A characterization of locally compact fields of zero characteristic

by

W. Więsław (Wrocław)

0. In this note we shall give a characterization of locally compact fields of zero characteristic which seems to be new. Let us recall some definitions. A field topology $\mathcal{T}$ is said to be locally bounded if there exists a bounded neighbourhood A of zero, i.e. if for every neighbourhood U of zero there exists another one, V, such that $AV \subset U$. For any topological field F we write $G(F)$ for the group of all its continuous automorphisms. Moreover, $\mathcal{T}$ is called a full topology if the completion $\hat{F}$ of F in it is a field. It is well known (see [8], [10]) that the only full, locally bounded, non-trivial topologies on a field are topologies of type V, that is topologies induced by Krull-valuations (i.e. valuations taking values in linearly ordered groups instead of the reals).

I am indebted to W. Narkiewicz for valuable remarks concerning this paper.

1. The aim of this paper is to prove the following

THEOREM. *Let K be a non-discrete topological field. Then the following conditions are equivalent*:

(1) *K is a locally bounded, complete field and for every closed subfield F of K, $G(F)$ is finite.*

(2) *K is a locally compact field of characteristic zero.*

(3) *K is a finite extension of the reals $\boldsymbol{R}$ or of some p-adic number field Q_p with the usual locally compact topologies.*

Proof of the theorem. The equivalence (2) $\Leftrightarrow$ (3) is the classical theorem of Pontryagin–Kowalsky–van Dantzig (see [4], [7], [15]). (3) $\Rightarrow$ (1). Since every automorphism of $\boldsymbol{R}$ and Q_p is trivial, $G(K)$ is finite as a subgroup of the Galois group $G(K/\boldsymbol{R})$ or $G(K/Q_p)$. Moreover, K is complete in the locally bounded field topology induced by an absolute value $|a|$ or by a p-adic norm $|a|_p$.

It remains to show that (1) $\Rightarrow$ (3).

Case I. K is not algebraically closed.

A. Suppose K is connected. Every locally bounded, complete and connected topological field is topologically isomorphic to $\boldsymbol{R}$ or $\boldsymbol{C}$ (see [9], [16]). This gives $K \simeq \boldsymbol{R}$ topologically.

B. Suppose that K is disconnected and of characteristic zero. Then K is totally disconnected (see [2], Theorem 1). Let L be a fixed field of $G(K)$. From Lemma 2 of [16] it follows that L is closed and the topology $\mathfrak{T}$ of K is the product topology induced from L; moreover, L is complete. The completeness of K implies that $\mathfrak{T}$ is a full topology and the local boundedness of $\mathfrak{T}$ implies that $\mathfrak{T}$ is induced by a suitable Krull valuation (see [8], [10]).

Suppose at first that the topology

(a) $$\mathfrak{T}_1 = \mathfrak{T}|Q \quad \text{is non-discrete}.$$

Since K is totally disconnected, its topology is given by the open subgroups of Q, i.e. by open $\boldsymbol{Z}$-submodules in Q. But Q is the quotient field of the principal ideal domain $\boldsymbol{Z}$, and so we can apply the following.

Lemma 1 (see [3]). *Let A be a principal ideal domain and K the quotient field of A. If $\mathfrak{T}$ is a non-discrete field topology on K, then $(K, \mathfrak{T})$ is a topological field for which the open A-submodules form a fundamental system of neighbourhoods of zero if and only if $\mathfrak{T}$ is the supremum of a family of p-adic topologies (p is an irreducible element in A).*

Lemma 1 implies now that $\mathfrak{T}_1$ is the supremum of a family of p-adic topologies. But the supremum of a family of locally bounded topologies is locally bounded if and only if that family is finite [6]. Moreover, $\mathfrak{T}$ is a full topology, and so $\mathfrak{T}_1$ is also full. We claim that our family of topologies consists of a single element which is a p-adic topology. Indeed, let $\mathfrak{T}_1$ be the supremum of p_i-adic topologies for $i = 1, 2, \ldots, m$. The approximation theorem for valuations implies that the completion $\hat{Q}$ of Q in $\mathfrak{T}_1$ is a direct sum of fields $Q_{p_1}, \ldots, Q_{p_m}$:

$$\hat{Q} \simeq Q_{p_1} \oplus \ldots \oplus Q_{p_m}.$$

But $\mathfrak{T}_1$ is a full topology and so $\hat{Q}$ is a field; thus $m = 1$.

We are going to prove that L is an algebraic extension of Q_p. Suppose the contrary. Let t be transcendental over Q_p. Denote by L_1 the closure of $Q_p(t)$ in L. We define an automorphism of $Q_p(t)$ by the formula:

$$\varphi_\varepsilon\left(\frac{f(t)}{g(t)}\right) = \frac{f(\varepsilon t)}{g(\varepsilon t)},$$

where ε is a fixed unit in Q_p, and so $|\varepsilon|_p = 1$. Let us remark that the topology $\mathfrak{T}$ is induced in L by a non-Archimedean valuation. Indeed, since $Q_p \subset L$ topologically and $p^n \to 0$ in $\mathfrak{T}$ as $n \to \infty$, the set T of all topological nilpotents in L is non-void, whence open (see [16], Lemma 5).

Since $\mathfrak{T}$ is induced by a Krull valuation v, $(L \setminus T)^{-1}$ is bounded. From the Šafarevič Theorem [15] it follows now that v is a valuation taking values in an Archimedean ordered group, i.e. v can be assumed to be a real valuation. Let us denote this valuation by $|a|$. We have

$$|t| = |\varepsilon|_p |t| = |\varepsilon| |t| = |\varepsilon t| \quad \text{and so} \quad |\varphi_\varepsilon(a)| = |a|$$

for every $a \in Q_p(t)$. It follows that $\varphi_\varepsilon \in G(Q_p(t))$ since φ_ε is an isometry. Let us extend φ_ε to an automorphism $\bar{\varphi}_\varepsilon \in G(L_1)$ by putting, for every sequence $x_n \to x_0 \in L_1$, $x_n \in Q_p(t)$

$$\bar{\varphi}_\varepsilon(x_0) = \lim_{n\to\infty} \varphi_\varepsilon(x_n).$$

It is not difficult to see (by using the completeness of L_1) the independence of this definition from the choice of $\{x_n\}$. Moreover, one easily sees that $\bar{\varphi}_\varepsilon \in G(L_1)$. In this way we should have for every unit ε an automorphism $\bar{\varphi}_\varepsilon \in G(L_1)$ and distinct ε's would generate distinct automorphisms, whence $G(L_1)$ would be infinite, contrary to our assumptions.

Finally we will need the following

Lemma 2. *Let E be a separable algebraic extension of F. Moreover, if E and F are both complete and real-valued fields and the valuations agree on F, then E is a finite extension of F, i.e. $[E:F] < \infty$.*

Proof of the lemma. If $[E:F] = \infty$, then there would exist a sequence $a_1, a_2, \ldots \in E$ with

$$a_{j+1} \notin F(a_1, a_2, \ldots, a_j) \quad \text{for} \quad j = 1, 2, \ldots$$

The separability assumption implies that with a suitable $b_j \in E$ we have $F(b_j) = F(a_1, a_2, \ldots, a_j)$ and, in view of the obvious inequalities

$$[F(b_1):F] < [F(b_2):F] < \ldots,$$

we infer that E contains elements of an arbitrary large degree over E, against a theorem of Ostrowski (see [12], Theorem 3). (If E/F is algebraic, E and F being valued complete fields, then the degrees over F of elements of E are bounded.)

From Lemma 2 we have $[L:Q_p] < \infty$, whence K is a finite extension of the p-adic number field Q_p.

Now we consider the case

(b) $$\mathfrak{T}_1 = \mathfrak{T}|Q \quad \text{is discrete}.$$

Then there exists an $x \in L$, transcendental over Q, since otherwise the extension L/Q would be algebraic and, as the topology $\mathfrak{T}$ is discrete on Q, it would remain discrete on every finite (algebraic) extension of Q, and so on L, which gives a contradiction. If $\mathfrak{T}$ were discrete on $Q(x)$, then the closed subfield $Q(x)$ of L would have infinitely many (continuous) automorphisms of the form

$$x \to \frac{ax+b}{cx+d},$$

where $a, b, c, d \in Q$, $ad \neq bc$, which is a contradiction. Hence $\mathcal{T}$ is non-discrete on $Q(x)$. But the local boundedness of $\mathcal{T}$ implies that $\mathcal{T}$ is induced on $Q(x)$ by a real valuation. This results from the following lemma:

LEMMA 3. *Let $F(x)$ be a transcendental extension of a field F and $\mathcal{T}$ a non-discrete, locally bounded, full topology on $F(x)$, discrete on F. Then $\mathcal{T}$ is induced by one of the following valuations: $|a|_{p(x)}$, where $p(x) \in F[x]$ is an irreducible polynomial, or $|a|_\infty$.*

(We recall the definitions of these valuations. Let $\frac{f(x)}{g(x)} \in F(x)$ be any non-zero element. We put

$$\left|\frac{f(x)}{g(x)}\right|_\infty = e^{\deg g - \deg f} \quad \text{and} \quad \left|\frac{f(x)}{g(x)}\right|_{p(x)} = e^{-N},$$

where $\frac{f(x)}{g(x)} = p(x)^N \frac{f_1(x)}{g_1(x)}$ and $(p, f_1) = (p, g_1) = 1$.)

Proof of the lemma. As the topology $\mathcal{T}$ is full and locally bounded, it is induced on $F(x)$ by a Krull valuation $v\colon F(x) \to \Gamma$, where Γ is a multiplicative linearly ordered group with added 0. Denote by e the unit element of Γ. If $v(x) > e$, then

$$v(cx^k) = v(x)^k > v(x)^l = v(dx^l) \quad \text{for all } k > l$$

and $c, d \in F$, $cd \neq 0$, since $v(x) > e$ implies $v(x)^N = v(x^N) > e$ for every $N \in \mathbf{N}$. This valuation v is non-Archimedean since it extends a trivial valuation. As for $v(\alpha) \neq v(\beta)$, we have

$$v(\alpha+\beta) = \max\big(v(\alpha), v(\beta)\big),$$

it follows that

$$v\big(f(x)\big) = v(a_M x^M + \ldots + a_0) = v(x^M) = v(x)^{\deg f} \quad \text{for every } f(x) \in F[x].$$

Thus

$$v\left(\frac{f(x)}{g(x)}\right) = v(x)^{\deg f - \deg g}.$$

However, if $0 < v(x) \leqslant e$ ($0 \leqslant \gamma$ for every $\gamma \in \Gamma$ by definition), then $v\big(h(x)\big) \leqslant e$ for every $h(x) \in F[x]$. Let

$$R_v = \{f(x) \in F[x]\colon v\big(f(x)\big) < e\}.$$

Observe that we must have

$$v\left(\frac{f(x)}{g(x)}\right) = v\big(p(x)\big)^N, \quad \text{where} \quad \frac{f(x)}{g(x)} = p(x)^N \frac{f_1(x)}{g_1(x)}$$

and f_1, g_1 are prime to $p(x)$; $p(x) \in F[x]$ is a suitable irreducible polynomial. Indeed, since R_v is a prime ideal in $F[x]$, it is generated by an irreducible polynomial $p(x)$. So $v\big(p(x)\big) = e$ if and only if $(k, p) = 1$ and

$$v\left(\frac{f(x)}{g(x)}\right) = v\left(p(x)^N \frac{f(x)}{g(x)}\right) = v\big(p(x)\big)^N = \gamma^N, \quad \gamma = v\big(p(x)\big) \leqslant e,\ \gamma \in \Gamma.$$

In both cases the value group consists of powers of a fixed element of Γ.

Since Γ is cyclic, its ordering must be Archimedean and so Γ can be regarded as a subgroup of the reals with the usual ordering; hence we may assume that v is a real valuation. This proves Lemma 3.

If $\mathcal{T}$ is discrete on Q but non-discrete on $Q(x)$, then Lemma 3 shows that $\mathcal{T}$ is induced on $Q(x)$ by a real valuation. Then the closure of $Q(x)$ in L in the topology $\mathcal{T}$ has infinitely many continuous automorphisms. In fact, let us extend the mapping $x \to ax$, $a \neq 0$, $a \in Q$, to a continuous automorphism of $Q(x)$ and then to a continuous automorphism of the closure of $Q(x)$ in L (compare with (a)).

Hence (b) is impossible and $\mathcal{T}_1 = \mathcal{T}|Q$ is non-discrete.

It remains to consider the case

C. K is a disconnected field of a finite characteristic $p \neq 0$.

We will show that this case never arises. As before, let L be a fixed field of $G(K)$. Obviously L is complete in our topology. There exists an element $x \in L$ which is transcendental over the field $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$ since otherwise no locally bounded non-discrete field topology would exist in L (see [5], Theorem 6.1). An element $x \in L$, transcendental over $\mathbf{Z}_p$, can be choosen in such a way that the topology $\mathcal{T}_2 = \mathcal{T}|\mathbf{Z}_p(x)$ be non-discrete. In fact, let $L = \mathbf{Z}_p(\mathcal{B})(\mathcal{A})$ be the Steinitz decomposition of L, where $\mathcal{B} \neq \emptyset$ is the transcendental base of L and $\mathcal{A}$ is the set of all algebraic elements of L over $\mathbf{Z}_p(\mathcal{B})$. Suppose at first that $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ is finite and that the topology $\mathcal{T}$ is discrete on every $\mathbf{Z}_p(b_j)$ ($j = 1, 2, \ldots, m$). A discrete topology is induced by a trivial valuation v_0:

$$v_0(a) = 1 \quad \text{for all } a \neq 0,\ v_0(0) = 0.$$

Let $c \in \mathbf{Z}_p(\mathcal{B})$. Clearly, $c = \frac{r(b_1, b_2, \ldots, b_m)}{s(b_1, b_2, \ldots, b_m)}$, where r, s are polynomials over $\mathbf{Z}_p$. Since

$$v(db_1^{N_1} \ldots b_m^{N_m}) = v_0(d)\, v(b_1)^{N_1} \ldots v(b_m)^{N_m} = v_0(b_1)^{N_1} \ldots v_0(b_m)^{N_m} = 1 \quad \text{for} \quad d \in \mathbf{Z}_p,\ d \neq 0,$$

we have $v(r(b_1, b_2, \ldots, b_m)) = 1$ for all non-zero r, and finally $v(c) = 1$ for all non-zero $c \in Z_p(\mathfrak{B})$. This implies that $\mathcal{C}$ is discrete also on L, which is impossible.

If $\mathfrak{B}$ is infinite, then the discreteness of $\mathcal{C}$ on every $Z_p(b)$, $b \in \mathfrak{B}$, implies the discreteness on $Z_p(\mathfrak{B})$. This is again impossible since the closed subfield $Z_p(\mathfrak{B})$ of L would then have infinitely many (continuous) automorphisms induced by any permutation of elements $b \in \mathfrak{B}$.

Hence let $x \in L$ be transcendental over Z_p and such that $\mathcal{C}_2 = \mathcal{C}|Z_p(x)$ is a non-discrete topology. Lemma 3 implies that the topology $\mathcal{C}_2$ is induced by a real valuation on $Z_p(x)$. By the same lemma L must contain either the (closed) field $Z_p\langle x\rangle$ of formal power series over Z_p or the closure $Z_p\{x\}$ of $Z_p(x)$ in L with respect to the valuation $|a|_\infty$. Let us note, however, that for every unit ε from the valuation ring of our valuation the mapping $x \to \varepsilon x$ can be extended to a continuous automorphism of $Z_p\langle x\rangle$ (or $Z_p\{x\}$), which is impossible since $G(Z_p\langle x\rangle)$ or $G(Z_p\{x\})$ has to be finite by the assumption.

Case II. K is algebraically closed.

In [16] it was shown that if K is a locally bounded, complete topological field with torsion and a non-trivial $G(K)$, then K is topologically isomorphic to the complex number field.

We will show that $G(K)$ is always non-trivial. The topology $\mathcal{C}$ is induced in K by a non-trivial Krull valuation since otherwise $G(K)$ would be infinite (see [16], Theorem 3). If K is of characteristic p, then the previous remark implies the existence of an element $x \in K$ which is transcendental over Z_p and such that our valuation v is non-trivial on $Z_p(x)$. As in C, case I, it can be shown that this is impossible. Hence K is of characteristic zero.

But then there is an involution in a group $\mathrm{Aut}(K)$ of all automorphisms of K (see [1], Theorem 1), i.e. an element $g \neq 1$, $g^2 = 1$. Let L be the fixed field of the group generated by g. Obviously $K = L(i)$. If L is complete in our topology, then the topology $\mathcal{C}$ is the product topology induced from L and g is continuous in it since $g(a+ib) = a \pm ib$; $a, b \in L$. Indeed, if $z_a = x_a + iy_a \to x + iy = z$, then $x_a \to x$ and $y_a \to y$ and so $g(z_a) \to g(z)$. It follows from a theorem of Mutylin ([11], Theorem 3) that K must contain topologically either $\boldsymbol{R}$ or Q_p (for some prime p) because Q is a non-discrete subfield of K. In the first case $K \simeq \boldsymbol{C}$ since $\boldsymbol{R}$ and $\boldsymbol{C}$ are the only locally bounded extensions of $\boldsymbol{R}$ (see [11], Theorem 5). In the second case the degree $[K:Q_p]$ must be finite since otherwise there would be a closed subfield M of K with infinite $G(M)$ (compare B, case I). But no finite extension of Q_p is algebraically closed. If v is discrete on Q, we obtain a contradiction, just as in (b). Hence $G(K)$ is always non-trivial. So the proof is achieved.

References

[1] R. Baer, *Die Automorphismengruppe eines algebraisch abgeschlossenen Körpers der Charakteristik* 0, Math. Z. 117 (1970), pp. 7–17.

[2] — und H. Hasse, *Zusammenhang und Dimension topologischer Körperräume*, J. Reine Angew. Math. 167 (1932), pp. 40–45.

[3] E. Correl, *Topologies on quotient fields*, Duke Math. J. 35 (1968), pp. 175–178.

[4] M. Endo, *A note on locally compact division rings*, Coment. Math. Univ. St. Paul 14 (1966), pp. 57–64.

[5] J. O. Kiltinen, *Inductive ring topologies*, Trans. Amer. Math. Soc. 134 (1968), pp. 149–169.

[6] H. J. Kowalsky, *Beiträge zur topologischen Algebra*, Math. Nachr. 11 (1954), pp. 143–185.

[7] — *Zur topologischen Kennzeichnung von Körpern*, ibidem 9 (1953), pp. 261–268.

[8] — und H. J. Dürbaum, *Arithmetische Kennzeichnung von Körpertopologien*, J. Reine Angew. Math., 191 (1953), pp. 135–152.

[9] А. Ф. Мутылин, *Связные локально ограниченные поля. Полные не локально ограниченные поля*, Мат. Сб. 76 (118) (1968), pp. 454–472.

[10] — *Вполне простые топологические коммутативные кольца*, Мат. Заметки 5 (1969), pp. 161–171.

[11] — *Пример нетривиальной топологизации поля рациональных чисел. Полные локально ограниченные поля*, ИАН СССР, сер. матем. 30 (1966), pp. 873–890.

[12] A. Ostrowski, *Über sogenannte perfekte Körper*, J. Reine Angew. Math. 47 (1917), pp. 191–204.

[13] — *Über einige Lösungen der Funktionengleichung $\varphi(xy) = \varphi(x)\varphi(y)$*, Acta Math. 41 (1918), pp. 271–284.

[14] O. F. G. Schilling, *The theory of valuations*, Math. Surveys, No 4, 1950.

[15] И. Р. Шафаревич, *О нормируемости топологических полей*, ДАН СССР, 40 (1943), pp. 133–135.

[16] W. Więsław, *On some characterizations of the complex number field*, Colloq. Math. 24 (2) (1972), pp. 13–19.

[17] O. Zariski and P. Samuel, *Commutative algebra*, vol. I, II, Van Nostrand 1960.

MATHEMATICAL INSTITUTE OF THE WROCŁAW UNIVERSITY

Reçu par la Rédaction le 24. 6. 1971