Product proximities

by

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Introduction. Let $\{X_i: i \in I\}$ be a family of sets and let X represent the Cartesian product of these sets. If $\mathcal{T}_i$ is a topology on X_i, then the product topology $\mathcal{T}$ on X can be defined as the least topology which makes the projection maps continuous. If $\mathcal{U}_i$ is a uniformity on X_i, then the product uniformity $\mathcal{U}$ on X can be defined as the least uniformity which makes the projection maps uniformly continuous. If δ_i is a proximity relation on X_i it is natural to define the product proximity δ on X as the least proximity relation on X which makes the projection maps p-continuous. For this definition to be a good one the topology induced by the product proximity should be identical to the product of the topologies induced by the coordinate proximity relations. Also the proximity relation induced by the product uniformity should be identical to the product of the proximity relations induced by the coordinate uniformities. The definition of the product proximity relation suggested above satisfies the former criterion but fails to satisfy the latter. It is possible, for example, that there exist uniformities $\mathcal{U}$ and $\mathcal{V}$ on X which induce the same proximity relation δ on X and yet the product uniformities $\mathcal{U}\times\mathcal{U}$ and $\mathcal{V}\times\mathcal{V}$ induce different proximity relations on $X\times X$. This paper examines this phenomenon.

§ 1. Proximity classes of Product Uniformities. Isbell [4] observes that the products of two non-totally bounded uniformities $\mathcal{U}$ and $\mathcal{V}$ induce a proximity relation distinct from the relation induced by the totally bounded uniformities $\mathcal{U}'$ and $\mathcal{V}'$ from the same proximity classes as $\mathcal{U}$ and $\mathcal{V}$ respectively. We generalize this result to $\mathfrak{m}$-bounded uniformities.

DEFINITION 1.1. Let $\mathfrak{m}$ be an infinite cardinal number. A uniformity $\mathcal{U}$ on X is $\mathfrak{m}$-*bounded* iff for any $U \in \mathcal{U}$ there exists a set A of cardinality $<\mathfrak{m}$ such that $U(A) = X$.

DEFINITION 1.2. Let $(X, \mathcal{U})$ be a uniform space. A subset A of X is *uniformly discrete* iff there exists $U \in \mathcal{U}$ such that $(x, y) \notin U$ for any $x, y \in A$, where $x \neq y$.

LEMMA 1.3. *A uniform space* $(X, \mathfrak{U})$ *is* $\mathfrak{m}$*-bounded iff every uniformly discrete subset of* X *has cardinality* $<\mathfrak{m}$.

Proof. See, for example, Reed and Thron [7], Theorem 1.2.

LEMMA 1.4. *Let* $\{(X_i, \mathfrak{U}_i)\colon i \in I\}$ *be a family of uniform spaces and let* $X = \times\{X_i\colon i \in I\}$. *Then the product uniformity* $\mathfrak{U}$ *on* X *has a subbase consisting of sets of the form* $H_{U(i)} = \{(x, y) \in X \times X\colon (x_i, y_i) \in U(i)\}$, *where* $U(i) \in \mathfrak{U}_i$, $i \in I$ *and* x_i *and* y_i *denote the* i*-th coordinate of* x *and* y, *respectively.*

In order to avoid awkward notation we shall employ the following unorthodox but simplified notation. We let $\langle\mathfrak{U}\rangle$ denote the proximity relation induced by the uniformity $\mathfrak{U}$, $\langle\mathfrak{U}, \mathfrak{V}\rangle$ denote the proximity induced by $\mathfrak{U}\times\mathfrak{V}$ and, in general $\langle\mathfrak{U}_i\colon i \in I\rangle$ denote the proximity induced by the product of the uniformities $\{\mathfrak{U}_i\colon i \in I\}$.

THEOREM 1.5. *Let* $\mathfrak{U}$ *and* $\mathfrak{V}$ *be uniformities on* X *and* $\mathfrak{U}'$ *and* $\mathfrak{V}'$ *be uniformities on* X'. *If* $\mathfrak{U}$ *or* $\mathfrak{U}'$ *is* $\mathfrak{m}$*-bounded and* $\mathfrak{V}$ *and* $\mathfrak{V}'$ *are not* $\mathfrak{m}$*-bounded, then* $\langle\mathfrak{U}, \mathfrak{U}'\rangle \neq \langle\mathfrak{V}, \mathfrak{V}'\rangle$.

Proof. We prove this in a more general setting in Theorem 3.10.

COROLLARY 1.6. *Let* $\mathfrak{U}$ *and* $\mathfrak{V}$ *be uniformities on* X *and* $\mathfrak{U}'$ *and* $\mathfrak{V}'$ *be uniformities on* X'. *If* $\mathfrak{U}$ *or* $\mathfrak{U}'$ *is* $\mathfrak{m}$*-bounded,* $\mathfrak{V}$ *and* $\mathfrak{V}'$ *are not* $\mathfrak{m}$*-bounded and* $\langle\mathfrak{U}\rangle = \langle\mathfrak{U}'\rangle$ *and* $\langle\mathfrak{V}\rangle = \langle\mathfrak{V}'\rangle$, *then* $\langle\mathfrak{U}, \mathfrak{U}'\rangle \neq \langle\mathfrak{V}, \mathfrak{V}'\rangle$.

It is not the case that boundedness is the only key to the formation of distinct product proximities. The following theorem shows, for example, that the product of uniformities of the same bound may also induce distinct relations. First we recall the standard ordering of uniformities and proximity relations on a set X. If $\mathfrak{U}$ and $\mathfrak{V}$ are uniformities on X we write $\mathfrak{U} < \mathfrak{V}$ iff $\mathfrak{U} \subset \mathfrak{V}$; if δ and δ' are proximity relations on X we write $\delta < \delta'$ iff $\delta \supset \delta'$. Now it is easily shown (and will be shown in a more general setting in Theorem 3.1) that if $\mathfrak{U} \leqslant \mathfrak{V}$, then $\langle\mathfrak{U}\rangle \leqslant \langle\mathfrak{V}\rangle$. It is not necessarily true, however, that $\mathfrak{U} < \mathfrak{V}$ implies $\langle\mathfrak{U}\rangle < \langle\mathfrak{V}\rangle$; it is possible that $\mathfrak{U} < \mathfrak{V}$ and $\langle\mathfrak{U}\rangle = \langle\mathfrak{V}\rangle$. For products we have the following theorems.

THEOREM 1.7. *If* $\mathfrak{U}$, $\mathfrak{U}'$ *and* $\mathfrak{V}$ *are uniformities on* X *such that* $\mathfrak{U}$ *or* $\mathfrak{U}' < \mathfrak{V}$, *then* $\langle\mathfrak{U}, \mathfrak{U}'\rangle \neq \langle\mathfrak{V}, \mathfrak{V}\rangle$.

Proof. Without loss of generality we assume that $\mathfrak{U} < \mathfrak{V}$.

Since $\mathfrak{U} < \mathfrak{V}$ there exists a symmetric member $V \in \mathfrak{V}$ such that $U \not\subseteq V$ for any $U \in \mathfrak{U}$. Let K be a symmetric entourage such that $K \circ K \circ K \subset V$. Letting $M = (X\times X) - V$ we shall show that $(K, M) \in \langle\mathfrak{U}, \mathfrak{U}'\rangle$ and $(K, M) \notin \langle\mathfrak{V}, \mathfrak{V}\rangle$. This will imply that $\langle\mathfrak{U}, \mathfrak{U}'\rangle \neq \langle\mathfrak{V}, \mathfrak{V}\rangle$.

Let $W \in \mathfrak{U}\times\mathfrak{U}'$. Thus $W = H_U \cap H_{U'}$ for some $U \in \mathfrak{U}$ and $U' \in \mathfrak{U}'$. Now let $(x, y) \in U$ be such that $(x, y) \notin V$. Such an (x, y) exists because $U \not\subseteq V$. Now $(x, x) \in U'$ so $\big((x, x), (y, x)\big) \in W$. Since $(x, x) \in K$ we have $(y, x) \in W(K)$. But $(y, x) \notin V$ because $(x, y) \notin V$ and V is symmetric so $W(K) \not\subset V$. Thus $W(K) \cap M \neq \emptyset$. Since W was arbitrary $(K, M) \in \langle\mathfrak{U}, \mathfrak{U}'\rangle$.

Let $W = H_K \cap H_K$ and let $(x, y) \in W(K)$. Then there exists $(a, b) \in K$ such that $\big((a, b), (x, y)\big) \in W$. Thus (a, b), (a, x) and $(b, y) \in K$. Since K is symmetric we have (x, a), (a, b) and $(b, y) \in K$ and therefore $(x, y) \in K \circ K \circ K \subset V$. Since (x, y) was an arbitrary member of $W(K)$ we have $W(K) \subset V$. We may conclude that $W(K) \cap M = \emptyset$ and so $(K, M) \notin \langle\mathfrak{V}, \mathfrak{V}\rangle$.

COROLLARY 1.8. *If* $\mathfrak{U}$, $\mathfrak{U}'$ *and* $\mathfrak{V}$ *are uniformities on* X *such that* $\mathfrak{U}$ *and* $\mathfrak{U}' < \mathfrak{V}$, *then* $\langle\mathfrak{U}, \mathfrak{U}'\rangle < \langle\mathfrak{V}, \mathfrak{V}\rangle$.

COROLLARY 1.9. *If* $\mathfrak{U}$, $\mathfrak{U}'$ *and* $\mathfrak{V}$ *are* $\mathfrak{m}$*-bounded uniformities on* X *such that* $\mathfrak{U}$ *or* $\mathfrak{U}' < \mathfrak{V}$, *then* $\langle\mathfrak{U}, \mathfrak{U}'\rangle \neq \langle\mathfrak{V}, \mathfrak{V}\rangle$.

If $\mathfrak{m} = \aleph_0$ the hypotheses of 1.9 cannot be satisfied because there is one and only one totally bounded uniformity compatible with δ. However, Reed and Thron [7] have shown that if δ has a compatible $\mathfrak{m}$-bounded uniformity $\mathfrak{U}$ and $\mathfrak{m} \neq \aleph_0$, then there is an infinite decreasing sequence of compatible $\mathfrak{m}$-bounded uniformities less than $\mathfrak{U}$.

§ 2. Product Proximities.

As explained in the introduction there is a natural definition of a product proximity.

DEFINITION 2.1. Let $\{(X_i, \delta_i)\colon i \in I\}$ be a family of proximity spaces and let $X = \times\{X_i\colon i \in I\}$. The *proximity relation* $\delta(1)$ is the least proximity on X such that each projection map is p-continuous.

Leader [5] and Csaszar [3] offer a different definition of a product proximity relation. We give Leader's version here.

DEFINITION 2.2. Let $\{(X_i, \delta_i)\colon i \in I\}$ be a family of proximity spaces and let $X = \times\{X_i\colon i \in I\}$. The *proximity relation* $\delta(2)$ on X is defined as follows. If $A, B \subset X$, then $(A, B) \in \delta(2)$ iff for every finite cover $\mathfrak{C} = \{C_j\colon j = 1, \ldots, n\}$ of A and $\mathfrak{D} = \{D_k\colon k = 1, \ldots, m\}$ of B, there exists $C_{j_0} \in \mathfrak{C}$ and $D_{k_0} \in \mathfrak{D}$ such that $\big(p_i(C_{j_0}), p_i(D_{k_0})\big) \in \delta_i$ for all $i \in I$ (where p_i denotes the projection map from X to X_i).

Csaszar's definition is essentially the same as Leader's, the only difference being that Csaszar considers finite decompositions rather than finite covers of sets A and B.

There is a third definition which arises from a product of uniformities definable on (X_i, δ_i). Each proximity relation has at least one compatible uniformity, the totally bounded uniformity whose base is generated by sets of the form $\{A_k \times A_k\colon k = 1, \ldots, n\}$, where $A_k \Supset B_k$ and $B_1, \ldots, B_n$ is an arbitrary finite cover of the space ($\Supset$ is the relation associated with δ, where $A \Supset B$ iff $\big((X-A), B\big) \notin \delta$).

DEFINITION 2.3. Let $\{(X_i, \delta_i): i \in I\}$ be a family of proximity spaces, let $X = \times\{X_i: i \in I\}$ and let $\mathfrak{U}_i$ be the totally bounded uniformity associated with δ_i. The *proximity relation* $\delta(3)$ on X is the proximity relation induced by the product of the uniformities $\mathfrak{U}_i$; i.e., $\delta(3) = \langle \mathfrak{U}_i: i \in I\rangle$.

These three definitions are, in fact, equivalent. Leader [5] shows for his definitions (Theorem 9 ii) that the topology induced by the product proximity is identical to the product of the topologies induced by the coordinate proximity relations. Thus all three definitions satisfy the first criterion of a good definition for a product proximity. We show these facts in a more general setting.

DEFINITION 2.4. Let $\mathfrak{m}$ be an infinite cardinal number. A proximity relation δ on X is called an $\mathfrak{m}$-*proximity relation* iff $(A, \bigcup\{B_i: i \in I\}) \in \delta$ implies that $(A, B) \in \delta$ for at least one $i \in I$, where I is of cardinality $< \mathfrak{m}$.

DEFINITION 2.5. Let $\{(X_i, \delta_i): i \in I\}$ be a family of $\mathfrak{n}_i$-proximity spaces, where $\mathfrak{n}_i \geqslant \mathfrak{m}$ for all $i \in I$ and some infinite cardinal number $\mathfrak{m}$. Let $X = \times\{X_i: i \in I\}$. Then the *proximity relation* $\delta^{\mathfrak{m}}(1)$ on X is the least $\mathfrak{m}$-proximity relation on X such that each projection map is p-continuous.

DEFINITION 2.6. Let $\{(X_i, \delta_i): i \in I\}$ and X be defined as in 2.5. The *proximity relation* $\delta^{\mathfrak{m}}(2)$ on X is defined as follows. If $A, B \subset X$, then $(A, B) \in \delta^{\mathfrak{m}}(2)$ iff for every cover $\{C_\alpha: \alpha < \mathfrak{n}' < \mathfrak{m}\}$ of A and $\{D_\beta: \beta < \mathfrak{n}'' < \mathfrak{m}\}$ of B there exists α_0 and β_0 such that $\big(p_i(C_{\alpha_0}), p_i(D_{\beta_0})\big) \in \delta_i$ for all $i \in I$.

These two definitions are obvious generalizations of $\delta(1)$ and $\delta(2)$ respectively. We may generalize $\delta(3)$ because every $\mathfrak{n}$-proximity has a unique $\mathfrak{m}$-bounded, $\mathfrak{m}$-uniformity compatible with δ for $\mathfrak{m} \leqslant \mathfrak{n}$.

DEFINITION 2.7. A uniformity $\mathfrak{U}$ on X is called an $\mathfrak{m}$-*uniformity* iff $\bigcap\{U_i: i \in I\} \in \mathfrak{U}$, where I is of cardinality $< \mathfrak{m}$.

THEOREM 2.8. *If δ is an $\mathfrak{n}$-proximity on X, then for each $\mathfrak{m} \leqslant \mathfrak{n}$ there exists a unique $\mathfrak{m}$-bounded, $\mathfrak{m}$-uniformity $\mathfrak{U}$ on X such that $\langle\mathfrak{U}\rangle = \delta$. This uniformity has as its base sets of the form $\{A_k \times A_k: k \in I\}$, where $A_k \supseteq B_k$, $\bigcup\{B_k: k \in I\} = X$, and I is of cardinality $< \mathfrak{m}$.*

Proof. The proof is a straight forward generalization of the proof of Theorem 21.20 in Thron [8].

DEFINITION 2.9. Let $\mathfrak{U}_i$ be an $\mathfrak{n}_i$ uniformity on X_i, where $\mathfrak{n}_i \leqslant \mathfrak{m}$ and $i \in I$, let $X = \times\{X_i: i \in I\}$, let $H_{U(i)}$ be as in Lemma 1.4, and let $\mathfrak{B}$ consist of sets of the form $\bigcap\{H_{U(i)}: i \in J \subset I\}$, where J is of cardinality $< \mathfrak{m}$. The $\mathfrak{m}$-*product* $\mathfrak{U}$ on X is defined to be $\{U: U \supseteq B, B \in \mathfrak{B}\}$. Thus $\mathfrak{B}$ is a base for $\mathfrak{U}$.

DEFINITION 2.10. Let $\{(X_i, \delta_i): i \in I\}$ be a family of $\mathfrak{n}_i$-proximity spaces, where $\mathfrak{n}_i \geqslant \mathfrak{m}$, let $X = \times\{X_i: i \in I\}$, let $\mathfrak{U}_i$ be the $\mathfrak{m}$-bounded uniformity associated with δ_i and let $\mathfrak{U}$ be the $\mathfrak{m}$-product uniformity of $\{\mathfrak{U}_i: i \in I\}$. The *proximity relation* $\delta^{\mathfrak{m}}(3)$ on X is defined to be the proximity induced on X by $\mathfrak{U}$.

THEOREM 2.11. *Let $\{(X_i, \delta_i): i \in I\}$ be a family of $\mathfrak{n}_i$-proximity spaces, where $\mathfrak{n}_i \geqslant \mathfrak{m}$ for all $i \in I$, and let $X = \times\{X_i: i \in I\}$; then $\delta^{\mathfrak{m}}(1) = \delta^{\mathfrak{m}}(2) = \delta^{\mathfrak{m}}(3)$ on X.*

Proof. We prove this theorem by showing that $\delta^{\mathfrak{m}}(1) \subset \delta^{\mathfrak{m}}(2) \subset \delta^{\mathfrak{m}}(3) \subset \delta^{\mathfrak{m}}(1)$.

First we show that $\delta^{\mathfrak{m}}(1) \subset \delta^{\mathfrak{m}}(2)$. Let $(A, B) \in \delta^{\mathfrak{m}}(1)$. It is easily shown that $\big(p_i(A), p_i(B)\big) \in \delta_i$ for all $i \in I$. Suppose that $\{C_\alpha: \alpha < \mathfrak{n}' < \mathfrak{m}\}$ and $\{D_\beta: \beta < \mathfrak{n}'' < \mathfrak{m}\}$ are coverings of A and B respectively. Now $(\bigcup\{C_\alpha: \alpha < \mathfrak{n}'\}, B) \in \delta^{\mathfrak{m}}(1)$ so $\big(p_i(\bigcup C_\alpha), p_i(B)\big) \in \delta_i$ for all $i \in I$. Hence for at least one α, call it $\alpha_0 < \mathfrak{n}'$, we have $\big(p_i(C_{\alpha_0}), p_i(B)\big) \in \delta_i$ for all $i \in I$. Similarly $(C_{\alpha_0}, \bigcup\{D_\beta: \beta < \mathfrak{n}''\}) \in \delta^{\mathfrak{m}}(1)$ so there exists $\beta_0 < \mathfrak{n}''$ such that $\big(p_i(C_{\alpha_0}), p_i(D_{\beta_0})\big) \in \delta_i$ for all $i \in I$. Thus $(A, B) \in \delta^{\mathfrak{m}}(2)$.

Next we show that $\delta^{\mathfrak{m}}(2) \subset \delta^{\mathfrak{m}}(3)$. Suppose that $(A, B) \notin \delta^{\mathfrak{m}}(3)$. Letting $\mathfrak{U}$ denote the $\mathfrak{m}$-product of the $\mathfrak{m}$-bounded $\mathfrak{m}$-uniformities associated with the δ_i we have $U \in \mathfrak{U}$ such that $U(A) \cap U(B) = \emptyset$. Now $U = \bigcap\{H_{U(i)}: i \in J \subset I\}$, where J has cardinality $< \mathfrak{m}$. Also

$$U(i) = \bigcup\{C(i, k) \times C(i, k): k \in K(i)\},$$

where $K(i)$ has cardinality $< \mathfrak{m}$, $C(i, k) \supseteq D(i, k)$ and $\bigcup\{D(i, k): k \in K(i)\} = X_i$. Let

$$M(i) = \{k \in K(i): D(i, k) \cap p_i(A) \neq \emptyset\}$$
$$\text{and} \quad N(i) = \{k \in K(i): D(i, k) \cap p_i(B) \neq \emptyset\}.$$

Now cover A with elements of the form $\bigcap\{D(i, k)': k \in M(i), i \in J\}$, where $D(i, k)' = \{x \in X: x_i \in D(i, k)\}$ and cover B with elements of the form $\{D(i, s)': s \in N(i), i \in J\}$. Note that there are $< \mathfrak{m}$ elements in these covers. Now suppose there exists elements E and F from the covers of A and B respectively such that $\big(p_i(E), p_i(F)\big) \in \delta_i$ for all $i \in I$. We complete the proof by showing that this is impossible. Letting $p_i(E) = D(i, k)$ and $p_i(F) = D(i, s)$ we have $\big(D(i, k), D(i, s)\big) \in \delta_i$ for $i \in J$. Since $D(i, k) \Subset C(i, k)$ and $D(i, s) \Subset C(i, s)$ we have $C(i, k) \cap C(i, s) \neq \emptyset$; denote a point in this intersection by $x(i)$. Now let $x \in X$ be such that $p_i(x) = x(i)$ for all $i \in J$ and $p_i(x)$ be arbitrary for $i \notin J$. Noting that $E \cap A \neq \emptyset$ and $F \cap B \neq \emptyset$ we let $y \in E \cap A$ and $z \in F \cap B$. Now $x_i, y_i \in C(i, k)$ so we have $(y_i, x_i) \in U(i)$ and hence $(y, x) \in U$ and $x \in U(A)$. Similarly $(z, x) \in U$ so $x \in U(B)$. But this contradicts the fact that $U(A) \cap U(B) = \emptyset$ and so we may conclude that $(A, B) \notin \delta^{\mathfrak{m}}(2)$.

Finally we show that $\delta^{\mathfrak{m}}(3) \subset \delta^{\mathfrak{m}}(1)$. Clearly $\mathfrak{U}$ is an $\mathfrak{m}$-uniformity so $\delta^{\mathfrak{m}}(3)$ is an $\mathfrak{m}$-proximity. Furthermore the projection maps p_j are uniformly continuous from (X, δ) to (X_j, δ_j) and hence p_j is p-continuous from $\big(X, \delta^{\mathfrak{m}}(3)\big)$ to (X_j, δ_j). Now since $\delta^{\mathfrak{m}}(1)$ is the least $\mathfrak{m}$-proximity on X such that the projection maps are p-continuous we have $\delta^{\mathfrak{m}}(1) < \delta^{\mathfrak{m}}(3)$; i.e., $\delta^{\mathfrak{m}}(3) \subset \delta^{\mathfrak{m}}(1)$.

Letting $\mathfrak{m} = \aleph_0$ we have the promised equivalence of the different definitions of a product proximity.

COROLLARY 2.12. *If* $\{(X_i, \delta_i): i \in I\}$ *is a family of proximity spaces, then the product proximity relations* $\delta(1)$, $\delta(2)$, *and* $\delta(3)$ *are all equivalent on* $\times\{X_i: i \in I\}$.

The topological counterpart to the $\mathfrak{m}$-uniformity and the $\mathfrak{m}$-proximity is the $\mathfrak{m}$-additive topology. A topology $\mathcal{T}$ is $\mathfrak{m}$-additive iff $\bigcap\{G_i: i \in I, G_i \in \mathcal{T}\} \in \mathcal{T}$, where I is of cardinality $< \mathfrak{m}$. It is clear that if δ is an $\mathfrak{m}$-proximity, then $\mathcal{T}_\delta$ is $\mathfrak{m}$-additive and if $\mathfrak{U}$ is an $\mathfrak{m}$-uniformity, then $\langle\mathfrak{U}\rangle$ is an $\mathfrak{m}$-proximity and $\mathcal{T}_{\mathfrak{U}}$ is $\mathfrak{m}$-additive. If $\{(X_i, \mathcal{T}_i): i \in I\}$ is a family of $\mathfrak{n}_i$-additive spaces for $\mathfrak{n}_i \geqslant \mathfrak{m}$ the $\mathfrak{m}$-product topology on $\times\{X_i: i \in I\}$ has as its base sets of the form $\bigcap\{G(i)': i \in J \subset I, G(i) \in \mathcal{T}_i\}$, where $G(i)' = \{x: x_i \in G(i)\}$ and J has cardinality $< \mathfrak{m}$. It is easily shown that $\delta^{\mathfrak{m}}(3)$ and hence $\delta^{\mathfrak{m}}(2)$ and $\delta^{\mathfrak{m}}(1)$ satisfy the first criterion of a good definition of the $\mathfrak{m}$-product proximity.

THEOREM 2.13. *Let* $\{(X_i, \delta_i): i \in I\}$ *be a family of* $\mathfrak{n}_i$*-proximity spaces for* $\mathfrak{n}_i \geqslant \mathfrak{m}$. *The* $\mathfrak{m}$*-product of the associated* $\mathfrak{n}_i$*-additive spaces* $\mathcal{T}_{\delta_i}$ *is the same as* $\mathcal{T}_{\delta^{\mathfrak{m}}(3)}$.

§ 3. Weak and Strong Product Proximities. Letting $\{(X_i, \delta_i): i \in I\}$ be a family of proximity spaces, letting $X = \times\{X_i: i \in I\}$ and letting $\mathfrak{U}_i$ be an arbitrary uniformity X_i compatible with δ_i we now consider the proximity relation $\langle\mathfrak{U}_i: i \in I\rangle$ on X. In § 1 we observed that $\langle\mathfrak{U}_i: i \in I\rangle$ and $\langle\mathfrak{V}_i: i \in I\rangle$ may be different for different choices of $\mathfrak{U}_i$ and $\mathfrak{V}_i$ on X_i. In § 2 we concentrated on the particular relation $\langle\mathfrak{U}_i: i \in I\rangle$, where $\mathfrak{U}_i$ is the totally bounded uniformity on X_i compatible with δ_i. Recalling the ordering of proximities and the ordering of uniformities on X we prove the following convenient relationship between them. As in § 2 we generalize to arbitrary cardinality. For notation we shall use $\langle\mathfrak{U}_i: i \in I\rangle^{\mathfrak{m}}$ to mean the $\mathfrak{m}$-proximity relation generated by the $\mathfrak{m}$-product uniformity on $\{\mathfrak{U}_i: i \in I\}$.

THEOREM 3.1. *If* $\mathfrak{U}_i$ *and* $\mathfrak{V}_i$ *are* $\mathfrak{n}_i$*-uniformities on* X_i *respectively, where* $\mathfrak{U}_i \leqslant \mathfrak{V}_i$ *and* $\mathfrak{n}_i \geqslant \mathfrak{m}$ *for all* $i \in I$, *then* $\langle\mathfrak{U}_i: i \in I\rangle^{\mathfrak{m}} \leqslant \langle\mathfrak{V}_i: i \in I\rangle^{\mathfrak{m}}$.

Proof. We shall show that $\langle\mathfrak{V}_i: i \in I\rangle^{\mathfrak{m}} \subseteq \langle\mathfrak{U}_i: i \in I\rangle^{\mathfrak{m}}$. Suppose that $(A, B) \in \langle\mathfrak{V}_i: i \in I\rangle^{\mathfrak{m}}$ and let W be an entourage of the $\mathfrak{m}$-product uniformity on $\{\mathfrak{U}_i: i \in I\}$. Thus $W = \bigcap\{H_{U(i)}: i \in J \subset I, U(i) \in \mathfrak{U}_i\}$, where J is of cardinality $< \mathfrak{m}$. Let $V(i) \subseteq U(i)$, where $V(i) \in \mathfrak{V}_i$, $i \in J$ and let $W' = \{H_{V(i)}: i \in J\}$. Now W' is an entourage of the $\mathfrak{m}$-product uniformity on $\{\mathfrak{V}_i: i \in I\}$. Also $W'(A) \cap B \neq \emptyset$ since $(A, B) \in \langle\mathfrak{V}_i: i \in I\rangle^{\mathfrak{m}}$. Therefore $W(A) \cap B \neq \emptyset$ and since W was arbitrary we conclude that $(A, B) \in \langle\mathfrak{U}_i: i \in I\rangle^{\mathfrak{m}}$.

COROLLARY 3.2. *If* $\mathfrak{U}_i$ *and* $\mathfrak{V}_i$ *are uniformities on* X_i *and* $\mathfrak{U}_i \leqslant \mathfrak{V}_i$ *for all* $i \in I$, *then* $\langle\mathfrak{U}_i: i \in I\rangle \leqslant \langle\mathfrak{V}_i: i \in I\rangle$.

Because of 3.2 we can legitimately speak of "weak" and "strong" product proximity relations. The weak product would naturally be the proximity induced by the product of the weakest (smallest, coarsest) uniformities on the coordinate proximities δ_1. Since the weakest uniformity is the totally bounded uniformity generated by δ_i the weak product proximity on $\times\{X_i: i \in I\}$ would be $\delta(3)$. The strong proximity would be the proximity induced by the strongest (largest, finest) uniformity on the coordinate proximities δ_i. Such uniformities do not always exist, however, as Theorem 3.12 indicates. Alfsen and Fenstad [1] constructed the first example of a proximity relation which does not admit a strongest compatible uniformity. Below we formalize the concept of weak and strong product of proximity spaces for arbitrary cardinality.

DEFINITION 3.3. Let $\{(X_i, \delta_i): i \in I\}$ be a family of $\mathfrak{n}_i$-proximity spaces such that $\mathfrak{n}_i \geqslant \mathfrak{m}$ for all $i \in I$ and let $X = \times\{X_i: i \in I\}$. Letting $\mathfrak{U}_i$ be the smallest $\mathfrak{m}$-uniformity compatible with δ_i on X_i and $\mathfrak{V}_i$ be the largest $\mathfrak{m}$-uniformity (assuming it exists) compatible with δ_i, then $\langle\mathfrak{U}_i: i \in I\rangle^{\mathfrak{m}}$ is the $\mathfrak{m}$-*weak product proximity* and $\langle\mathfrak{V}_i: i \in I\rangle^{\mathfrak{m}}$ is the $\mathfrak{m}$-*strong product proximity*.

A uniformity is called total if it is the largest member of its proximity class. For example, a uniformity which has a linearly ordered base is total (see [2]). In fact if the least cardinality of a linearly ordered base is $\mathfrak{m}$, then the uniformity $\mathfrak{U}$ is an $\mathfrak{m}$-uniformity and is largest in the proximity class of the $\mathfrak{m}$-proximity $\langle\mathfrak{U}\rangle$. For the sake of convenience rather than for the sake of confusion (we hope) we use the word total for proximities also.

DEFINITION 3.4. A proximity relation is an $\mathfrak{m}$-*total proximity relation* iff it admits a largest $\mathfrak{m}$-uniformity. This largest $\mathfrak{m}$-uniformity we shall call an $\mathfrak{m}$-*total uniformity*.

THEOREM 3.5. *Let* δ_i *be an* $\mathfrak{n}_i$*-total proximity on* X_i, *where* $\mathfrak{n}_i \geqslant \mathfrak{m}$ *and* $i \in I$, *let* $\mathfrak{U}_i$ *be the* $\mathfrak{n}_i$*-total uniformity compatible with* δ_i, *and let* $X = \times\{X_i: i \in I\}$. *Then* $\langle\mathfrak{U}_i: i \in I\rangle^{\mathfrak{m}}$ *is an* $\mathfrak{m}$*-total proximity on* X.

Proof. We prove this by showing that $\mathfrak{U}$, the $\mathfrak{m}$-product of the uniformities $\mathfrak{U}_i$ is an $\mathfrak{m}$-total uniformity on X. Suppose that there exists an $\mathfrak{m}$-uniformity $\mathfrak{V}$ on X such that $\mathfrak{V} \subseteq \mathfrak{U}$. Therefore there exists $V \in \mathfrak{V}$ such that $W \not\subset V$ for any $W \in \mathfrak{U}$. Letting $V^j = \{(x_j, y_j): (x, y) \in V\}$ we see that we have two cases: 1) There exists $j \in I$ such that $U \not\subseteq V^j$ for all $U \in \mathfrak{U}_j$, and 2) There exists a set $K \subseteq I$ such that $V^k \neq X_k \times X_k$ for all $k \in K$, where K has cardinality $\geqslant \mathfrak{m}$. Now in both cases we shall find $\langle\mathfrak{V}\rangle \neq \langle\mathfrak{U}\rangle$ thus completing the proof that $\langle\mathfrak{U}\rangle$ is $\mathfrak{m}$-total.

For case 1) this may be seen as follows. Since $U \subseteq V^j$ for all $U \in \mathfrak{U}_j$ we have $\mathfrak{V}_j \nsubseteq \mathfrak{U}_j$ (where $\mathfrak{V}_j = \{V^j\colon V \in \mathfrak{V}\}$). Now $\mathfrak{V}_j$ is a uniformity on X_j and since $\mathfrak{U}_j$ is an $\mathfrak{m}$-total proximity it follows that $\langle\mathfrak{V}_j\rangle \neq \langle\mathfrak{U}_j\rangle$. Thus there exists subsets A and B of X_j such that $(A, B) \in \langle\mathfrak{U}_j\rangle$ and $(A, B) \notin \langle\mathfrak{V}_j\rangle$. Letting $C = \{x \in X\colon x_j \in A\}$ and $D = \{x \in X\colon x_j \in B\}$ we have $(C, D) \in \langle\mathfrak{U}\rangle$ and $(C, D) \notin \langle\mathfrak{V}\rangle$. Hence $\langle\mathfrak{U}\rangle \neq \langle\mathfrak{V}\rangle$.

For case 2) we need only note that the topology induced by $\mathfrak{V}$ on X is not the $\mathfrak{m}$-product topology and so $\langle\mathfrak{U}\rangle \neq \langle\mathfrak{V}\rangle$.

COROLLARY 3.6. *Let $\mathfrak{U}_i$ be an $\mathfrak{n}_i$-total uniformity on X_i, where $\mathfrak{n}_i \geqslant \mathfrak{m}$ and $i \in I$. Then the $\mathfrak{m}$-product of the $\mathfrak{n}_i$-uniformities is an $\mathfrak{m}$-total uniformity on $\times\{X_i\colon i \in I\}$.*

COROLLARY 3.7. *Let $\mathfrak{U}_i$ be a uniform space with a linearly ordered base of least cardinality $\mathfrak{n}_i$ on X_i, where $\mathfrak{n}_i \geqslant \mathfrak{m}$ and $i \in I$. Then the $\mathfrak{m}$-product of the uniformities is an $\mathfrak{m}$-total uniformity on $\times\{X_i\colon i \in I\}$.*

COROLLARY 3.8. *Let $\mathfrak{U}_i$ be a uniform space with a linearly ordered base on X_i. Then the product of the uniformities is a total uniformity on $\times\{X_i\colon i \in I\}$.*

The $\mathfrak{m}$-weak product proximity is clearly the proximity $\delta^{\mathfrak{m}}(3)$ (and hence $\delta^{\mathfrak{m}}(2)$ and $\delta^{\mathfrak{m}}(1)$) studied in § 2. Here we show that this proximity is, in general, not an $\mathfrak{m}$-total proximity. More precisely, if a least two $\mathfrak{m}$-proximities, δ_j and δ_k of a family $\{\delta_i\colon i \in I\}$ of proximities admit compatible non $\mathfrak{m}$-bounded $\mathfrak{m}$-uniformities, then the $\mathfrak{m}$-weak product proximity is not $\mathfrak{m}$-total. This fact was stated in Isbell [4] for the special case of the product of two ordinary proximity spaces which admit non-totally bounded uniformities.

Our first theorem in this regard is a straight forward generalization of the analogous theorem for total bounded uniformities. We omit its proof here.

THEOREM 3.9. *Let $\{(X_i, \mathfrak{U}_i)\colon i \in I\}$ be a family of $\mathfrak{n}_i$-uniform spaces, where $\mathfrak{n}_i \geqslant \mathfrak{m}$, let $X = \times\{X_i\colon i \in I\}$ and let $\mathfrak{U}$ denote the $\mathfrak{m}$-product $\mathfrak{m}$-uniformity on $\{\mathfrak{U}_i\colon i \in I\}$. The $\mathfrak{m}$-uniformity $(X, \mathfrak{U})$ is $\mathfrak{m}$-bounded iff $(X_i, \mathfrak{U}_i)$ is $\mathfrak{m}$-bounded for all $i \in I$.*

THEOREM 3.10. *Let $\mathfrak{U}_i$ and $\mathfrak{V}_i$ be $\mathfrak{n}_i$-uniformities on X_i, where $\mathfrak{n}_i \geqslant \mathfrak{m}$ and $i \in I$. If for some $j \in I$, $\mathfrak{U}_j$ is $\mathfrak{n}$-bounded and $\mathfrak{V}_j$ is not $\mathfrak{n}$-bounded and for some $k \neq j \in I$, $\mathfrak{V}_k$ is not $\mathfrak{n}$-bounded, where $\mathfrak{n} \geqslant \mathfrak{m}$, then $\langle\mathfrak{U}_i\colon i \in I\rangle^{\mathfrak{m}} \neq \langle\mathfrak{V}_i\colon i \in I\rangle^{\mathfrak{m}}$.*

Proof. Notice that this theorem is a generalization of Theorem 1.5. Since $\mathfrak{V}_j$ and $\mathfrak{V}_k$ are not $\mathfrak{n}$-bounded there exists uniformly discrete sets $A = \{x_a\colon a < \mathfrak{n}\}$ in $(X_j, \mathfrak{V}_j)$ and $B = \{y_a\colon a < \mathfrak{n}\}$ in $(X_k, \mathfrak{V}_k)$ of cardinality $\mathfrak{n}$. Letting $X = \times\{X_i\colon i \in I\}$, $C = \{c \in X\colon p_j(c) = x_a$ and $p_k(c) = y_a$ for some $a < \mathfrak{n}\}$ and $D = \{d \in X\colon p_j(d) = x_a$ and $p_k(d) = y_\beta$ for some a, β, where $a \neq \beta\}$ we show that $(C, D) \in \langle\mathfrak{U}_i\colon i \in I\rangle^{\mathfrak{m}}$ and $(C, D) \notin \langle\mathfrak{V}_i\colon i \in I\rangle^{\mathfrak{m}}$.

Let $\mathfrak{U}$ denote the $\mathfrak{m}$-product of $\{\mathfrak{U}_i\colon i \in I\}$ and let W be an arbitrary member of $\mathfrak{U}$. Thus $W = \bigcap\{H_{U(i)}\colon U(i) \in \mathfrak{U}_i$ and $i \in J \subset I\}$, where J is of cardinality $< \mathfrak{m}$. Since $\mathfrak{U}_j$ is $\mathfrak{n}$-bounded there exists $a, a' < \mathfrak{n}$ such that $(x_a, x_{a'}) \in U(j)$ (if $j \notin J$, then $U(j) = X_j$). Now we have $(a, b) \in W$, where $p_j(a) = x_a$, $p_j(b) = x_{a'}$, and $p_k(a) = y_a = p_k(b)$. But $a \in C$ and $b \in D$ so $W(C) \cap D \neq \emptyset$. Since W was arbitrary we have $(C, D) \in \langle\mathfrak{U}_i\colon i \in I\rangle^{\mathfrak{m}}$.

Since A and B are uniformly discrete sets there exists $V(j) \in \mathfrak{V}_j$ and $V(k) \in \mathfrak{V}_k$ such that $(x_a, x_{a'}) \notin V(j)$ for all $a \neq a' < \mathfrak{n}$ and $(y_\beta, y_{\beta'}) \notin V(k)$ for all $\beta, \beta' < \mathfrak{n}$. Let $W = H_{V(j)} \cap H_{V(k)}$. Now suppose $b \in W(C)$. Thus there exists $a \in C$ such that $(a, b) \in W$. Since $a \in C$ we have $p_j(a) = x_a$ and $p_k(a) = y_a$ for some $a < \mathfrak{n}$. Since $(a, b) \in W$ we have $p_j(b) = x_a$ and $p_k(b) = y_a$ for the same a. Thus $b = a \notin D$. Therefore $W(C) \cap D = \emptyset$ and we have $(C, D) \notin \mathfrak{V}_i\colon i \in I\rangle^{\mathfrak{m}}$.

THEOREM 3.11. *If $\mathfrak{U}$ and $\mathfrak{U}'$ are $\mathfrak{m}$-uniformities on X_1 such that $\langle\mathfrak{U}\rangle = \langle\mathfrak{U}'\rangle$ and if $\mathfrak{V}$ is the $\mathfrak{m}$-bounded $\mathfrak{m}$-uniformity on X_2, then $\langle\mathfrak{U}, \mathfrak{V}\rangle = \langle\mathfrak{U}', \mathfrak{V}\rangle$.*

Proof. We shall show here that $\langle\mathfrak{U}, \mathfrak{V}\rangle \geqslant \langle\mathfrak{U}', \mathfrak{V}\rangle$. Since $\mathfrak{U}$ and $\mathfrak{U}'$ play a symmetric role in our proof we also have the result that $\langle\mathfrak{U}, \mathfrak{V}\rangle \leqslant \langle\mathfrak{U}', \mathfrak{V}\rangle$ and thus the equality we seek. We accomplish our proof that $\langle\mathfrak{U}, \mathfrak{V}\rangle \leqslant \langle\mathfrak{U}', \mathfrak{V}\rangle$ by showing that $\langle\mathfrak{U}, \mathfrak{V}\rangle \subseteq \langle\mathfrak{U}', \mathfrak{V}\rangle$; that is, if $(A, B) \notin \langle\mathfrak{U}', \mathfrak{V}\rangle$, then $(A, B) \notin \langle\mathfrak{U}, \mathfrak{V}\rangle$.

Suppose that $(A, B) \notin \langle\mathfrak{U}', \mathfrak{V}\rangle$. Thus there exists $W' \in \mathfrak{U}' \times \mathfrak{V}$ such that $W'(A) \cap W'(B) = \emptyset$. Now $W' = H_{U'} \cap H_V$ for some $U' \in \mathfrak{U}'$ and $V \in \mathfrak{V}$. We may assume that U' and V are symmetric entourages. Letting $A_2 = p_2(A)$, $B_2 = p_2(B)$ and $K \in \mathfrak{V}$ such that K is symmetric and $K \circ K \circ K \subset V$ there exists a set $C = \{y_a\colon a < \mathfrak{n} < \mathfrak{m}\}$ such that $K(C) \supseteq A_2 \cap B_2$. This is true because $(X_2, \mathfrak{V})$ is $\mathfrak{m}$-bounded. We denote $\{(x, y)\colon y \in V(y_a)$ and $(x, y) \in A\}$ by A^a and $\{(x, y)\colon y \in V(y_a)$ and $(x, y) \in B\}$ by B^a.

Letting $p_1(A^a) = A(a, 1)$ and $p_1(B^a) = B(a, 1)$ we first show that $\big(A(a, 1), B(a, 1)\big) \notin \langle\mathfrak{U}'\rangle$. We accomplished this by proving that $U'\big(A(a, 1)\big) \cap U'\big(B(a, 1)\big) = \emptyset$. If x is in this intersection we have $x \in U'\big(A(a, 1)\big)$ and $x \in U'\big(B(a, 1)\big)$. Thus there exists z and z' such that $z \in A(a, 1)$, $z' \in B(a, 1)$, $(z, x) \in U'$, $(z', x) \in U'$ and there exists $y, y' \in X_2$ such that $(z, y) \in A$, $(z', y') \in B$, $(y, y_a) \in V$ and $(y', y_a) \in V$. This means that $(x, y_a) \in W'(A) \cap W'(B)$ which contradicts the hypothesis that $W'(A) \cap W'(B) = \emptyset$.

We complete the proof by showing that $(A, B) \notin \langle\mathfrak{U}, \mathfrak{V}\rangle$. Since $\langle\mathfrak{U}'\rangle = \langle\mathfrak{U}\rangle$ we know that $\big(A(a, 1), B(a, 1)\big) \notin \langle\mathfrak{U}\rangle$ and so there exists $U_a \in \mathfrak{U}$ such that $U_a\big(A(a, 1)\big) \cap U_a\big(B(a, 1)\big) = \emptyset$. Letting $U = \{U_a\colon a < \mathfrak{n}\}$ (and recalling that $\mathfrak{U}$ is an $\mathfrak{m}$-uniformity so $U \in \mathfrak{U}$) we show that $W(A) \cap W(B) = \emptyset$, where $W = H_U \cap H_K$. This will show that

$(A, B) \notin \langle \mathfrak{U}, \mathfrak{V} \rangle$. Suppose that $(a, b) \in W(A) \cap W(B)$. Since $(a, b) \in W(A)$ there exists $(x, y) \in A$ such that $((a, b), (x, y)) \in W$. Hence $(a, x) \in U$ and $(b, y) \in K$. Similarly there exists $(x', y') \in B$ such that $(a, x') \in U$ and $(b, y') \in K$. Now since $(b, y), (b, y') \in K$ there exists $\alpha, \alpha' < \mathfrak{n}$ such that $(y, y_\alpha), (y', y_{\alpha'}) \in K$. Thus we have $(y', b), (b', y), (y, y_\alpha) \in K$ so $(y', y) \in K \circ K \circ K \subset V$. Since $(x, y) \in A$ and $(x', y') \in B$ we have $(x, y) \in A^\alpha$ and $(x', y') \in B^\alpha$. Now $x \in A^\alpha$, $x' \in B^\alpha$ and $(a, x), (a, x') \in U$ so $a \in U(A(\alpha, 1))$ and $a \in U(B(\alpha, 1))$. But $U \subset U_\alpha$ so we have $a \in U_\alpha(A(\alpha, 1)) \cap U_\alpha(B(\alpha, 1))$. This contradicts the hypothesis that $U_\alpha(A(\alpha, 1)) \cap U_\alpha(B(\alpha, 1)) = \emptyset$ and so we obtain $W(A) \cap W(B) = \emptyset$.

THEOREM 3.12. *If δ_i is a family of $\mathfrak{m}$-proximities on X_i respectively (where $i \in I$) such that at least two admit non- $\mathfrak{m}$-bounded $\mathfrak{m}$-uniformities, then the weak $\mathfrak{m}$-product proximity $\delta^{\mathfrak{m}}(3)$ is not an $\mathfrak{m}$-total proximity.*

Proof. Let δ_i and δ_j be two proximities which admit non-$\mathfrak{m}$-bounded uniformities. Let $\langle \mathfrak{U}_i \colon i \in I \rangle^{\mathfrak{m}} = \mathfrak{U}^j$, where $\mathfrak{U}_i$ is the $\mathfrak{m}$-bounded $\mathfrak{m}$-uniformity compatible with δ_i for all $i \neq j$ and $\mathfrak{U}_j = \mathfrak{V}_j$. Let $\mathfrak{U}^k$ be similarly defined with j replaced by k. It easily follows from Theorems 3.9 and 3.11 that $\langle \mathfrak{U}^j \rangle = \delta^{\mathfrak{m}}(3) = \langle \mathfrak{U}^k \rangle$. Suppose that $\mathfrak{U}'$ is compatible with $\delta^{\mathfrak{m}}(3)$. Now $\mathfrak{U}'$ is the $\mathfrak{m}$-product of $\{\mathfrak{U}'_i \colon i \in I\}$ and it follows from 3.10 that $\mathfrak{U}'_i$ must be $\mathfrak{m}$-bounded for all but at most one $i \in I$. If $i \neq k$, then $\mathfrak{U}' \not> \mathfrak{U}^k$; if $i = k$, then $\mathfrak{U}' \not> \mathfrak{U}^j$. Thus $\mathfrak{U}'$ is not an $\mathfrak{m}$-total proximity. We conclude that $\delta^{\mathfrak{m}}(3)$ is not an $\mathfrak{m}$-total proximity.

If $\mathfrak{m} = \aleph_0$ we have the following corollary.

COROLLARY 3.13. *If δ_i is a family of proximities on X_i respectively such that at least two admit more than one compatible uniformity, then the weak product proximity does not admit a strongest compatible uniformity.*

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On the topology of curves IV

by

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As is well-known, each arc is an acyclic and atriodic curve, and these two properties characterize arcs within some considerably large classes of curves, for instance the class of locally connected curves. The second author has proved that all acyclic Suslinian curves possess a decomposition property (see [5], Theorem 2.2). An analogue for atriodic curves is established in this paper (see § 1). Actually, we show the decomposition property to be possessed by all Suslinian curves which are locally atriodic in a weak sense, and we derive a stronger decomposition property for all atriodic Suslinian curves (see § 3). The latter property, however, is not necessarily possessed by all acyclic Suslinian curves (see § 4). Although the general question remains unsolved (see [6], Problem 10), it seems now to be answered almost completely for the class of atriodic curves, which comprises some interesting cases: a classical example of a plane curve constructed by G. T. Whyburn [10] as well as other curves obtained by means of the method of R. D. Anderson and Gustave Choquet [1]. The topological structure of atriodic hereditarily decomposable curves is essential in our approach (see § 2). Also, at the end of the paper, we provide an example of a chainable Suslinian curve that is not rational.

§ 1. Hereditarily discontinuous subsets. A space is called *hereditarily discontinuous* provided each continuum contained in it is degenerate ([1]). A curve X is called *atriodic* provided, for each three subcurves C_1, C_2, C_3 of X such that

$$C_0 = C_1 \cap C_2 = C_2 \cap C_3 = C_1 \cap C_3 \neq \emptyset$$

is connected, C_0 coincides with at least one of the curves C_1, C_2, C_3. We follow [3] to mean by a *clump* any non-degenerate collection $\boldsymbol{C}$ of continua whose union is a continuum and for which there exists a non-empty continuum C_0, called the *core* of $\boldsymbol{C}$, such that C_0 is a proper subset of every

([1]) Hereditarily discontinuous spaces were called "ponctiform" in [5] but now we adopt the terminology of [4] which seems to be more suitable for this paper.