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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Mapping properties of characters of LCA groups

by

D. L. Armacost (Amherst, Mass.)

In this paper we shall be concerned with some questions about the mapping behavior of characters in locally compact Abelian (LCA) groups. In the first section, we shall describe the classes of LCA groups determined by requiring that the ranges of the continuous characters satisfy certain conditions. In the second section, we shall give necessary and sufficient conditions for a continuous character to be an open mapping, while in the third and last section, we shall touch upon a property of the set of non-surjective characters of an LCA group.

A word about notation: All LCA groups under consideration are assumed to satisfy the T_2 separation axiom. If G is an LCA group, we denote its character group by $\hat{G}$. The trivial character is written as 1, and the kernel of a continuous character γ in $\hat{G}$ is denoted $\ker\gamma$. The LCA groups of which we make constant mention are the circle T, the real numbers R, the integers Z, the cyclic groups $Z(n)$ of order n, the quasicyclic groups $Z(p^\infty)$, where p is a prime, and the additive group of the rational numbers Q, taken discrete. Precise definitions of these groups may be found in [2].

1. Range properties of characters. In this section we seek to classify the LCA groups whose non-trivial continuous characters satisfy a certain property with respect to their ranges. This type of problem is not new; for example, Robertson [3] has investigated the significance of the properties "all $\gamma \neq 1$ in $\hat{G}$ have range contained in the torsion subgroup of the circle" (see [3], 3.15), and "all $\gamma \neq 1$ in $\hat{G}$ have range contained in $Z(p^\infty)$" (see [3], 3.17). We shall make use of some of Robertson's results in our investigations.

In [3], 3.25, Robertson obtained a characterization of the dual of a connected LCA group. Our starting point will be the characterization of connected LCA groups by means of the simplest range property of their characters (see Theorem 1.1). We shall then subject this property to certain natural variations and determine the LCA groups whose characters satisfy these new properties. We list these properties as follows:

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P_1: Each $\gamma \neq 1$ in $\hat{G}$ is surjective.
P_2: Each $\gamma \neq 1$ in $\hat{G}$ has uncountable range.
P_3: Each $\gamma \neq 1$ in $\hat{G}$ has countably infinite range.
P_4: The ranges of the $\gamma \neq 1$ in $\hat{G}$ are algebraically isomorphic.
P_5: The ranges of the $\gamma \neq 1$ in $\hat{G}$ are identical.
P_6: The ranges of the γ in $\hat{G}$ are totally ordered by inclusion.

We shall now describe the LCA groups determined by these properties. We begin with a simple lemma.

LEMMA 1.1. *Let G be a discrete Abelian group and let $x \neq 0$ be an element of G. Then there exists $\gamma \in \hat{G}$ such that $\gamma(x) \neq 1$ and γ has countable range.*

Proof. Let $E(G)$ be the minimal divisible extension of G (see [2], A. 15). Since $E(G)$ is a weak direct sum of groups of the form Q and $Z(p^\infty)$, both of which are isomorphic to subgroups of the circle, γ can be taken as the composition of the injection from G into $E(G)$ with a projection of $E(G)$ onto an appropriate direct summand.

This lemma, easy in itself, renders the determination of the groups with properties P_1 and P_2 almost immediate.

THEOREM 1.1. *Let G be an* LCA *group. The following are equivalent*:

(a) *Each $\gamma \neq 1$ in $\hat{G}$ is surjective* (P_1).

(b) *Each $\gamma \neq 1$ in $\hat{G}$ has uncountable range* (P_2).

(c) *G is connected.*

Proof. Obviously (a) $\Rightarrow$ (b). Assume (b). If G is not connected, then G contains a proper open subgroup U. Since the discrete quotient group G/U has, by the lemma, a non-trivial character with countable range, so does G, which contradicts (b). Hence (b) $\Rightarrow$ (c). Finally, since T has no connected subgroups other than $\{1\}$ and itself, (c) $\Rightarrow$ (a), completing the proof.

Remark 1.1. We pause to discuss the rôle of the circle T in this theorem. Let H be an LCA group which is not totally disconnected. Suppose that the statement "An LCA group G is connected if and only if every non-trivial continuous homomorphism from G into H is surjective" is true. Then we can show that H is topologically isomorphic with T. This fact should perhaps be compared with Pontrjagin's observation that the circle is the only LCA group to yield a duality theorem for all LCA groups (see, for example, [2], 25.36).

Remark 1.2. At this point, one might inquire about the property "all $\gamma \neq 1$ in $\hat{G}$ have infinite range". This is, of course, a purely group theoretic property, viz. "$\hat{G}$ is torsion-free". Thus an LCA group G has this property if and only if its Bohr compactification is connected, or, as is much more difficult to show, if and only if G has a dense divisible subgroup (see [3], 5.2). Groups satisfying the preceding condition are called *densely divisible*. We shall use this term in the sequel.

To expedite our discussion of the remaining properties P_3 through P_6 we reproduce a definition of Robertson [3], 3.1.

DEFINITION 1.1. Let G be an additively written LCA group.

(a) If $\lim_{n\to\infty}(n!)x = 0$ for every x in G, we call G a *topological torsion group*.

(b) Let p be a prime. If $\lim_{n\to\infty} p^n x = 0$ for every x in G, we call G a *topological p-group*.

Remark 1.3. In [1], II, 1, Braconnier gives a definition of a "groupe primaire associé à l'entier premier p". It can be shown that this definition is equivalent to Definition 1.1 (b).

THEOREM 1.2. *Let G be an* LCA *group. The following are equivalent*:

(a) *Each $\gamma \neq 1$ in $\hat{G}$ has countably infinite range* (P_3).

(b) *$G \cong D \times G_0$, where D is a weak direct sum of countably many copies of Q, taken discrete, and G_0 is a topological torsion group.*

Proof. Assume (a). Then certainly $\hat{G}$ is torsion-free and G is totally disconnected. It follows from [3], 3.27, that G has the form $R^n \times D \times G_0$, where D is a direct sum of copies of Q, taken discrete, and G_0 is a totally disconnected group consisting entirely of compact elements. It is clear that $n = 0$ here, and it follows from [3], 3.15, that G_0 is a topological torsion group. Finally, it is clear that D is the direct sum of at most countably many copies of Q, so that (a) $\Rightarrow$ (b). The converse implication follows again from [3], 3.15, so that the proof is complete.

Remark 1.4. Suppose that property P_3 is changed to read "Each $\gamma \neq 1$ in $\hat{G}$ has proper infinite range". The same argument as in the theorem above shows that a group G with this property is of the form given above, except that D is a direct sum of μ copies of Q, where μ is a cardinal number strictly less than the power of the continuum. Thus, if we assume the continuum hypothesis, we conclude that the new property is equivalent to P_3.

We are now ready to discuss P_4, P_5, and P_6. The determination of the groups satisfying these properties is probably more onerous than conceptually difficult, and the results are most likely not unexpected.

THEOREM 1.3. *Let G be an* LCA *group. The following are equivalent*:

(a) *The ranges of the $\gamma \neq 1$ in $\hat{G}$ are algebraically isomorphic* (P_4).

(b) *The ranges of the $\gamma \neq 1$ in $\hat{G}$ are identical* (P_5).

(c) *G is one of the following types of groups*:

(1) *A connected group.*

(2) *A densely divisible topological p-group.*

(3) *A group every element of which has order p, where p is a prime (see* [1], *Théorème* 2, p. 41, *or* [2], 25.29, *for a complete description of these groups*).

Proof. First assume that G is discrete and satisfies (a). We claim that G is a torsion group. For if x in G has infinite order, we could extend the isomorphism from the cyclic group generated by x onto Z to a homomorphism from G into Q. This would give rise to a character of G with torsion-free range. But since every discrete group has characters with range contained in the torsion subgroup of the circle, (a) could not hold if G were not a torsion group. Thus, since G is a torsion group, we can write it as a weak direct sum of its p-components (see [2], A.3). If (a) holds, then G must coincide with one of these p-components. Hence every character of G must have range contained in a subgroup of the form $Z(p^\infty) \subseteq T$. We further deduce from (a) that either every $\gamma \neq 1$ in $\hat{G}$ has range $Z(p)$ or else every $\gamma \neq 1$ in $\hat{G}$ has range $Z(p^\infty)$.

Now let G be a general LCA group satisfying (a). Of course, G could be connected. If not, then G has a proper open subgroup U, and the discrete group G/U also satisfies (a). Thus we conclude from the first paragraph that either every $\gamma \neq 1$ in $\widehat{(G/U)}$, and hence every $\gamma \neq 1$ in $\hat{G}$, has range $Z(p)$, or else every $\gamma \neq 1$ in $\widehat{(G/U)}$, and hence every $\gamma \neq 1$ in $\hat{G}$, has range $Z(p^\infty)$. In the former case, every element of $\hat{G}$, and hence every element of G, has order p. In the latter case, we conclude from [3], 3.17 and 5.2, that G is a densely divisible topological p-group. Hence (a) $\Rightarrow$ (c). On the other hand, it is clear that (c) $\Rightarrow$ (b) $\Rightarrow$ (a), which completes the proof.

THEOREM 1.4. *Let G be an* LCA *group. The following are equivalent*:

(a) *The ranges of the continuous characters of G are totally ordered by inclusion* (P_6).

(b) *G/C is a topological p-group, where C is the identity component of G, and p is a fixed prime.*

Proof. Assume (b). A character in G either annihilates C or it does not. If $\gamma(C) \neq 1$ then $\gamma(G) = T$. Otherwise, γ may be thought of as a character of the topological p-group G/C, so that the range of γ is a subset of $Z(p^\infty)$ by [3], 3.17. Since the subgroups of $Z(p^\infty)$ are totally ordered by inclusion, it is clear that (a) holds for G, so that (b) $\Rightarrow$ (a).

Conversely, suppose that (a) holds for the discrete group G. We conclude, just as in the proof of the previous theorem, that G is a torsion group coinciding with one of its p-components. Thus every character of G has its range contained in $Z(p^\infty)$. If G is totally disconnected, then we conclude from [2], 7.7, that $\ker\gamma$ is open in G for each γ in $\hat{G}$. Thus, if (a) holds for a totally disconnected group G, and if γ is in $\hat{G}$, then each character of the discrete group $G/\ker\gamma$ has range contained in $Z(p^\infty)$ for some prime p. We can conclude from this that every character of G has its range contained in the same group $Z(p^\infty)$. Finally, if G is a general group satisfying (a), we conclude from the preceding discussion that each character of the totally disconnected group G/C has range contained in a fixed subgroup $Z(p^\infty)$ of T, so that G/C is a topological p-group, by [3], 3.17. Hence (a) $\Rightarrow$ (b), which completes the proof.

COROLLARY 1.1. *Let G be a totally disconnected* LCA *group. The following are equivalent*:

(a) *The ranges of the continuous characters of G are totally ordered by inclusion.*

(b) *G is a topological p-group.*

2. Open characters. We might ask whether Theorem 1.1 has a companion at the other end of the spectrum, namely, is it true that G is totally disconnected if and only if no γ in $\hat{G}$ is surjective? It is easy to see that this is false, as an examination of the group R with the discrete topology readily shows. With sufficient restrictions we could obtain a statement of the desired sort. For example, it is easy to see that if G is σ-compact, then G is totally disconnected if and only if no γ in $\hat{G}$ is surjective. Similarly, an arbitrary LCA group G is totally disconnected if and only if there are sufficiently many non-surjective γ in $\hat{G}$ to separate the points of G.

To proceed in a different way, however, let us seek a mapping property P of the non-trivial characters of a group G such that G is connected if and only if all $\gamma \neq 1$ in $\hat{G}$ have property P, while G is totally disconnected if and only if no γ in $\hat{G}$ has property P. To see how to select P let us begin with the following result.

PROPOSITION 2.1. *Let G be an* LCA *group. Then G is connected if and only if each $\gamma \neq 1$ in $\hat{G}$ is an open mapping.*

Proof. If G is connected, it is σ-compact [2], 9.14. It then follows from [2], 5.29, that each $\gamma \neq 1$ in $\hat{G}$, being surjective, must be an open mapping. Conversely, since an open character must be surjective, a group G all of whose continuous characters are open must be connected, by Theorem 1.1.

We are now in a position to prove that an LCA group is totally disconnected if and only if none of its continuous characters is an open mapping. This will follow as a corollary of the following more general result.

THEOREM 2.1. *Let G be* LCA *and let γ be in $\hat{G}$. The following are equivalent*:

(a) *γ is an open mapping.*

(b) $\gamma(C) \neq \{1\}$, *where C is the identity component of G.*

(c) *γ is not a compact element of $\hat{G}$.*

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Proof. Assume (a). It then follows from [2], 7.12, that $\gamma(C)$ is dense in the identity component of T, so that $\gamma(C) \neq \{1\}$. Hence (a) $\Rightarrow$ (b). If (b) holds, then γ, restricted to C, is an open mapping from C onto T, by Proposition 2.1. It follows from this that γ is an open mapping from G onto T, so that (b) $\Rightarrow$ (a). The equivalence (b) $\Leftrightarrow$ (c) is shown in [2], 24.17, thus completing the proof.

We can now state the companion to Proposition 2.1. This is a direct consequence of the theorem.

COROLLARY 2.1. *Let G be* LCA. *Then G is totally disconnected if and only if no γ in $\hat{G}$ is an open mapping.*

COROLLARY 2.2. *Let G be* LCA. *Then G is not totally disconnected if and only if there are sufficiently many open characters in $\hat{G}$ to separate the points of G.*

Proof. If G has any open characters at all, it follows from Corollary 2.1 that G is not totally disconnected. Conversely, if G is not totally disconnected, $\hat{G}$ contains non-compact elements. It is not hard to see that every element of $\hat{G}$ can be written as a product of non-compact elements of $\hat{G}$. Hence every γ in $\hat{G}$ can be written as the product of open characters, whence it follows that the reare sufficiently many open characters of G to separate the points of G.

We conclude this section with another corollary of Theorem 2.1 which we shall use in the next section.

COROLLARY 2.3. *Let G be* LCA *and let γ be in $\hat{G}$. Then γ is open if and only if* $\ker\gamma$ *is not open in $\hat{G}$.*

Proof. If γ is an open mapping, then it is obvious that $\ker\gamma$ is not open in G. Conversely, if $\ker\gamma$ is not open in G, then its annihilator in $\hat{G}$ is not compact. But this annihilator is just the closed monothetic subgroup of $\hat{G}$ generated by γ, so that γ is not a compact element of $\hat{G}$. It follows from Theorem 2.1 that γ is an open mapping.

3. Non-surjective characters. In this section we consider the following question: When does the set of non-surjective characters γ in $\hat{G}$ form a subgroup of $\hat{G}$? If G is σ-compact, we answer the question affirmatively, since in that case every non-surjective character has countable range. But the answer is in general negative. In fact, it is not hard to see that the discrete circle possesses non-surjective characters whose product is surjective. We shall answer the question in the following way.

THEOREM 3.1. *Let G be* LCA *and let N denote the set of non-surjective γ in $\hat{G}$. The following are equivalent*:

(a) *N is a subgroup of $\hat{G}$.*
(b) *N is a closed subgroup of $\hat{G}$.*
(c) *Every surjective γ in $\hat{G}$ is open.*

Proof. Assume (a). Suppose that some surjective γ in $\hat{G}$ is not open. Then $\ker\gamma$ is open in G by Corollary 2.3, so that $G/\ker\gamma$ is topologically isomorphic with the discrete circle. Since, as we have mentioned, the discrete circle possesses non-surjective characters whose product is surjective, we can find non-surjective characters of G whose product is surjective, which violates (a). Thus (a) $\Rightarrow$ (c). From (c) we conclude that N consists precisely of the non-open characters of G, that is, N consists of the compact elements of $\hat{G}$. Since this set is a closed subgroup of $\hat{G}$ [2], 9.10, we conclude that (c) $\Rightarrow$ (b). Since it is obvious that (b) $\Rightarrow$ (a), the proof is complete.

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AMHERST COLLEGE
Amherst, Massachusetts

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