On compact classes of models

by

L. Pacholski and **Jan Waszkiewicz** (Wrocław)

In the recent paper [7] Omarov proved that for every compact class K of similar relational structures the class of all direct products of elements of K reduced by a Fréchet ideal is compact. He had sooner proved [8] a similar result for some special class of operations on classes of relational structures. The result of [8] extends an earlier result of Makkai [5] who proved that the class of all direct products of elements of a compact class is compact, too, as well as a result of Kogalovskii [4] which says that every regular (1) product preserves compactness.

The main result of this paper says that for every ideal $\mathfrak{J}$ of subsets of a set I such that $2^I_{\mathfrak{J}}$ is ω_1-universal, $\mathfrak{J}$-reduced products preserve compactness of classes (Th. 1). Moreover, if $\mathfrak{J}$ is also (ω, ω)-regular, then for every class K, the class of all $\mathfrak{J}$-reduced products of elements of K is compact (Th. 2). We also give several examples of ideals $\mathfrak{J}$ such that $2^I_{\mathfrak{J}}$ is ω_1-universal.

It turns out that the assumptions of Th. 1 an Th. 2 are necessary. A proof of this fact and some connected results will be published in [11].

We shall denote by L an arbitrary countable first order language, and by L_B the language of Boolean algebras. The sentences of L will be denoted by $\alpha, \beta, \gamma, \ldots$ (possibly with subscripts), and sets of sentences by $\Gamma, \Delta, \ldots$ For L_B we will use the following notation: v — for variables, τ — for terms, σ — for formulas (all with convenient subscripts) and Σ for sets of formulas. Relational structures of the type L will be denoted by $A, A_i, \ldots$ and classes of such structures by K.

If $\{A_i \colon i \in I\}$ is a family of relational structures then the direct product of this family will be denoted by $\mathbf{P}_{i\in I} A_i$. If $\mathfrak{J}$ is an ideal over I, then $\mathbf{P}_{i\in I} A_i/\mathfrak{J}$ is the direct product of this family reduced by $\mathfrak{J}$, or simply the $\mathfrak{J}$-reduced product. The $\mathfrak{J}$-reduced power of a given A will be denoted by $A^I_{\mathfrak{J}}$.

2 denotes the two-element Boolean algebra. 2^I is the power of 2 as

(1) This notion was introduced by A. I. Malcev in [6].

well as the power set of I. So elements of 2^I will be identified with subsets of I. Elements of $2^I_{\mathfrak{J}}$ will be denoted by $X/\mathfrak{J}$ (for $X \subseteq I$), or simply by X if it does not lead to a misunderstanding.

For any term τ of L_B we define $\varepsilon\tau = \tau$ for $\varepsilon = 1$, and $\varepsilon\tau = -\tau$ for $\varepsilon = 0$. Similar convention will be made for formulas of any language.

The symbol 2^∞ will denote the set of all finite sequences of 0's and 1's.

Slightly modifying the notation of [2], we denote the set $\{i \in I: A_i \models \alpha\}$ by $K[A, \alpha]$ for $A = \mathbf{P}_{i\in I} A_i$. Sometimes the simpler notation $K[\alpha]$ will be used. A sequence $\zeta = \langle \sigma, \alpha_1, \ldots, \alpha_m \rangle$ is called an *acceptable sequence* if σ is a formula of L_B with at most $v_1, \ldots, v_m$ as free variables, and $\alpha_1, \ldots, \alpha_m$ are sentences of L. The acceptable sequence is called *partitioning* if

$$\vdash \neg(\alpha_j \wedge \alpha_k) \text{ for } j \neq k \quad \text{and} \quad \vdash \bigvee \{\alpha_i : i \leqslant m\}.$$

We will use the following Weinstein modification [13] (see also [3]) of the theorem of Feferman and Vanght:

F. V. THEOREM. *For every sentence γ of L there is an acceptable partitioning sequence $\langle \sigma, \alpha_1, \ldots, \alpha_m \rangle$ such that $\mathbf{P}_{i\in I} A_i/\mathfrak{J} \models \gamma$ if and only if $2^I_{\mathfrak{J}} \models \sigma[K[\alpha_1]/\mathfrak{J}, \ldots, K[\alpha_m]/\mathfrak{J}]$.*

Such a partitioning sequence will be called the F.V.-*reduction of* γ.

We say that a set of formulas Σ is *finitely satisfiable in* A if for every finite $\Sigma_0 \subseteq \Sigma$, $A \models \exists v_0, \ldots, v_m \bigwedge \Sigma_0$, where $v_0, \ldots, v_m$ are free variables in Σ_0. A sequence $\langle a_i: i<\omega \rangle$ of elements of A satisfies Σ if, for every $\sigma \in \Sigma$, $A \models \sigma[a_0, \ldots, a_n]$ holds ($v_0, \ldots, v_n$ are free variables in σ). A set Σ of formulas is *satisfiable in* A if there exists a sequence $\langle a_i: i \in \omega \rangle$ of elements of A which satisfies Σ.

If Σ is finitely satisfiable in A, then we say that σ is *consistent with* Σ if $\Sigma \cup \{\sigma\}$ is finitely satisfiable in A.

A relational structure A is ω_1-*universal* if for every finitely satisfiable set of formulas Σ of the language of A, or, equivalently, if for every countable relational structure B such that $B \equiv A$, B is isomorphic to an elementary submodel of A.

An ideal $\mathfrak{J}$ of subsets of a set I is (ω, ω)-*regular* if I is the union of a countable subfamily of $\mathfrak{J}$.

If $\mathfrak{J}$ is an ideal over I and $I_0 \subseteq I$, then by $\mathfrak{J} \upharpoonright I_0$ we denote the ideal $\mathfrak{J}_0$ over I_0 such that $X \in \mathfrak{J}_0$ if for some $Y \in \mathfrak{J}$ $X = Y \cap I_0$.

If K_1 and K_2 are classes of similar relational structures, then $K_1 \equiv K_2$ (K_1 and K_2 are elementarily equivalent), if for every $A \in K_1$ there exists $B \in K_2$ such that $A \equiv B$, and conversely.

If Σ is a set of sentences, then we say that Σ is *satisfiable in* K if there exists $A \in K$ such that $A \models \Sigma$. Σ is *finitely satisfiable in* K if every finite subset of Σ is satisfiable in K. A class K is called to be *compact* if every set Σ of sentences which is finitely satisfiable in K, is satisfiable in K. Of course, if K_1 and K_2 are compact and every sentence α satisfiable in K_1 is satisfiable in K_2 and conversely, then $K_1 \equiv K_2$.

If $\mathfrak{J}$ is an ideal over I and K is a class of relational structures, then we denote by $\mathfrak{J}(K)$ the class of all $\mathfrak{J}$-reduced products of elements of K. The symbol $\mathfrak{J}^*(K)$ will denote the class of all $\mathfrak{J}$-reduced powers of elements of K and $P(K)$ will denote the class of all direct products of elements of K.

Let B be a Boolean algebra. We denote by $\mathfrak{J}(B)$ the ideal of all elements of B such that $x \in \mathfrak{J}$ if and only if $x = y \cup z$, where y is atomic and z is atomless. Let $B_0 = B$ and $B_n = B_{n-1}/\mathfrak{J}(B_{n-1})$ and let h_n be the natural homomorphism of B_{n-1} onto B_n. Let $g_1 = h_1$ and $g_n = h_n \circ g_{n-1}$. We put $\mathfrak{J}_n(B) = g_n^{-1}(0)$.

With every Boolean algebra we can connect the triple $\langle a, b, c \rangle$ with $a \leqslant \omega$, $b \leqslant \omega$, $c \leqslant 1$, where $a = \sup\{n: B_n \text{ is non-trivial}\}$ and if $a < \omega$, then $b = \sup\{n: B_a \text{ has at least } n \text{ atoms}\}$ and $c = 0$ for B_a atomic, $c = 1$ if B_a contains an atomless element. For $a = \omega$ we put $b = c = 0$. Eršov proved that $\langle a, b, c \rangle$ depends on the elementary type of B only. Moreover, there exists a 1–1 correspondence between elementary types of Boolean algebras and triples $\langle a, b, c \rangle$ (for $a \leqslant \omega$, $b \leqslant \omega$, $c \leqslant 1$). (For proof and details see [1].) We say that a Boolean algebra B is of the *type* $\langle a, b, c \rangle$ (in symbols $\mathrm{Th}(B) = \langle a, b, c \rangle$) if $\langle a, b, c \rangle$ is connected with B in the described way.

THEOREM 1. *If K is a compact class of similar relational structures and $2^I_{\mathfrak{J}}$ is ω_1-universal, then $\mathfrak{J}(K)$ is compact.*

Let $\Gamma = \{\gamma_j: j < \omega\}$ be a given set of sentences of L. For every j we consider the partitioning acceptable sequence $\xi_j = \langle \sigma_j, \alpha_{j1}, \ldots, \alpha_{j,m_j} \rangle$, the F.V.-reduction of γ_j.

We put

$$A_0 = \{\alpha_{01}, \ldots, \alpha_{0m_0}\}, \quad A_n = \{\beta \wedge \alpha_{nl}: \beta \in A_{n-1}, l \leqslant m_n\} \text{ and } \Delta(\Gamma) = \bigcup_{n<\omega} A_n.$$

For $\Delta(\Gamma)$ we have:

(i) if $\alpha, \beta \in A_n$, then either $\vdash \alpha \leftrightarrow \beta$ or $\vdash \neg(\alpha \wedge \beta)$;

(ii) if $\alpha \in A_n$, $\beta \in A_{n+1}$, then $\vdash \beta \rightarrow \alpha$ or $\vdash \neg(\alpha \wedge \beta)$;

(iii) for every $\gamma_n \in \Gamma$ there exists the partitioning F.V.-reduction ζ of γ_n such that $\zeta = \langle \sigma, \alpha_1, \ldots, \alpha_k \rangle$ and for every $\beta \in A_n$ there is a $j \leqslant k$ such that $\vdash \beta \leftrightarrow \alpha_j$;

(iv) $\Delta(\Gamma)$ is closed with respect to conjunctions;

(v) $\Delta(\Gamma)$ can be ordered in the type ω in such a way that from $\vdash \alpha_i \rightarrow \alpha_j$ follows $i \leqslant j$.

In the sequel, whenever consider a numbering of elements of $\Delta(\Gamma)$, we assume that it fulfils (v).

By $\Sigma(\Gamma)$ we will denote the smallest set of formulas of L_B such that:

(j) if $\langle\sigma, a_{j1}, \ldots, a_{jk}\rangle$ is the F.V.-reduction of some $\gamma \in \Gamma$, then $\sigma[v_{j1}/v_1, \ldots, v_{jk}/v_k] \in \Sigma(\Gamma)$.

(jj) if $K \models \neg\bigwedge\{\varepsilon_j a_j\colon j \leqslant k\}$, then $\bigcap\{\varepsilon_j v_j\colon j \leqslant k\} = 0 \in \Sigma(\Gamma)$.

Lemma 1. *If $2^I_{\mathfrak{J}}$ is ω_1-universal and Γ is finitely satisfiable in $\mathfrak{J}(K)$, then $\Sigma(\Gamma)$ is satisfiable in $2^I_{\mathfrak{J}}$.*

Proof. By the ω_1-universality of $2^I_{\mathfrak{J}}$ it suffices to prove the assertion for any finite $\Sigma \subseteq \Sigma(\Gamma)$. Such Σ is the sum of finite sets Σ_1 and Σ_2 of formulas of type (j) and (jj) respectively. By a suitable extension of Σ_1 we can assume that every variable from Σ_2 is free in Σ_1. But for every $\sigma_j \in \Sigma_1$ there exists the F.V.-reduction $\langle\sigma_j, a_{j1}, \ldots, a_{jk}\rangle$ of some $\gamma \in \Gamma$. Since Γ is finitely satisfiable in $\mathfrak{J}(K)$, so there is in $\mathfrak{J}(K)$ a model A of $\bigwedge\{\gamma_j\colon \sigma_j \in \Sigma_1\}$. So we take $K[A, a_i]/\mathfrak{J}$ as the elements of $2^I_{\mathfrak{J}}$ fulfilling Σ_1. The satisfaction of Σ_2 is obvious.

Lemma 2. *For every sequence $\langle b_n\colon n<\omega\rangle$ of elements of $2^I_{\mathfrak{J}}$ there is a sequence $\langle B_n\colon n<\omega\rangle$ of elements of $2^I_{\mathfrak{J}}$ such that $b_n = B_n/\mathfrak{J}$, and if $\bigcap\{\varepsilon_k b_k\colon k<n\} = 0$, then*

$$\bigcap\{\varepsilon_k B_k\colon k<n\} = \emptyset \quad \textit{for every } \langle\varepsilon_0, \ldots, \varepsilon_{n-1}\rangle \in 2^{\infty}.$$

Proof. A more general statement was proved in [9] (as a part of the proof of Theorem 1). We adapt this argumentation to our case. We will determine B_n by induction. Let $B_0, \ldots, B_{n-1}$ be determined. We put $b(\varepsilon) = \emptyset$ if $2^I_{\mathfrak{J}} \models \bigcap_{k<n} \varepsilon_k b_k = 0$ and $b(\varepsilon) = I$ otherwise.

Let us choose an arbitrary $B'_n \in b_n/\mathfrak{J}$ and define

$$B_n(\varepsilon) = \varepsilon_0 B_0 \cap \ldots \cap \varepsilon_{n-1}B_{n-1} \cap \big(B'_n \cap b(\varepsilon_0, \ldots, \varepsilon_{n-1}, 1)\big) \cup \big(I \setminus b(\varepsilon_0, \ldots, \varepsilon_{n-1}, 0)\big).$$

We put $B_n = \bigcup\{B_n(\varepsilon)\colon \varepsilon \in 2^n\}$.

The verification of the fact that B_n fulfil the assertion of the lemma is a matter of an easy computation.

Proof of Theorem 1. Let Γ be a set of sentences of L finitely satisfiable in $\mathfrak{J}(K)$. Let $\langle B_n\colon n<\omega\rangle$ be a sequence of subsets of I fulfilling Lemmas 1 and 2. For every $i \in I$ we define

$$\Phi_i = \{a_n \in \Delta(\Gamma)\colon i \in B_n\}.$$

Let us suppose that Φ_i has no model in K. Then some finite subset Φ'_i of Φ_i has no model in K, so $K \models \neg\bigwedge\Phi'_i$. Let $n_0 = \max\{n\colon a_n \in \Phi'_i\}$. Then $K \models \neg\bigwedge\{\varepsilon_j a_j\colon j \leqslant n_0\}$ for every $\varepsilon \in 2^{n_0}$ such that $\varepsilon_j = 1$ for $a_j \in \Phi'_i$.

Then, by the definition of $\Sigma(\Gamma)$, $\bigcap\{\varepsilon_j v_j\colon j \leqslant n_0\} = 0$ (for ε as above) is an element of $\Sigma(\Gamma)$.

By Lemma 1 and 2 we have:

$$\emptyset = \bigcup\{\bigcap_{j \leqslant n_0} \varepsilon_j B_j\colon \varepsilon \in 2^{n_0} \text{ and } \varepsilon_j = 1 \text{ for } a_j \in \Phi'_i\} = \bigcap\{B_j\colon a_j \in \Phi'_i\},$$

what contradicts the fact that i belongs to the right-hand side (by the definition of Φ'_i).

So every Φ_i has a model $A_i \in K$. Consider the structure $A = \mathbf{P}_{i\in I} A_i$. We shall prove that $B_n = K[A, a_n]$.

It is obvious that if $i \in B_n$, then $a_n \in \Phi_i$ and $A_i \models a_n$. So $B_n \subseteq K[A, a_n]$.

Now let $i \notin B_n$. By the properties of $\Delta(\Gamma)$ there is a partitioning sequence of elements of $\Delta(\Gamma)$, say $\langle a_{i,1}, \ldots, a_{i,k}\rangle$, such that a_n is an element of this sequence. By an argument analogous to those used in the proof of the consistency of Φ_i we can obtain that $B_{i_1}, \ldots, B_{i_k}$ is form a partition of I. So i is an element of some B_m such that $B_m \cap B_n = \emptyset$, and $A_i \models a_m$. Since $\vdash a_m \to \neg a_n$, we have that $i \notin K[A, a_n]$ whence follows $B_n = K[A, a_n]$.

Now we can complete the proof of the theorem, by showing that $\mathbf{P}_{i\in I} A_i/\mathfrak{J} \models \gamma$ for every $\gamma \in \Gamma$. By the definition of $\Delta(\Gamma)$, for some F.V.-reduction $\zeta = \langle\sigma, \beta_1, \ldots, \beta_k\rangle$ of γ we have $\sigma \in \Sigma(\Gamma)$ and $\beta_i = a_{n_i}$. By the definition of $\Sigma(\Gamma)$, Lemma 1 and Lemma 2, $2^I_{\mathfrak{J}} \models \sigma[B_{n_1}/\mathfrak{J}, \ldots, B_{n_k}/\mathfrak{J}]$. Since $B_{n_i} = K[A, a_{n_i}]$, so by the F.V.-theorem $\mathbf{P}_{i\in I} A_i/\mathfrak{J} \models \gamma$.

Theorem 2. *If $2^I_{\mathfrak{J}}$ is ω_1-universal and $\mathfrak{J}$ is (ω, ω)-regular, then for every class K of similar relational structures $\mathfrak{J}(K)$ is compact.*

Proof. Let Γ be finitely satisfiable in $\mathfrak{J}(K)$ and let $I_0, \ldots, I_n, \ldots$ be a partition of I such that $I_n \in \mathfrak{J}$ for every n. Let $\Delta(\Gamma)$ be ordered in such a way that property (v) of $\Delta(\Gamma)$ holds, and let $\langle B_n\colon n<\omega\rangle$ be a sequence of subsets of I determined by Lemmas 1 and 2.

We define $\Phi_i = \{a_n\colon i \in B_n\}$. As previously we can show that every finite subset of Φ_i has a model. But now it does not imply that Φ has a model in K.

We put $m(i) = k$ if $i \in I_k$ and $k(i) = \max_{j<m(i)}\{j\colon i \in B_j\}$. Finally let $A_i \models a_{k(i)}$. For $A = \mathbf{P}_{i\in I} A_i$ we will write K_i for $K[A, a_i]$.

As soon as we show that $K_i/\mathfrak{J} = B_i/\mathfrak{J}$, the proof of theorem will be completed in the identical way as the proof of Theorem 1.

Let $i \in K_j - B_j$. Then $A_i \models a_j \wedge a_{k(i)}$. By the properties of $\Delta(\Gamma)$ we have two possibilities:

a) $\vdash a_{k(i)} \to a_j$, and then $B_{k(i)} \subseteq B_j$, and $i \in B_j$ (contradiction);

b) $\vdash a_j \to a_{k(i)}$, then $j \geqslant m(i)$ and $i \in \bigcup_{s<j} I_s$. So

$$K_j - B_j \subseteq \bigcup\{I_s\colon s<j\} \in \mathfrak{J}.$$

Let $i \in B_j - K_j$. Then $A_i \models \alpha_j \wedge \neg\alpha_{k(i)}$. If $j \leqslant m(i)$, then $j \leqslant k(i)$ and, since $i \in B_j$, $B_j \supseteq B_{k(i)}$. So $\vdash \alpha_{k(i)} \to \alpha_j$ (contradiction). If $j > m(i)$, then $i \in B_j - K_j \subseteq \bigcup \{I_s\colon s < j\} \in \mathfrak{I}$.

This finishes the proof of the equality $B_j/\mathfrak{I} = K_j/\mathfrak{I}$.

As a consequence of our Theorem 2 and Theorem 2 of [9] we obtain

COROLLARY 1. *For every Boolean algebra B there is an ideal I of subsets of a set I such that $2^I_{\mathfrak{I}} \equiv B$ and for every class K of similar relational structures $\mathfrak{I}(K)$ is compact.*

PROPOSITION 1. *Every atomless Boolean algebra is ω_1-universal.*

Proof. Let Σ be a finitely satisfiable set of formulas of L_B. First of all we shall prove that one can assume that

$$\Sigma_0 = \{\varepsilon_0 v_0 \cap \ldots \cap \varepsilon_n v_n = 0\colon \langle \varepsilon_0, \ldots, \varepsilon_n\rangle \in S_1\} \cup \{\varepsilon_0 v_0 \cap \ldots \cap \varepsilon_n v_n \neq 0\colon \langle \varepsilon_0, \ldots, \varepsilon_n\rangle \in 2^{\infty} - S_1\} \tag{1}$$

for some $S_1 \subseteq 2^{\infty}$.

By a theorem of Skolem [12] we may assume that every element of Σ is in a conjunctive normal form. Also we may assume that every element of Σ is of the form

$$\bigwedge \{\bigvee \{\eta_{ij}(\tau_{ij} = 0)\colon i \leqslant n\}\colon j \leqslant n\}, \tag{2}$$

where each τ_{ij} is a meet of variables and their complements and every η_{ij} is 0 or 1. In fact, every atomic formula and the negation of every atomic formula in the theory of Boolean algebras is equivalent to a Boolean combination of formulas of the form $\tau = 0$ and $\tau \neq 0$. Moreover, if $\tau = \bigcup \{\tau_j\colon j \leqslant n\}$, then $\tau = 0$ if and only if $\bigwedge \{\tau_j = 0\colon j \leqslant n\}$ and $\tau \neq 0$ if and only if $\bigvee \{\tau_j \neq 0\colon j \leqslant n\}$. Of course, every formula in Σ which is of the form (2) can be replaced in Σ by a finite set of formulas

$$\{\bigvee \{\eta_{ij}(\tau_{ij} = 0)\colon i \leqslant n\}\colon j < m\}.$$

Now let

$$\Sigma = \{\bigvee \{\eta_{ij}(\tau_{ij} = 0)\colon i \leqslant i_j\}\colon j < \omega\}$$

be the set of formulas finitely satisfiable in B, and let $X = \prod_{n<\omega} \{i\colon i \leqslant i_n\}$ ($\prod$ denotes the Cartesian product of topological spaces with the product topology). Let for $n < \omega$

$$X_n = \{x \in X\colon B \models \exists v_0 \ldots \exists v_m \bigwedge \{\varphi_{x_j,j}\colon j < n\}\},$$

where free variables of $\bigwedge \{\varphi_{x_j,j}\colon j < n\}$ are among $v_0, \ldots, v_m$. Since Σ is finitely satisfiable, X_n is a non-empty closed set and since X is compact, there is an element $x \in \bigcap_{n<\omega} X_n$. It is a matter of simple calculation to check that for $x \in \bigcap_{n<\omega} X_n$,

$$\Sigma_1 = \{\eta_j(\tau_{x_j,j} = 0)\colon \eta_j = \eta_{x_j,j},\ j < \omega\}$$

is finitely satisfiable in B. If, for some τ, $\tau = 0$ and $\tau \neq 0$ do not occur in Σ we may add $\tau = 0$ or $\tau \neq 0$ to Σ without a loss of finite satisfiability. (If e.g. $\tau = 0$ is not consistent with Σ then clearly $\tau \neq 0$ is), what finishes the proof of the fact that we can assume that Σ is of the form (1). The proof of the fact, that any finitely satisfiable in B set of formulas of the form (1) is satisfiable in B, is easy.

PROPOSITION 2. *If a Boolean algebra $2^I_{\mathfrak{I}}$ is atomic, then it is ω_1-universal.*

Proof. We assume that $2^I_{\mathfrak{I}}$ is infinite. Let Σ be a finitely satisfiable set of formulas of L_B. Using arguments similar to that in the proof of Proposition 1 we may assume that every formula in Σ is of the form $\alpha_i(\tau_i)$ or $\beta_i(\tau_i)$, where $\alpha_i(x)$ means that x has exactly i atoms and $\beta_i(x)$ means that x has at least i atoms (cf. [10]). Let $\gamma(x)$ be an infinite formula saying that x has infinitely many atoms. We add $\gamma(\tau)$ to Σ if for every $i < \omega$ $\beta_i(\tau)$ is consistent with Σ. If for some $i < \omega$ $\beta_i(\tau)$ is not consistent with Σ, we add $\neg\beta_i(\tau)$ to Σ. In such a way we can obtain a set Σ_1 of formulas such that (i) for every α in Σ_1 there is a formula β in Σ_1 such that $\beta \vdash \beta \to \alpha$, (ii) Σ_1 finitely satisfiable, and (iii) for every term τ, $\gamma(\tau) \in \Sigma_1$ or for some $i < \omega$ $\alpha_i(\tau) \in \Sigma_1$.

Now we shall define a sequence $\langle x_i\colon i < \omega\rangle$ satisfying Σ_1 in $2^I_{\mathfrak{I}}$. For $i = 0$ we have (i) for some $i < \omega$, $\alpha_i(v_0) \in \Sigma_1$ and $\gamma(-v_0) \in \Sigma_1$, or (ii) for some $i < \omega$, $\alpha_i(-v_0) \in \Sigma_1$ and $\gamma(v_0) \in \Sigma$, or (iii) $\gamma(-v_0) \in \Sigma$ and $\gamma(v_0) \in \Sigma$. In the last case we select a sequence $\{Y_i\colon i \in \omega\}$ such that $Y_i/\mathfrak{I}$ is an atom in $2^I_{\mathfrak{I}}$ and $Y_i \cap Y_j = \emptyset$ for $i \neq j$. Of course, $\bigcup_{i<\omega} Y_i/\mathfrak{I}$ has infinitely many atoms. We put $X_0 = \bigcup_{i<\omega} Y_{2i}$. If $X_0, \ldots, X_{n-1}$ are defined, we define X_n in a similar way restricting the computation to the sets $\varepsilon_0 X_0 \cap \ldots \cap \varepsilon_{n-1} X_{n-1}$.

COROLLARY 2. *If $\mathrm{Th}(2^I_{\mathfrak{I}}) = \langle 0, m, n\rangle$ for some $m \leqslant \omega$, $n < 2$, then $2^I_{\mathfrak{I}}$ is ω_1-universal.*

Proof. We devide I into two sets I_0, I_1 such that $2^{I_0}_{\mathfrak{I}_0}$ is atomless and $2^{I_1}_{\mathfrak{I}_1}$ is atomic, where $\mathfrak{I}_0 = \mathfrak{I} \upharpoonright I_0$, $\mathfrak{I}_1 = \mathfrak{I} \upharpoonright I_1$.

An example which shows that the assumption $\mathrm{Th}(2^I_{\mathfrak{I}}) = \langle 0, m, n\rangle$ is necessary will be given in [11].

COROLLARY 3. *For every ideal $\mathfrak{I}$ if $\mathrm{Th}(2^I_{\mathfrak{I}}) = \langle 0, m, n\rangle$ for $m \leqslant \omega$, $n < 2$, then $\mathfrak{I}$ preserves compactness. Moreover, if $\mathfrak{I}$ is (ω, ω)-regular, then for every class K, $\mathfrak{I}(K)$ is compact.*

As an immediate consequence of Corollary 2 and Theorem 4 of [10] we obtain

COROLLARY 4. *If* $\mathrm{Th}(2^I_{\mathfrak{I}}) = \langle 0, n, m\rangle$ *and* $\mathfrak{I}$ *is* (ω, ω)*-regular, then for every relational structure* A, *the reduced power* $A^I_{\mathfrak{I}}$ *is* ω_1*-universal.*

THEOREM 3. *If a Boolean algebra* $2^I_{\mathfrak{I}}$ *is atomless, then for every compact class* K *of relational structures* $\mathfrak{I}(K) \equiv P\big(\mathfrak{I}^*(K)\big)$.

Proof. The proof will be devided into several steps.

a) Every sentence which is true in some structure belonging to $\mathfrak{I}(K)$, is true in some structure belonging to $P\big(\mathfrak{I}^*(K)\big)$.

Let $A = \mathbf{P}_{i\in I} A_i/\mathfrak{I}$ and $A \models \alpha$. By the F.V.-theorem we have

$$2^I_{\mathfrak{I}} \models \sigma(K[A, \theta_1], \ldots, K[A, \theta_n])$$

for some partitioning acceptable sequence $\langle \sigma, \theta_1, \ldots, \theta_n\rangle$. Let

$$S_1 = \{k \leqslant n\colon\ K[A, \theta_k] \notin \mathfrak{I}\}$$

and

$$S_2 = \{k \leqslant n\colon\ K[A, \theta_k] \in \mathfrak{I}\}\,.$$

For $k \in S_1$ we select a structure $A'_k \in \{A_i\colon\ i \in k[A, \theta_k]\}$. Let

$$I_k = \begin{cases} I & \text{if} \quad k \in S_1\,, \\ \emptyset & \text{if} \quad k \in S_2\,. \end{cases}$$

It is easy to check that

$$(2^I_{\mathfrak{I}})^{S_1} \models \sigma(I_1, \ldots, I_k)\,,$$

hence by the F.V.-theorem we have $\mathbf{P}_{k\in S_1}\big((A'_k)^I_{\mathfrak{I}}\big) \models \alpha$ and, of course, $\mathbf{P}_{k\in S_1}\big((A'_k)^I_{\mathfrak{I}}\big)$ belongs to $P\big(\mathfrak{I}^*(K)\big)$.

b) By a theorem of Omarov [8] the class $\mathfrak{I}^*(K)$ is compact and by a theorem of Makkai [5] $P\big(\mathfrak{I}^*(K)\big)$ is compact, too. Hence, by a) for every relational structure A in $\mathfrak{I}(K)$ there is a relational structure B in $P\big(\mathfrak{I}^*(K)\big)$ such that $A \equiv B$.

c) Now we shall prove that every relational structure from $P\big(\mathfrak{I}^*(K)\big)$ is elementarily equivalent to some structure in $\mathfrak{I}(K)$. By Corollary 2 it is enough to show that for every sentence α which is true in some element of $P\big(\mathfrak{I}^*(K)\big)$ there is a relational structure in $\mathfrak{I}(K)$ in which α is true.

Let $\mathbf{P}_{i\in J}\big((A_i)^I_{\mathfrak{I}}\big) \models \alpha$. By Theorem 6.6 of Feferman and Vaught (see [2], p. 83), for some finite $J_0 \subseteq J$ we have $\mathbf{P}_{i\in J_0}\big((A_i)^I_{\mathfrak{I}}\big) \models \alpha$. Let $\langle I_i\colon\ i \in J_0\rangle$ be a partition of I into sets outside I. Then $(A_i)^I_{\mathfrak{I}} \equiv (A_i)_{\mathfrak{I}_j}$, where $\mathfrak{I}_j = \mathfrak{I} \upharpoonright I_j$. Consequently, if $B = \mathbf{P}_{i\in J_0}\big((A_i)^{I_i}_{\mathfrak{I}_i}\big)$, then $B \models \alpha$. On the other hand, B is a structure from $\mathfrak{I}(K)$.

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