Semisimple A-semigroups and semirings

by

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In an earlier paper [2], we investigated the class of completely 0-simple semirings, i.e. 0-simple semirings which are unions of their 0-minimal left ideals and also unions of their 0-minimal right ideals. This suggested the present approach to the study of semisimplicity for A-semigroups and semirings, which differs from the classical approach in that (direct) sums have been replaced by (disjoint) unions.

A (*left*) A-*semigroup*, as defined in [3], is a semigroup S on which a semiring A acts so that $a(x+y) = ax+ay$, $(a+b)x = ax+bx$, $a(bx) = (ab)x$ for all $a, b \in A$ and $x, y \in S$. If A has a zero, we also require that S be a semigroup with an identity element 0 and that $0x = a0 = 0$ for all $x \in S$ and $a \in A$. Right A-semigroups are defined similarly. An A-semigroup is simple in case it has no proper A-subsemigroup and semisimple if it is the [disjoint] union of its simple A-subsemigroups. With this definition, we obtain all expected properties of semisimple A-semigroups; in particular semisimplicity is inherited by A-subsemigroups and homomorphic images. We find that the semirings A for which all left A-semigroups and right A-semigroups are semisimple are precisely the completely simple semirings.

If now A is a semiring with zero, we call an A-semigroup 0-simple if $AS \neq 0$ and S has no non-trivial A-subsemigroups, 0-semisimple if it is the union of its 0-simple A-subsemigroups. Then all results above extend to 0-semisimple A-semigroups. We also obtain the following structure theorem which is our main result: Let A be a semiring with zero. Then the following properties are equivalent: (i) A is 0-semisimple; (ii) A is the union of a family of non-nilpotent 0-minimal left ideals and the union of a family of non-nilpotent 0-minimal right ideals; (iii) A is the union of a family of two sided ideals of A which are completely 0-simple; (iv) A is a completely 0-simple semiring.

Throughout this paper, semigroups will be denoted additively (unless otherwise specified), althrough we shall not assume them to be commutative in general. Also by a semiring we mean a set together with two associative operations called addition and multiplication such that the multiplication is distributive with respect to the addition.

Let S be an A-semigroup. If A^0 is the semiring obtained by adjoining a zero to A and S^0 the semigroup obtained by adjoining an identity to S, then we can extend the action of A on S to an action of A^0 on S^0 by setting: $a0 = 0$ and $0x = 0$ for all $a \in A^0$ and $x \in S^0$ so that trivially S^0 becomes an A^0-semigroup. This remark will allow us to concentrate on the case when A has a zero in most of the following study.

An *A-subsemigroup* of a left (right) A-semigroup S is a subsemigroup T of S which admits the action of A. If A has a zero, we also require that T be non-empty, or equivalently that T contain 0. We shall denote by $\langle A \rangle$ the smallest additive semigroup of S containing a subset A of S. If T, T' are A-subsemigroups of S, then $\langle T+T' \rangle$ is also an A-subsemigroup of S; if L is a left ideal of A and T an A-subsemigroup of S, then $L \cdot T = \langle LS \rangle$ is an A-subsemigroup of S. If $x \in S$, we call the set $A^1x = \langle Ax \cup \{x\} \rangle$ the *principal A-subsemigroup of S generated by x*; observe that in general $x \notin Ax$ whence $A^1x \neq Ax$.

Finally we call a *homomorphism* of an A-semigroup S into an A-semigroup T an additive homomorphism f of S into T such that $f(ax) = af(x)$ for all $a \in A$ and $x \in S$. Note that a homomorphism of an A-semigroup S into an A-semigroup T trivially extends to a homomorphism of S^0 into T^0.

1. Simple A-semigroups. Let A be a semiring and S be an A-semigroup. We say that S is *simple* if it has no proper A-subsemigroup. If A is a semiring with zero, we say that S is *0-simple* if $AS \neq 0$ and S contains no non-trivial A-subsemigroup. Simple and 0-simple right A-semigroups are defined similarly.

The (0-) minimal left ideals L of A (such that $AL \neq 0$) are important examples of (0-)simple A-semigroups; in particular A itself is (0-)simple if and only if it is a left (0-)simple semiring (i.e. A contains no non-trivial left ideals (and $A^2 \neq 0$)).

Clearly S is a simple A-semigroup if and only if S^0 is a 0-simple A^0-semigroup. However, the concepts of simple A-semigroup and 0-simple A^0-semigroup are in general different in the sense that a 0-simple A^0-semigroup is not always obtained by adjoining an identity to some simple A-semigroup. An interesting case when all A^0-semigroups are obtained by adjoining an identity to some A-semigroup is as follows:

PROPOSITION 1. *Let A be a left simple semiring. Then, for any 0-simple A^0-semigroup S, $S - \{0\}$ is a simple A-semigroup.*

Proof. We first show that $S^* = S - \{0\}$ admits the action of A, i.e. that for all $a \in A$ and $x \in S^*$, $ax \neq 0$ in S. Suppose that $ax = 0$ for some $a \in A$ and $x \in S^*$. Then the set of all $b \in A$ such that $bx = 0$ is a non-empty left ideal of A; therefore it coincides with A so that $Ax = 0$. Then the set of all $y \in S$ such that $A^0y = 0$ is different from zero since it contains x and it also is clearly an A^0-subsemigroup of S; this set must coincide with S, since S is 0-simple. Therefore we get a contradiction since this would implies $A^0S = 0$. It follows that $ax \in S^*$ for all $a \in A$ and $x \in S^*$.

To show that S^* is closed under addition, observe first that, if $x \in S^*$, then A^0x is an A^0-subsemigroup of S which is different from zero by the above; thus $A^0x = S$. Let now $y, z \in S^*$; then there exist $a, b \in A$ such that $y = ax$ and $z = bx$. Since $a+b \in A$, it follows that $y+z = ax+bx = (a+b)x \in S^*$. This proves that S^* is closed under addition. It is then clear that S^* is a simple A-semigroup, which completes the proof.

The following results give elementary properties of 0-simple A-semigroups. They obviously also apply to simple A-semigroups via the adjunction of an identity.

PROPOSITION 2. *Let S be a 0-simple A-semigroup. Then $Ax = S$ for all non-zero elements x of S.*

Proof. Assume that S is a 0-simple A-semigroup and let x be any non-zero element of S. If $Ax = 0$, then the set N of all elements $y \in S$ such that $Ay = 0$ would be an A-subsemigroup of S which is different from zero; since S is 0-simple, this would imply $N = S$, which is impossible since $AN = 0$ and $AS \neq 0$. Therefore $Ax \neq 0$ and, since Ax is clearly an A-subsemigroup of S, we conclude that $Ax = S$.

LEMMA 3. *Let S be a 0-simple A-semigroup and f be a homomorphism of S onto an A-semigroup T. Then T is either reduced to zero or 0-simple.*

Proof. Let f be a homomorphism of a 0-simple A-semigroup S onto an A-semigroup $T \neq 0$. Then first, since $AS \neq 0$, the A-subsemigroup $A \cdot S$ generated by AS must be equal to S; this implies that $T = f(S) = f(A \cdot S) = A \cdot f(S) = A \cdot T \neq 0$. Therefore $AT \neq 0$. Also, for every A-subsemigroup T' of T, the set $S' = \{x \in S;\ f(x) \in T'\}$ is clearly an A-subsemigroup of S; since S is 0-simple, S' is a trivial A-subsemigroup of S; it follows that T' is a trivial A-subsemigroup of T. Therefore T is 0-simple.

An obvious consequence of this result is the following:

PROPOSITION 4. *Let f be a homomorphism of an A-semigroup S into an A-semigroup T. If S' is a 0-simple A-subsemigroup of S, then either $f(S') = 0$ or $f(S')$ is a 0-simple A-subsemigroup of T.*

COROLLARY 5. *Let L be a 0-simple A-subsemigroup of A, and S be an A-semigroup. Then for every element x of S, either $Lx = 0$ or Lx is a 0-simple A-subsemigroup of S.*

Proof. Let $x \in S$. Then a homomorphism f of L onto Lx is defined by: $f(a) = ax$ for all $a \in L$. The result follows, by Lemma 3.

2. Semisimple A**-semigroups.** We recall that a semigroup S (with identity 0) is a (0-)disjoint union of a family $(S_i)_{i\in I}$ of subsemigroups (with identity) in case $\bigcup_{i\in I} S_i = S$ and the intersection of any two elements of the family $(S_i)_{i\in I}$ is empty (reduced to the identity).

Let A be a semiring (with zero) and S be an A-semigroup. We say that S is (0-)*semisimple* if S is the (0-)disjoint union of a family of (0-)simple A-subsemigroups.

The next theorem gives equivalent definitions of (0-)semisimple A-semigroups.

THEOREM 6. *Let A be a semiring (with zero) and S be an A-semigroup. Then the following properties are equivalent:*

(i) *S is (0-)semisimple;*

(ii) *S is the union of a family of (0-)simple A-subsemigroups of S;*

(iii) *For every (non-zero) element x of S, the principal A-subsemigroup of S generated by x is (0-)simple;*

(iv) *$AS = S$ and Ax is (0-)simple for every (non-zero) element x of S.*

Proof. We need only consider the case when A is a semiring with zero. Trivially (i) implies (ii).

Assume that (ii) holds so that S is the union of a family $(S_i)_{i\in I}$ of 0-simple A-subsemigroups of S. Then, for every non-zero element x of S, $x \in S_j$ for some $j \in I$, whence A^1x is a non-zero A-subsemigroup of S_j; since S_j is 0-simple, it follows that $S_j = A^1x$ so that A^1x is 0-simple. Thus (iii) holds.

To show that (iii) implies (iv), let x be any non-zero element of S. If $Ax = 0$, then the set N of all elements y of S such that $Ay = 0$ is a non-zero A-subsemigroup of S; since $N \cap A^1x \neq 0$ and A^1x is 0-simple, $A^1x \subseteq N$. This implies that $A(A^1x) = 0$ which contradicts the fact that A^1x is 0-simple. Therefore $Ax \neq 0$ and again by 0-simplicity of A^1x, $Ax = A^1x$. In particular Ax is 0-simple. Also $S = \bigcup_{x\in S} A^1x = \bigcup_{x\in S} Ax = AS$ which shows that (iv) holds.

Finally if (iv) holds, then first $S = AS$ implies that $S = \bigcup_{x\in S} Ax$. Also for every non-zero element x of S, Ax is 0-simple; it follows that the intersection of any two different Ax's is reduced to zero. Therefore S is the 0-disjoint union of the family of all different Ax's, which shows that (i) holds.

An important property of (0-)semisimple A-semigroups that we have incidentaly proved in the proof of Theorem 6 is that $A^1x = Ax$ for all $x \in S$, $x \neq 0$. In particular $x \in Ax$ for all $x \in S$ so that there exists $a \in A$ with $ax = x$.

Our next result shows that semisimplicity is a property inherited by A-subsemigroups and homomorphic images.

PROPOSITION 7. *Let S be a (0-)semisimple A-semigroup. Then each A-subsemigroup of S and each homomorphic image of S are also (0-)semisimple.*

Proof. Again it is enough to prove the result in the case when S is 0-semisimple. Assume that S is the union of a family $(S_i)_{i\in I}$ of 0-simple A-subsemigroups of S.

First let S' be an A-subsemigroup of S. Let $J = \{j \in J;\ S_j \cap S' \neq 0\}$. Then, since S_j is 0-simple for all $j \in J$, $S_j \subseteq S'$ for all $j \in J$. Also $S_i \cap S' = 0$ for all $i \in I \setminus J$. It follows that $S' = \bigcup_{j\in J} S_j$ ($= 0$ if $J = \emptyset$), which shows that S' is 0-semisimple.

Let now f be a homomorphism of S onto an A-semigroup T. Then clearly $T = \bigcup_{i\in I} f(S_i)$; also by Proposition 4, $f(S_i)$ is either reduced to zero or 0-simple for all $i \in I$. It follows that T is the union of the non-zero $f(S_i)$'s which are 0-simple and thus T is 0-semisimple.

3. Semisimple semirings. A semiring A (with zero) is *left* (0-)*semisimple* if it is (0-)semisimple as a left A-semigroup. *Right* (0-)*semisimple* semirings are defined similarly.

Contrary to what happens for semisimple rings with minimum condition in the classical sense, the concepts of left (0-)semisimple semiring and right (0-)semisimple semiring are different, as it is well known that a semiring may be the union of its minimal left ideals without being the union of its minimal right ideals. We say that a semiring is (0-)*semisimple* in case it is both left (0-)semisimple and right (0-)semisimple.

An important property of left (right) (0-)semisimple semirings is the following:

THEOREM 8. *Let A be a left (0-)semisimple semiring (with zero). Then any left A-semigroup S such that $AS = S$ is (0-)semisimple. A similar result holds when A is right (0-)semisimple.*

Proof. Let A be a left 0-semisimple semiring and S be an A-semigroup such that $AS = S$. Then, for every non-zero element x of S, there exist $a \in A$ and $y \in S$ such that $ay = x$. Since A is left 0-semisimple, a belongs to some 0-simple A-subsemigroup L of A. Then by Corollary 5, Ly is a 0-simple A-subsemigroup of S which contains x. Since each of its elements belongs to some 0-simple A-subsemigroup, S is 0-semisimple. The result concerning right 0-semisimple semirings is proved similarly.

It is easy to describe semisimple semirings. Indeed a semiring is semisimple if and only if it is the union of its minimal left ideals an the union of its minimal right ideals. In view of Theorem 2.2 of [2], this means exactly the following:

THEOREM 9. *A semiring is semisimple if and only if it is completely simple.*

We now proceed to investigate the structure of 0-semisimple semirings. We start with some results on left 0-semisimple semirings which we shall need later together with the similar properties for right 0-semisimple semirings.

PROPOSITION 10. *Let A be a left 0-semisimple semiring. Then $A^2 = A$ and each 0-minimal left ideal of A is a 0-simple A-semigroup.*

Proof. That $A^2 = A$ results trivially from (iv) of Theorem 6. Let L be a 0-minimal left ideal of A. To show that L is a 0-simple A-semigroup, we need only check that $AL \neq 0$. Since $L \neq 0$, there exists some non-zero element x of L. Then $Ax \neq 0$ by (iv) of Theorem 6 and, since $Ax \subseteq AL$, it follows that $AL \neq 0$.

In any semiring A with zero, $\mathcal{A}_r = \{a \in A;\ Aa = 0\}$ and $\mathcal{A}_l = \{a \in A;\ aA = 0\}$ are clearly two sided ideals of A; they are called *right annihilator of A and left annihilator of A* respectively.

PROPOSITION 11. *Let A be a left 0-semisimple semiring. Then $\mathcal{A}_r = 0$ and $\mathcal{A}_l$ is the union of all nilpotent 0-minimal left ideals of A.*

Proof. Again by (iv) of Theorem 6, $Ax \neq 0$ for every non-zero element x of A; thus $\mathcal{A}_r = 0$. To show the second part of the statement, let first L be a 0-minimal left ideal of A which is contained in $\mathcal{A}_l$. Then $LA = 0$ and in particular $L^2 = 0$. Therefore L is nilpotent. Now assume that L is a 0-minimal left ideal of A such that $L^2 = 0$ (which is equivalent to $L^n = 0$ for some $n > 1$). Since A is left 0-semisimple, A is the union of a family $(L_i)_{i \in I}$ of 0-minimal left ideals of A. If $LL_i \neq 0$ for some $i \in I$, then $L \cdot L_i = L_i$ so that $L_i = L \cdot L_i = \langle L^2 L_i \rangle = 0$ which is impossible. Therefore $LL_i = 0$ for all $i \in I$ and it follows that $LA = 0$, whence L is contained in $\mathcal{A}_l$. Thus $\mathcal{A}_l$, which as a left ideal of A is clearly a union of 0-minimal left ideals of A, must be the union of all nilpotent 0-minimal left ideals of A.

If now A is a 0-semisimple semiring, then by Proposition 11 and the dual result, $\mathcal{A}_r = \mathcal{A}_l = 0$ so that the following is immediate:

COROLLARY 12. *Let A be a 0-semisimple semiring. Then all 0-minimal left ideals of A and all 0-minimal right ideals of A are non-nilpotent.*

We now give our main result which characterizes 0-semisimple semirings.

THEOREM 13. *Let A be a semiring with zero. Then the following properties are equivalent:*

(i) *A is 0-semisimple;*

(ii) *A is the union of a family of non-nilpotent 0-minimal left ideals and the union of a family of non-nilpotent 0-minimal right ideals;*

(iii) *A is the union of a family of two-sided ideals of A which are completely 0-simple semirings;*

(iv) *A is a completely 0-simple semiring.*

Note that "union" may be replaced by "disjoint union" in the statements of (ii) and (iii).

Proof. Trivially (iv) implies (i). In view of Corollary 12, (i) implies (ii). Therefore we need only show that (ii) implies (iii) and (iii) implies (iv).

Assume first that (ii) holds. Then A is clearly 0-semisimple, whence by Corollary 12, all 0-minimal left ideals of A are non-nilpotent. Define a binary relation $\equiv$ on the set Γ of all 0-minimal left ideals of A in the following way: if $L, L' \in \Gamma$,

$$L \equiv L' \text{ if and only if } LL' \neq 0 .$$

Note that, since L' is a 0-minimal left ideal, $L \equiv L'$ if and only if $L \cdot L' = L'$. The relation $\equiv$ is an equivalence relation on Γ: indeed first $L^2 \neq 0$ for all $L \in \Gamma$ implies that $\equiv$ is reflexive; also if $LL' \neq 0$ for some $L, L' \in \Gamma$, then $L \cdot L' = L'$ so that $\langle LL'LL' \rangle = L'^2 \neq 0$, whence $L'L \neq 0$ and $\equiv$ is symmetric; finally, if $LL' \neq 0$ and $L'L'' \neq 0$ for some $L, L', L'' \in \Gamma$, then $L \cdot L'' = \langle LL'' \rangle = \langle LL'L'' \rangle = \langle L'L'' \rangle = L' \cdot L'' = L''$, whence $LL'' \neq 0$ which shows that $\equiv$ is transitive.

We now choose a complete system of representatives $(L_k)_{k \in K}$ of $\equiv$, denote by Γ_k the equivalence class of L_k, and by R_k the union of all 0-minimal left ideals of A which are equivalent to L_k. Clearly A is the union of the family $(R_k)_{k \in K}$. Note also that a 0-minimal left ideal L of A is contained in R_k if and only if it is equivalent to L_k modulo $\equiv$: by definition of R_k, if $L \in \Gamma_k$, then $L \subseteq R_k$; furthermore, if $L \subseteq R_k$, then there exists some $L' \in \Gamma_k$ such that $L \cap L' \neq 0$ so that by 0-minimality of L and L', $L = L' = L \cap L'$; thus $L \in \Gamma_k$.

We proceed to show that, for every $k \in K$, R_k is a two-sided ideal of A which is completely 0-simple.

To show that R_k is closed under addition, let $a, b \in R_k$. Then there exist some $L, L' \in \Gamma_k$ such that $a \in L$ and $b \in L'$. As a non-zero ideal of A, $\langle L+L' \rangle$ is the union of a family $(L_j)_{j \in J}$ of 0-minimal left ideals of A. Then for every $j \in J$, $L_j^2 \neq 0$, whence $L_j(\langle L+L' \rangle) \neq 0$, since it contains L_j^2. It follows that either $L_j L \neq 0$ or $L_j L' \neq 0$ which means that either $L_j \equiv L$ or $L_j \equiv L'$. In both cases, since $L, L' \in \Gamma_k$, we obtain $L_j \equiv L_k$ so that $L_j \subseteq R_k$. Therefore $\langle L+L' \rangle \subseteq R_k$, and in particular $a+b \in R_k$. Thus R_k is closed under addition.

Clearly, as a union of left ideals of A, R_k is a multiplicative left ideal of A. Also, for every $L \in \Gamma_k$, LA is the union of the family $(LL')_{L' \in \Gamma}$. Since $LL' = 0$ if $L' \notin \Gamma_k$, and $LL'' \subseteq L \cdot L'' = L''$ if $L'' \in \Gamma_k$, it is clear that $LA \subseteq R_k$. It follows that $R_k A$, which is the union of the family $(LA)_{L \in \Gamma_k}$, is also contained in R_k. This shows that R_k is a multiplicative right ideal of A, and therefore a two-sided ideal of A.

To show that R_k is a 0-simple semiring, it is enough by Proposition 1.5 of [2] to prove that R_k is a 0-minimal two-sided ideal of A.

Let I be a two-sided of A which is non-zero and contained in R_k. Then $I \cap L \neq 0$ for some $L \in \Gamma_k$. Then, by 0-minimality of L, $L \subseteq I$. Also, for every $L' \in \Gamma_k$, $L' = L \cdot L' = \langle LL' \rangle \subseteq \langle LA \rangle \subseteq I$. This implies that $R_k \subseteq I$, whence $R_k = I$. Therefore R_k is a 0-minimal two-sided ideal of A, and is 0-simple.

As an ideal of A, R_k is the union of a family of 0-minimal left (right) ideals of A. Also by Proposition 1.6 of [2], since R_k is a 0-minimal two-sided ideal of A. The 0-minimal left (right) ideals of R_k coincide with the 0-minimal left (right) ideals of A contained in R_k. It follows that R_k is the union of its 0-minimal left ideals and the union of its 0-minimal right ideals. Thus R_k is completely 0-simple, which proves that (ii) implies (iii).

Let now A be the union of a family $(R_k)_{k \in K}$ of two-sided ideals of A which are completely 0-simple semirings. To prove that A is completely 0-simple, we shall show that no two semirings of the family $(R_k)_{k \in K}$ can possibly be distinct.

Let $i, j \in K$ be such that $R_i \neq R_j$. Then $R_i \neq 0$, $R_j \neq 0$ (since R_i and R_j are 0-simple), and also $R_i \cap R_j = 0$ so that, in particular, $R_i R_j = 0$. Let a, b be any two non-zero elements of A in R_i and R_j respectively. Since R_i is completely 0-simple, by (iv) of Theorem 2.2 of [2], there exists $a' \in R_i$ such that $aa' \neq 0$. Then $(a+b)a' = aa' + ba' = aa' \neq 0$ which implies that $a+b \in R_i$. Similarly $a+b \in R_j$. Therefore $a+b = 0$ which is impossible since $(a+b)a' \neq 0$. Thus there exists no $i, j \in K$ such that $R_i \neq R_j$ which completes the proof.

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A note on monics and epics in varieties of categories

by

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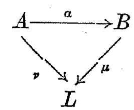
In memory of Professor Hanna Neumann

§ 1. Introduction. This work is motivated by the question, given any variety $\mathfrak{B}$ (see below) of a category $\mathfrak{C}$, are the monomorphisms, epimorphisms in $\mathfrak{B}$, same as those in $\mathfrak{C}$, to answer this question we choose certain axioms on a category, so as to extend our study in any category of multioperator groups [3]; and answer the question, by showing that in such categories the monomorphisms and normal epimorphisms in any variety is same, as in the original category. These categories have also been studied by Suliński [7], Szász and Wiegandt [8], for various other purposes. In the sequel, we are using the results of [6], [4] and [5] whenever necessary, without reference.

§ 2. Axioms and Main results. Let $\mathfrak{C}$ be a category equipped with the following axioms:

C_1: *$\mathfrak{C}$ has a null object.*

C_2: *Every morphism a in $\mathfrak{C}$, admits a factorization as in the diagram*



i.e. $a = \nu\mu$, where ν is a normal epimorphism and μ is a monomorphism.

C_3: *$\mathfrak{C}$ has product and coproduct for any arbitrary family of objects.*

C_4: *The sub-objects and normal factor objects of any object form a set.*

C_5: *If a is a monomorphism and β is a normal epimorphism such that $a\beta$ admits image $a\beta = \nu\mu$, then a normal implies μ is normal.*

DEFINITION 2.1. *A variety $\mathfrak{B}$* is a full subcategory of $\mathfrak{C}$, satisfying the following axioms:

B_1: *If μ: $A \to B$ be a monomorphism in $\mathfrak{C}$ and $B \in \mathfrak{B} \Rightarrow A \in \mathfrak{B}$.*

B_2: *If ν: $A \to B$ be a normal epimorphism in $\mathfrak{C}$ and $A \in \mathfrak{B} \Rightarrow B \in \mathfrak{B}$.*

B_3: *If $(A_i)_{i \in I}$ be a family of objects of $\mathfrak{B}$, then their product $\prod_i A_i \in \mathfrak{B}$.*