Let I be a two-sided of A which is non-zero and contained in R_k. Then $I \cap L \neq 0$ for some $L \in \Gamma_k$. Then, by 0-minimality of L, $L \subseteq I$. Also, for every $L' \in \Gamma_k$, $L' = L \cdot L' = \langle LL' \rangle \subseteq \langle LA \rangle \subseteq I$. This implies that $R_k \subseteq I$, whence $R_k = I$. Therefore R_k is a 0-minimal two-sided ideal of A, and is 0-simple.

As an ideal of A, R_k is the union of a family of 0-minimal left (right) ideals of A. Also by Proposition 1.6 of [2], since R_k is a 0-minimal two-sided ideal of A. The 0-minimal left (right) ideals of R_k coincide with the 0-minimal left (right) ideals of A contained in R_k. It follows that R_k is the union of its 0-minimal left ideals and the union of its 0-minimal right ideals. Thus R_k is completely 0-simple, which proves that (ii) implies (iii).

Let now A be the union of a family $(R_k)_{k \in K}$ of two-sided ideals of A which are completely 0-simple semirings. To prove that A is completely 0-simple, we shall show that no two semirings of the family $(R_k)_{k \in K}$ can possibly be distinct.

Let $i, j \in K$ be such that $R_i \neq R_j$. Then $R_i \neq 0$, $R_j \neq 0$ (since R_i and R_j are 0-simple), and also $R_i \cap R_j = 0$ so that, in particular, $R_i R_j = 0$. Let a, b be any two non-zero elements of A in R_i and R_j respectively. Since R_i is completely 0-simple, by (iv) of Theorem 2.2 of [2], there exists $a' \in R_i$ such that $aa' \neq 0$. Then $(a+b)a' = aa' + ba' = aa' \neq 0$ which implies that $a+b \in R_i$. Similarly $a+b \in R_j$. Therefore $a+b = 0$ which is impossible since $(a+b)a' \neq 0$. Thus there exists no $i, j \in K$ such that $R_i \neq R_j$ which completes the proof.

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A note on monics and epics in varieties of categories

by

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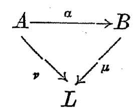
In memory of Professor Hanna Neumann

§ 1. **Introduction.** This work is motivated by the question, given any variety $\mathfrak{B}$ (see below) of a category $\mathfrak{C}$, are the monomorphisms, epimorphisms in $\mathfrak{B}$, same as those in $\mathfrak{C}$, to answer this question we choose certain axioms on a category, so as to extend our study in any category of multioperator groups [3]; and answer the question, by showing that in such categories the monomorphisms and normal epimorphisms in any variety is same, as in the original category. These categories have also been studied by Suliński [7], Szász and Wiegandt [8], for various other purposes. In the sequel, we are using the results of [6], [4] and [5] whenever necessary, without reference.

§ 2. **Axioms and Main results.** Let $\mathfrak{C}$ be a category equipped with the following axioms:

C_1: *$\mathfrak{C}$ has a null object.*

C_2: *Every morphism α in $\mathfrak{C}$, admits a factorization as in the diagram*



i.e. $\alpha = \nu\mu$, where ν is a normal epimorphism and μ is a monomorphism.

C_3: *$\mathfrak{C}$ has product and coproduct for any arbitrary family of objects.*

C_4: *The sub-objects and normal factor objects of any object form a set.*

C_5: *If α is a monomorphism and β is a normal epimorphism such that $\alpha\beta$ admits image $\alpha\beta = \nu\mu$, then α normal implies μ is normal.*

DEFINITION 2.1. *A variety $\mathfrak{B}$* is a full subcategory of $\mathfrak{C}$, satisfying the following axioms:

B_1: *If μ: $A \to B$ be a monomorphism in $\mathfrak{C}$ and $B \in \mathfrak{B} \Rightarrow A \in \mathfrak{B}$.*

B_2: *If ν: $A \to B$ be a normal epimorphism in $\mathfrak{C}$ and $A \in \mathfrak{B} \Rightarrow B \in \mathfrak{B}$.*

B_3: *If $(A_i)_{i \in I}$ be a family of objects of $\mathfrak{B}$, then their product $\prod_i A_i \in \mathfrak{B}$.*

We recall the concept of a variety functor [4], [5].

Definition 2.2. A functor $V\colon \mathfrak{C}\to\mathfrak{C}$ is called a *variety functor* if it satisfies

V_1: V is a normal subfunctor of $I_{\mathfrak{C}}$.

V_2: V preserves normal epimorphisms.

V being a subfunctor of the identity functor, naturally preserves monomorphisms.

As exhibited in [5], the varieties of the category $\mathfrak{C}$ are in biunique correspondence with the variety functors, and the variety functor $V_{\mathfrak{B}}$ associated with a variety $\mathfrak{B}$, being a normal subfunctor defines a quotient functor $U_{\mathfrak{B}}\colon \mathfrak{C}\to\mathfrak{B}$, cf. [1], [2], and indeed we have immediately.

Proposition A. *The quotient functor associated with a variety $\mathfrak{B}$ is a coadjoint to the inclusion functor $I\colon \mathfrak{B}\to\mathfrak{C}$.*

Proof. We notice that for any object A, we have a maximal normal factor object $\big(p_A, U_{\mathfrak{B}}(A)\big)\in\mathfrak{B}$. Let $f\colon A\to B$ be a morphism with $B\in\mathfrak{B}$. Then if f admits image (ν, L, μ), then $L\in\mathfrak{B}$ so $(\nu, L)\leqslant\big(p_A, U_{\mathfrak{B}}(A)\big)$. Thus there exists a unique θ, such that $p_A\theta=\nu$. Then $\theta\mu\colon U_{\mathfrak{B}}(A)\to B$ is the unique map so that $p_A\theta\mu=f$. This proves our assertion.

Corollary. *If an object C is $\mathfrak{C}$-projective, then $U_\beta(C)$ is $\mathfrak{B}$-projective.*

The proposition though simple leads to the information:

Proposition B. *The monomorphisms and normal epimorphisms in a variety $\mathfrak{B}$ of $\mathfrak{C}$, do coincide with monomorphisms and normal epimorphisms in $\mathfrak{C}$, respectively.*

Proof. The monomorphisms and normal epimorphisms of $\mathfrak{C}$, that are in $\mathfrak{B}$, are indeed monomorphisms and normal epimorphisms in $\mathfrak{B}$. Conversely if μ is a monomorphism in $\mathfrak{B}$; let α,β be two maps $C\to A$, in $\mathfrak{C}$, such that $\alpha\mu=\beta\mu$. Then we have the diagram

$$\begin{array}{ccccc} C & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & A & \xrightarrow{\mu} & B \\ \downarrow{\scriptstyle p_C} & & & & \\ U_\beta(C) & & & & \end{array}$$

where p_C is the canonical normal epimorphism; obviously since $A\in\mathfrak{B}$, $\exists\,\alpha',\beta'\colon U(C)\to A$, such that $p_C\alpha'=\alpha$ and $p_C\beta'=\beta$.

Thus $\alpha\mu=\beta\mu\Rightarrow\alpha'\mu=\beta'\mu$ i.e. $\alpha'=\beta'$ so $\alpha=\beta$.

Next let $\nu\colon A\to B$ be a normal epimorphism in $\mathfrak{B}\subset\mathfrak{C}$, and (K,μ) $=$ kernel ν in $\mathfrak{C}$. Immediately $(K,\mu)\in\mathfrak{B}$, so the sequence

$$K\overset{\mu}{\to}B\overset{\nu}{\to}C$$

is exact in $\mathfrak{B}$. Next suppose that the morphism μ admit a cokernel (ν^*, C^*) in $\mathfrak{C}$. Then we have a θ, such that $\nu^*\theta=\nu$. Also there exists a ϱ such that $\nu\varrho=\nu^*p_C$. Since (θ, B) is a normal factor object of C, with $B\in\mathfrak{B}$ and $\big(p_C, U_{\mathfrak{B}}(C)\big)$ is the maximal such one, ϱ is an equivalence; so $\nu=\nu^*p_C\varrho^{-1}$, hence a normal epimorphism, being composition of normal epimorphisms (in view of axiom C_2).

Proposition C. (i) *The terminal object of $\mathfrak{C}$, if any, must belong to $\mathfrak{B}$,* (ii) *If A is an initial object of $\mathfrak{C}$, then $U_{\mathfrak{B}}(A)$ is initial in $\mathfrak{B}$.*

Proof. Since A is terminal, we have a unique $\theta\colon U_{\mathfrak{B}}(A)\to A$, such that $p_A\theta=1$ i.e. θ is a retraction and hence a normal epimorphism, i.e. $A\in\mathfrak{B}$. (cf. [4], proof of Proposition 3.1.10).

We notice that for any object $C\in\mathfrak{B}$, $\exists\,\theta\colon A\to C\in\mathfrak{C}$. If θ admits image (ν, L, μ), then $L\in\mathfrak{B}$, thus $(\nu, L)\leqslant\big(p_A, U_{\mathfrak{B}}(A)\big)$; hence $\exists\,\theta'\colon U_{\mathfrak{B}}(A)\to C$, such that $p_A\theta'=\theta$; that this θ' is unique satisfying $p_A\theta'=\theta$ is easy to check.

Added in proof. After the acceptance of this paper, it came to the author's attention that Livšic, Calenko and Sulgeifer, also obtained Proposition B in a different way. However, our proof is more general in the sense that this holds for epicoreflective subcategories of any category, instead of varieties in categories with present set of axioms.

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