: i \in I \), then \( \langle \mathcal{U}_\delta: i \in I \rangle \subseteq \langle \mathcal{V}_\delta: i \in I \rangle \).

**Proof.** We shall show that \( \langle \mathcal{U}_\delta: i \in I \rangle \subseteq \langle \mathcal{V}_\delta: i \in I \rangle \). Suppose that \( \langle A, B \rangle \in \langle \mathcal{V}_\delta: i \in I \rangle \) and let \( W \) be an entourage of the \( m \)-product uniformity on \( \langle \mathcal{U}_\delta: i \in I \rangle \). Thus \( W = \bigcap \{ H: i \in I \}, \mathcal{H}(i) \subseteq \mathcal{U}_\delta \rangle \), where \( J \) is of cardinality \( < m \). Let \( V(i) \subseteq U(i) \), where \( V(i) \subseteq \mathcal{V}_\delta \rangle \), \( i \in I \) and let \( W^o \) be an entourage of the \( m \)-product uniformity on \( \langle \mathcal{U}_\delta: i \in I \rangle \). Also \( W(A) \subseteq B \neq \emptyset \) since \( \langle A, B \rangle \in \langle \mathcal{U}_\delta: i \in I \rangle \). Therefore \( W(A) \cap B \neq \emptyset \) and since \( W \) was arbitrary we conclude that \( \langle A, B \rangle \in \langle \mathcal{U}_\delta: i \in I \rangle \).

**Corollary 3.2.** If \( \mathcal{U}_\delta \) and \( \mathcal{U}_\epsilon \) are uniformities on \( X_i \) and \( \mathcal{U}_\epsilon \subseteq \mathcal{U}_\delta \) for all \( i \in I \), then \( \langle \mathcal{U}_\epsilon: i \in I \rangle \subseteq \langle \mathcal{U}_\delta: i \in I \rangle \).

Because of 3.2 we can legitimately speak of “weak” and “strong” product proximity relations. The weak product would naturally be the proximity induced by the product of the weakest (smallest, coarsest) uniformities on the coordinate proximities \( \delta_i \). Since the weakest uniformity is the totally bounded uniformity generated by \( \delta_i \) the weak product proximity on \( \times X_i: i \in I \) would be \( \delta(3) \). The strong proximity would be the proximity induced by the strongest (largest, finest) uniformity on the coordinate proximities \( \delta_i \). Such uniformities do not always exist, however, as Theorem 3.12 indicates. Aitken and Fenstad [1] constructed the first example of a proximity relation which does not admit a strongest compatible uniformity. Below we formalize the concept of weak and strong product of proximity spaces for arbitrary cardinalities.

**Definition 3.3.** Let \( \langle X_i, \mathcal{G}_i \rangle: i \in I \) be a family of \( m \)-proximity spaces such that \( n \geq m \) for all \( i \) and let \( X = \times X_i: i \in I \). Letting \( \mathcal{U}_\delta \) be the smallest \( m \)-uniformity compatible with \( \delta_i \) on \( X_i \) and \( \mathcal{U}_\epsilon \) be the largest \( m \)-uniformity (assuming it exists) compatible with \( \delta_i \), then \( \langle \mathcal{U}_\epsilon: i \in I \rangle \) is the \( m \)-weak product proximity and \( \langle \mathcal{U}_\delta: i \in I \rangle \) is the \( m \)-strong product proximity.

A uniformity is called total if it is the largest member of its proximity class. For example, a uniformity which has a linearly ordered base is total (see [2]). In fact if the least cardinality of a linearly ordered base is \( m \), then the uniformity \( \mathcal{G}_\delta \) is an \( m \)-uniformity and is largest in the proximity class of the \( m \)-proximity \( \langle \mathcal{G}_\delta \rangle \). For the sake of convenience rather than for the sake of confusion (we hope) we use the word total for proximities also.

**Definition 4.3.** A proximity relation is an \( m \)-total proximity relation iff it admits a largest \( m \)-uniformity. This largest \( m \)-uniformity we shall call an \( m \)-total uniformity.

**Theorem 3.5.** Let \( \delta_i \) be an \( m \)-total proximity on \( X_i \), where \( n \geq m \) and \( i \in I \), let \( \mathcal{U}_\epsilon \) be the \( m \)-total uniformity compatible with \( \delta_i \), and let \( X = \times X_i: i \in I \). Then \( \langle \mathcal{U}_\delta: i \in I \rangle \) is an \( m \)-total proximity on \( X \).

**Proof.** We prove this by showing that \( \mathcal{U}_\epsilon \) the \( m \)-product of the uniformities \( \mathcal{U}_\delta \) is an \( m \)-total uniformity on \( X \). Suppose that there exists an \( m \)-uniformity \( \mathcal{V}_\delta \) on \( X \) such that \( \mathcal{V}_\delta \subseteq \mathcal{U} \). Therefore there exists \( V \subseteq \mathcal{V} \) such that \( \emptyset \subseteq \mathcal{V}_\delta \) for any \( W \subseteq \mathcal{U}_\delta \). Setting \( \mathcal{V}_\delta = \{ (x, y): (x, y) \in V \} \) we see that we have two cases: 1) There exists \( j \in I \) such that \( U \subseteq \mathcal{V} \) for all \( U \subseteq \mathcal{U}_\delta \), and 2) There exists a set \( K \subseteq I \) such that \( \mathcal{V} \neq \emptyset \times \mathcal{V} \) for all \( k \in K \), where \( K \) has cardinality \( \geq m \). Now in both cases we shall find \( \langle \mathcal{V} \rangle \neq \langle \mathcal{U} \rangle \) thus completing the proof that \( \langle \mathcal{U} \rangle \) is m-total.
For case 1) this may be seen as follows. Since $U \subseteq V$ for all $U \in \mathcal{U}_I$, we have $\mathcal{V}_I \subseteq \mathcal{U}_I$ (where $\mathcal{U}_I = \{V \subseteq X : V \subseteq U\}$). Now $\mathcal{U}_I$ is a uniformity on $X_I$ and since $\mathcal{U}_I$ is an $m$-total proximity it follows that $\langle \mathcal{U}_I \rangle = \langle \mathcal{U}_I \rangle^m$. Thus there exists a subset $A$ of $X_I$ such that $(A, B) \in \langle \mathcal{U}_I \rangle$ and $(A, B) \in \langle \mathcal{U}_I \rangle$. Letting $C = \{x \in X : a \in A\}$ and $D = \{x \in X : a \in B\}$ we have $(C, D) \in \langle \mathcal{U}_I \rangle$ and $(C, D) \in \langle \mathcal{U}_I \rangle$. Hence $\langle \mathcal{U}_I \rangle \neq \langle \mathcal{U}_I \rangle$.

For case 2) we need only note that the topology induced by $\mathcal{U}_I$ on $X_I$ is not the $m$-product topology and so $\langle \mathcal{U}_I \rangle \neq \langle \mathcal{U}_I \rangle$.

**Corollary 3.6.** Let $\mathcal{U}_I$ be an $m$-total uniformity on $X_I$, where $n_I \geq m$ and $i \in I$. Then the $m$-product of the $n_I$-uniformities is an $m$-total uniformity on $X_I$.

**Corollary 3.7.** Let $\mathcal{U}_I$ be a uniform space with a linearly ordered base of least cardinality $n_I$ on $X_I$, where $n_I \geq m$ and $i \in I$. Then the $m$-product of the uniformities is an $m$-total uniformity on $X_I$.

**Theorem 3.8.** Let $\mathcal{U}_I$ be a uniform space with a linearly ordered base on $X_I$. Then the product of the uniformities is a uniform space on $X_I$.

The $m$-weak product proximity is clearly the proximity $\delta^{m}(3)$ and hence $\delta^{m}(1), \delta^{m}(2)$ and $\delta^{m}(1)$ studied in § 2. Here we show that this proximity is in general, not an $m$-total proximity. More precisely, if a least two $m$-proximities, $\delta_I$ and $\delta_J$, of $\langle \mathcal{U}_I : i \in I \rangle$ of proximities admit non $m$-bounded $m$-uniformities, then the $m$-weak product proximity is not $m$-total. This fact was stated in Isbell [4] for the special case of the product of two ordinary proximity spaces which admit non-total bounded uniformities.

Our first theorem in this regard is a straightforward generalization of the analogous theorem for total bounded uniformities. We omit its proof here.

**Theorem 3.9.** Let $\{X_I, \mathcal{U}_I : i \in I\}$ be a family of $n_I$-uniform spaces, where $n_I \geq m$, let $X = \prod (X_I : i \in I)$ and let $\mathcal{U}_X$ be the $m$-product of $\mathcal{U}_I$ on $X$. Then $\mathcal{U}_X$ is a uniformity on $X$. The uniformity on $X$, $\mathcal{U}_X$, is $m$-bounded against $\mathcal{U}_I$.

**Theorem 3.10.** Let $\mathcal{U}_I$ and $\mathcal{U}_J$ be $n_I$-uniformities on $X_I$, where $n_I \geq m$ and $i \in I$. If for some $j \in I$, $\mathcal{U}_J$ is $m$-bounded and $\mathcal{U}_J$ is not $m$-bounded, and for some $k \neq j \in I$, $\mathcal{U}_K$ is not $m$-bounded, where $n_I \geq m$, then $\langle \mathcal{U}_I : i \in I \rangle$.

Proof. Notice that this theorem is a generalization of Theorem 1.5. Since $\mathcal{U}_I$ and $\mathcal{U}_J$ are not $m$-bounded there exists uniformly discrete sets $A = \{x_x : a \in n\}$ in $X_I$ and $B = \{y_x : a \in n\}$ in $X_J$. Since $\mathcal{U}_I$ and $\mathcal{U}_J$ are $n_I$-uniformities of cardinality $m$, let $X = \prod (X_I : i \in I)$, $C = \{x : a \in A\}$ and $D = \{x : a \in B\}$ for some $a \in n$ and $i \neq j \in I$, where $a \neq \beta$. We show that $(C, D) \in \langle \mathcal{U}_I : i \in I \rangle$.

Let $\langle \mathcal{U} \rangle$ denote the $m$-product of $\langle \mathcal{U}_I : i \in I \rangle$ and let $W$ be an arbitrary member of $\mathcal{U}$. Thus $W = \cap \{U \subseteq X : U \subseteq W\}$, where $\mathcal{U}$ is a uniformity on $X_I$, and since $\mathcal{U}$ is an $m$-total proximity it follows that $\langle \mathcal{U}_I \rangle = \langle \mathcal{U} \rangle^m$. Thus there exists a subset $A$ of $X_I$ such that $(A, B) \in \langle \mathcal{U}_I \rangle$ and $(A, B) \in \langle \mathcal{U}_I \rangle$. Letting $C = \{x \in X : a \in A\}$ and $D = \{x \in X : a \in B\}$ we have $(C, D) \in \langle \mathcal{U}_I \rangle$ and $(C, D) \in \langle \mathcal{U}_I \rangle$. Hence $\langle \mathcal{U}_I \rangle \neq \langle \mathcal{U}_I \rangle$.
On the topology of curves IV

by

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As is well-known, each arc is an acyclic and atriodic curve, and these two properties characterize arcs within some considerably large classes of curves, for instance the class of locally connected curves. The second author has proved that all acyclic Suslinian curves possess a decomposition property (see [5], Theorem 2.2). An analogue for atriodic curves is established in this paper (see §1). Actually, we show the decomposition property to be possessed by all Suslinian curves which are locally atriodic in a weak sense, and we derive a stronger decomposition property for all atriodic Suslinian curves (see §3). The latter property, however, is not necessarily possessed by all acyclic Suslinian curves (see §4). Although the general question remains unsolved (see [6], Problem 10), it seems now to be answered almost completely for the class of atriodic curves, which comprises some interesting cases: a classical example of a plane curve constructed by G. T. Whyburn [10] as well as other curves obtained by means of the method of R. D. Anderson and Gustave Choquet [1]. The topological structure of atriodic hereditarily decomposable curves is essential in our approach (see §2). Also, at the end of the paper, we provide an example of a chainable Suslinian curve that is not rational.

§1. Hereditarily discontinuous subsets. A space is called hereditarily discontinuous provided each continuum contained in it is degenerate (2). A curve $X$ is called atriodic provided, for each three subcurves $C_1$, $C_2$, $C_3$ of $X$ such that

$$C_3 = C_1 \cap C_2 = C_1 \cap C_3 = C_2 \cap C_3 \neq \emptyset$$

is connected, $X_0$ coincides with at least one of the curves $C_2$, $C_1$, $C_3$. We follow [3] to mean by a *chump* any non-degenerate collection $C$ of continua whose union is a continuum and for which there exists a non-empty continuum $C_0$, called the core of $C$, such that $C_0$ is a proper subset of every

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(2) Hereditarily discontinuous spaces were called "pseudotopological" in [6] but now we adopt the terminology of [4] which seems to be more suitable for this paper.