A characterization of locally compact fields of zero characteristic

by

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0. In this note we shall give a characterization of locally compact fields of zero characteristic which seems to be new. Let us recall some definitions. A field topology \( \mathcal{C} \) is said to be locally bounded if there exists a bounded neighbourhood \( A \) of zero, i.e. if for every neighbourhood \( U \) of zero there exists another one, \( V \), such that \( AV \subseteq U \). For any topological field \( F \) we write \( G(F) \) for the group of all its continuous automorphisms. Moreover, \( \mathcal{C} \) is called a full topology if the completion \( \hat{F} \) of \( F \) in it is a field. It is well known (see [8], [10]) that the only full, locally bounded, non-trivial topologies on a field are topologies of type \( V \), that is topologies induced by Krull-valuations (i.e. valuations taking values in linearly ordered groups instead of the reals).

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1. The aim of this paper is to prove the following

**Theorem.** Let \( K \) be a non-discrete topological field. Then the following conditions are equivalent:

1. \( K \) is a locally bounded, complete field and for every closed subfield \( F \) of \( K \), \( G(F) \) is finite.
2. \( K \) is a locally compact field of characteristic zero.
3. \( K \) is a finite extension of the reals \( R \) or of some \( p \)-adic number field \( Q_p \) with the usual locally compact topologies.

**Proof of the theorem.** The equivalence \((2) \iff (3)\) is the classical theorem of Pontryagin–Kowalsky–van Dantzig (see [4], [7], [15]). \((3) \implies (1)\). Since every automorphism of \( R \) and \( Q_p \) is trivial, \( G(K) \) is finite as a subgroup of the Galois group \( G(K/R) \) or \( G(K/Q_p) \). Moreover, \( K \) is complete in the locally bounded field topology induced by an absolute value \(|\cdot|\) or by a \( p \)-adic norm \(|\cdot|_p\).

It remains to show that \((1) \implies (3)\).
Case I. $K$ is not algebraically closed.

A. Suppose $K$ is connected. Every locally bounded, complete and connected topological field is topologically isomorphic to $R$ or $C$ (see [9], [16]). This gives $K = R$ topologically.

B. Suppose that $K$ is disconnected and of characteristic zero. Then $K$ is totally disconnected (see [2], Theorem 1). Let $L$ be a fixed field of $G(K)$. From Lemma 2 of [16] it follows that $L$ is closed and the topology $\mathcal{T}$ of $K$ is the product topology induced from $L_1$ moreover, $L$ is complete. The completeness of $\mathcal{T}$ implies that $\mathcal{T}$ is a full topology and the local boundedness of $\mathcal{T}$ implies that $\mathcal{T}$ is induced by a suitable Krull valuation (see [8], [10]).

Suppose at first that the topology
\[ \mathcal{T}_1 = \mathcal{T}(Q) \text{ is non-discrete.} \]

Since $K$ is totally disconnected, its topology is given by the open subgroups of $Q$, i.e., by open $Z$-submodules in $Q$. But $Q$ is the quotient field of the principal ideal domain $Z$ and so we can apply the following.

**Lemma 1** (see [3]). Let $A$ be a principal ideal domain and $K$ the quotient field of $A$. If $\mathcal{T}$ is a non-discrete field topology on $K$, then $(K, \mathcal{T})$ is a topological field for which the open $A$-submodules form a fundamental system of neighborhoods of zero if and only if $\mathcal{T}$ is the supremum of a family of $p$-adic topologies ($p$ is an irreducible element in $A$).

Lemma 1 implies now that $\mathcal{T}_1$ is the supremum of a family of $p$-adic topologies. But the supremum of a family of locally bounded topological fields is locally bounded if and only if that family is finite (§). Moreover, $\mathcal{T}$ is a full topology, and so $\mathcal{T}_1$ is also full. We claim that our family of topologies consists of a single element which is a $p$-adic topology. Indeed, let $\mathcal{T}_p$ be the supremum of $p$-adic topologies for $i = 1, 2, ..., m$. The approximation theorem for valuations implies that the completion $\hat{Q}$ of $Q$ in $\mathcal{T}_1$ is a direct sum of fields $Q_{p_1}, ..., Q_{p_m}$:
\[ \hat{Q} \cong Q_{p_1} \oplus ... \oplus Q_{p_m}. \]

But $\mathcal{T}$ is a full topology and so $\hat{Q}$ is a field; thus $m = 1$.

We are going to prove that $L$ is an algebraic extension of $Q_p$. Suppose the contrary. Let $t^i$ be transcendental over $Q_p$. Denote by $L_i$ the closure of $Q_p(t)$ in $L$. We define an automorphism of $Q_p(t)$ by the formula:
\[ \varphi_p(f(t)) = (\varphi(t))^i (\varphi(t))^{i-1} \]
where $t$ is a fixed unit in $Q_p$, and so $|t|_{p} = 1$. Let us remark that the topology $\mathcal{T}$ is induced in $L$ by a non-Archimedean valuation. Indeed, since $Q_p \subset L$ topologically and $p^n \to 0$ in $\mathcal{T}$ as $n \to \infty$, the set $T$ of all topological nilpotents in $L$ is non-void, whence open (see [16], Lemma 5).

Since $\mathcal{T}$ is induced by a Krull valuation $v$, $(L, \mathcal{T})^{-1}$ is bounded. From the Safronov Theorem (15) it follows now that $v$ is a valuation taking values in an Archimedean ordered group, i.e., $v$ can be assumed to be a real valuation. Let us denote this valuation by $|a|$. We have
\[ |t|_{p} = |t|^i |t|_{p} = |t|^i |t| = |t| \]
and so $|t|^i = |t|_{p}$ for every $a \in Q_p(t)$. It follows that $\varphi_p \in G(Q_p(t))$ since $\varphi_p$ is an isometry.

Let us extend $\varphi$ to an automorphism $\varphi_p \in G(L_1)$ by putting, for every sequence $x_n \to x \in L_1$, $x_n \in Q_p(t)$
\[ \varphi_p(x_n) = \lim_{n \to \infty} \varphi_p(x_n). \]

Prove the lemma. If $[E:F] = \infty$, then there would exist a sequence $a_1, a_2, ... \in E$ with
\[ a_j \neq F(a_1, a_2, ..., a_j) \text{ for } j = 1, 2, ... \]
The separability assumption implies that with a suitable base $b_j \in E$ we have $F(b_j) = F(a_1, a_2, ..., a_j)$ and, in view of the obvious inequalities
\[ [F(b_j):F] < [F(b_i):F] < ... \]
we infer that $E$ contains elements of an arbitrary large degree over $E$, against a theorem of Ostrowski (see [12], Theorem 3). (If $E/F$ is algebraic, $E$ and $F$ being valued complete fields, then the degree over $F$ of elements of $F$ are bounded.)

From Lemma 2 we have $[L:Q_p] < \infty$, whence $K$ is a finite extension of the $p$-adic number field $Q_p$.

Now we consider the case
\[ \mathcal{T}_1 = \mathcal{T}(Q) \text{ is discrete.} \]
Then there exists an $a \in L_1$ transcendental over $Q_p$ since otherwise otherwise the extension $L/Q_p$ would be algebraic and, as the topology $\mathcal{T}$ is discrete on $Q_p$, it would remain discrete on every finite (algebraic) extension of $Q_p$ and so on $L_1$ which gives a contradiction. If $\mathcal{T}$ were discrete on $Q(a)$, then the
closed subfield \( Q(x) \) of \( L \) would have infinitely many (continuous) automorphisms of the form

\[
x \mapsto \frac{ax + b}{cx + d},
\]

where \( a, b, c, d \in Q, \ ad \neq bc \), which is a contradiction. Hence \( \mathcal{C} \) is non-discrete on \( Q(x) \). But the local boundedness of \( \mathcal{C} \) implies that \( \mathcal{C} \) is induced on \( Q(x) \) by a real valuation. This results from the following lemma:

**Lemma 3.** Let \( F(x) \) be a transcendental extension of a field \( F \) and \( \mathcal{C} \) a non-discrete, locally bounded, full topology on \( F(x) \), discrete on \( F \). Then \( \mathcal{C} \) is induced by one of the following valuations: \( [a]_{p(x)} \), where \( p(x) \in F[x] \) is an irreducible polynomial, or \( [a]_{\infty} \).

(We recall the definitions of these valuations. Let \( \frac{f(x)}{g(x)} \in F(x) \) be any non-zero element. We put

\[
\frac{f(x)}{g(x)} = e^{o_{\mathcal{C}}(f(x)g(x))} \quad \text{and} \quad \frac{f(x)}{g(x)} = e^{-N},
\]

where \( f(x) = p(x)^N f_1(x) \) and \( (p, f_1) = (p, g_1) = 1 \.)

Proof of the lemma. As the topology \( \mathcal{C} \) is full and locally bounded, it is induced on \( F(x) \) by a Krull valuation \( v : F(x) \to \Gamma \), where \( \Gamma \) is a multiplicative linearly ordered group with added \( 0 \). Denote by \( e \) the unit element of \( \Gamma \). If \( v(x) > e \), then

\[
v(a^k) = v(x^k) = v(ax^k) = v(dx^k) \quad \text{for all} \ k > 1
\]

and \( e, d \in E, cd \neq 0, \) since \( v(x) > e \) implies \( v(x)^N = v(x^N) > e \) for every \( N \in N \). This valuation \( v \) is non-Archimedean since it extends a trivial valuation. As for \( v(a) = v(\beta) \), we have

\[
v(a + \beta) = \max(v(a), v(\beta))
\]

if it follows that

\[
\frac{f(x)}{x} = v(a_1 x^N + \ldots + a_0) = v(x^N) = v(ax^N)\quad \text{for every} \ f(x) \in F(x).
\]

Thus

\[
\frac{f(x)}{g(x)} = v(ax^N - bxy^N).
\]

However, if \( 0 < v(x) < e \ (0 < v(x) < e \) for every \( \gamma \in \Gamma \) by definition), then \( v(x) < e \) for every \( h(x) \in F(x) \). Let

\[
R_v = \{ f(x) \in F(x) : v(f(x)) < e \}.
\]

Observe that we must have

\[
v \left( \frac{f(x)}{g(x)} \right) = v(p(x))^N, \quad \text{where} \quad \frac{f(x)}{g(x)} = p(x)^N \frac{f_1(x)}{g_1(x)}
\]

and \( f_1, g_1 \) are prime to \( p(x); \) \( p(x) \in F[x] \) is a suitable irreducible polynomial. Indeed, since \( K \) is a prime ideal in \( F[x] \), it is generated by an irreducible polynomial \( p(x) \). So \( v(p(x)) = e \) if and only if \( (k, p) = 1 \) and

\[
v \left( \frac{f(x)}{g(x)} \right) = v(p(x)^N \frac{f_1(x)}{g_1(x)}) = v(p(x))^N = v(p(x))^N, \quad \gamma = v(p(x)) \leq 0, \ \gamma \in \Gamma.
\]

In both cases the value group consists of powers of a fixed element of \( \Gamma \).

Since \( \Gamma \) is cyclic, its ordering must be Archimedean and so \( \Gamma \) can be regarded as a subgroup of the reals with the usual ordering; hence we may assume that \( v \) is a real valuation. This proves Lemma 3.

If \( \mathcal{C} \) is discrete on \( Q \) but non-discrete on \( Q(x) \), then Lemma 3 shows that \( \mathcal{C} \) is induced on \( Q(x) \) by a real valuation. Then the closure of \( Q(x) \) in \( L \) is the topology \( \mathcal{C} \) has infinitely many continuous automorphisms. In fact, let us extend the mapping \( x \mapsto ax^2, \ a \neq 0, \ a \in Q \), to a continuous automorphism of \( Q(x) \) and then to a continuous automorphism of the closure of \( Q(x) \) in \( L \) (compare with (a)).

Hence \( \mathcal{C} \) is impossible and \( \mathcal{C}_L = \mathcal{C}Q \) is non-discrete.

It remains to consider the case

\[ C: K \text{ is a disconnected field of a finite characteristic} \ p \neq 0. \]

We will show that this case never arises. As before, let \( L \) be a fixed field of \( G(K) \). Obviously \( L \) is complete in our topology. There exists an element \( x \in L \) which is transcendental over the field \( Z_0 = Z/pZ \) since otherwise no locally bounded non-discrete field topology would exist in \( L \) (see [5], Theorem 6.1). An element \( x \in L \), transcendental over \( Z_0 \), can be chosen in such a way that the topology \( \mathcal{C}_0 = \mathcal{C}_0\mathcal{Z}_0(x) \) is non-discrete. In fact, let \( L = Z_0(b) \) the Steinitz decomposition of \( L \), where \( \mathcal{S} \neq \emptyset \) is the transcendental base of \( L \) and \( \mathcal{A} \) is the set of all algebraic elements of \( L \) over \( Z_0(b) \). Suppose at first that \( \mathcal{S} = (\beta_1, \beta_2, ..., \beta_m) \) is finite and that the topology \( \mathcal{C} \) is discrete on every \( Z_0(b) \) (\( j = 1, 2, ..., m \)). A discrete topology is induced by a trivial valuation \( v_0; \)

\[ v_0(a) = 1 \quad \text{for all} \ a \neq 0, \ v_0(0) = 0. \]

Let \( c \in Z_0(b) \). Clearly, \( v = v_0(b_1, b_2, ..., b_m) \), where \( r, s \) are polynomials over \( Z_0 \). Since

\[ v(d b_1^{N_1} ... b_m^{N_m}) = v_0(d) v_0(b_1)^{N_1} ... v_0(b_m)^{N_m} = v_0(b_1)^{N_1} ... v_0(b_m)^{N_m} = 1 \]

for \( d \in Z_0, d \neq 0, \)
we have \( v(b_1, b_2, \ldots, b_n) = 1 \) for all non-zero \( v \), and finally \( v(0) = 1 \) for all non-zero \( e \in \mathbb{Z}_p(\mathfrak{p}) \). This implies that \( \mathfrak{p} \) is discrete also on \( L \), which is impossible.

If \( \mathfrak{p} \) is infinite, then the discreteness of \( \mathfrak{p} \) on every \( \mathbb{Z}_p(L) \), \( b \in \mathfrak{p} \), implies the discreteness on \( \mathbb{Z}_p(\mathfrak{p}) \). This is again possible since the closed subfield \( \mathbb{Z}_p(\mathfrak{p}) \) of \( L \) would then have infinitely many (continuous) automorphisms induced by any automorphism of \( b \in \mathfrak{p} \).

Hence let \( x \in L \) be transcendental over \( \mathbb{Z}_p(\mathfrak{p}) \) such that \( \mathbb{Z}_p(\mathfrak{p}) = \mathbb{Z}_p(x) \) is a non-discrete topology. Lemma 3 implies that the topology \( \mathfrak{p}_x \) is induced by a real valuation on \( \mathbb{Z}_p(x) \). By the same lemma \( L \) must contain either the (closed) field \( \mathbb{Z}_p(x) \) of formal power series over \( \mathbb{Z}_p \) or the closure \( \mathbb{Z}_p[x] \) of \( \mathbb{Z}_p(x) \) in \( L \) with respect to the valuation \( \mathfrak{p}_x \). Let us note, however, that for every unit \( e \) from the valuation ring of our valuation the mapping \( x \to ex \) can be extended to a continuous automorphism of \( \mathbb{Z}_p(x) \) (or \( \mathbb{Z}_p(x) \)), which is impossible since \( G(\mathbb{Z}_p(x)) \) or \( G(\mathbb{Z}_p(x)) \) has to be finite by the assumption.

Case II. \( K \) is algebraically closed.

In [16] it was shown that if \( K \) is a locally bounded, complete topological field with torsion and a non-trivial \( G(K) \), then \( K \) is topologically isomorphic to the complex number field.

We will show that \( G(K) \) is always non-trivial. The topology \( \mathfrak{p} \) is induced in \( K \) by a non-trivial Krull valuation since otherwise \( G(K) \) would be infinite (see [16], Theorem 3). If \( K \) is of characteristic \( p \), then the previous remark implies the existence of an element \( x \in K \) which is transcendental over \( \mathbb{Z}_p \) and such that our valuation \( v \) is non-trivial on \( \mathbb{Z}_p(\mathfrak{p}) \). As in Case I, it can be shown that this is impossible. Hence \( K \) is of characteristic zero.

But then there is an involution in a group \( \text{Aut}(K) \) of all automorphisms of \( K \) (see [1], Theorem 1), i.e. an element \( g \neq 1 \), \( g^2 = 1 \). Let \( L \) be the fixed field of the group generated by \( g \). Obviously \( K = L(i) \). If \( L \) is complete in our topology, then the topology \( \mathfrak{p} \) is the product topology induced from \( L \) and \( g \) is continuous in it since \( g(a+b) = a \pm b; a, b \in L \). Indeed, if \( x_k \to x + iy \to x + iy = z \), then \( x_k \to x \) and \( y_k \to y \) and so \( g(x) \to g(x) \). It follows from a theorem of Matyús ([11], Theorem 5) that \( K \) must contain topologically either \( R \) or \( Q_p \) (for some prime \( p \)) because \( G \) is a non-discrete subfield of \( K \). In the first case \( K \cong \mathbb{C} \) since \( R \) and \( C \) are the only locally bounded extensions of \( R \) (see [11], Theorem 5). In the second case the degree \( [K : Q_p] \) must be finite since otherwise there would be a closed subfield \( M \) of \( K \) with infinite \( G(M) \) (compare B, Case I). But no finite extension of \( Q_p \) is algebraically closed. If \( \nu \) is discrete on \( Q_p \), we obtain a contradiction, just as in (b). Hence \( G(K) \) is always non-trivial. So the proof is achieved.