Fixed point theorems
for non-compact approximative ANR’s

by

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1. Introduction. Compact approximative ANR’s were first introduced by H. Noguchi and in 1968 A. Granas proved that they were Lefschetz spaces. More recently J. Jaworowski proved the Lefschetz fixed point theorem for upper semi-continuous, acyclic multi-valued maps of these compact approximative ANR’s.

On the other hand there is much recent interest in Lefschetz fixed point theory for compact maps of non-compact spaces. A space $X$ is a $A$-space if for every compact continuous map $f : X \to X$, the Lefschetz number $L(f)$ exists and $f$ has a fixed point whenever $L(f) \neq 0$. For example, ANR’s are $A$-spaces ([2],[4]). The corresponding concepts, $M$-Lefschetz space and $MA$-space, for certain multi-valued maps have been studied. (The maps used need not be acyclic; it suffices to require that they be compositions of acyclic maps.) ANR’s are known to be $MA$-spaces. (See [7].)

In this note, it is shown that (non-compact) approximative ANR’s are $A$-spaces. It is also proved that a second related class of spaces are $MA$-spaces.

2. Preliminary definitions. In this section we recall the pertinent facts about multi-valued maps, establish the homology theories under which we will be working, and recall the definitions of $A$-space and $MA$-space. The reader is referred to [7] for the details of this section.

A map is said to be compact if its image is contained in a compact set. A multi-valued map $F : X \to Y$ is upper semi-continuous (usc) if

(i) $F(x)$ is compact for each $x$ in $X$

(ii) for each $x$ in $X$ and each open set $V$ containing $F(x)$, there is an open neighborhood $U$ of $x$ such that $F(U) \subseteq V$.

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If \( F : X \to Y \) and \( G : Y \to Z \) are multi-valued maps, the composition of \( F \) and \( G \) is denoted \( G \circ F : X \to Z \) and is defined by \( G \circ F(x) = \bigsqcup_{y \in F(x)} G(y) \).

A point \( x \) is a fixed point for \( F : X \to X \) if \( x \in F(x) \).

Next with regard to the homology theories, let \( J \) denote the category of Hausdorff spaces and continuous maps and \( I \) the category of graded vector spaces and homomorphisms of degree zero. \( X : J \to I \) can be any covariant functor which has compact support, satisfies the homotopy axiom, and agrees with the Cech homology functor \( \tilde{H} \) on the full subcategory of compact spaces. In particular, when working with multi-valued maps we will use the functor \( \tilde{H} \). \((\tilde{H}X) \) is the direct limit of groups \( \tilde{H}(X) \) taken over the compact subsets \( K \) of \( X \). See [12]. Let \( F : X \to Y \) be an u.s.c. multi-valued map of spaces in \( J. \) \( \tilde{H}(F) = \{ F \} \) is defined \((\tilde{H}F) = \{ \tilde{H}F \} \) if for each \( x \in X, F(x) \) is an acyclic subset of \( Y \). When \( F \) is acyclic the induced homomorphism \( \tilde{H}(F) = \{ F \} \) is defined \((\tilde{H}F) = \{ F \} \).

Finally, we recall the definitions of \( \Lambda \)-space and \( MA \)-space. \( \Lambda \) will denote a subcategory of \( J \). Refer to § 2 of [7] for the definitions of generalized trace, finite type, and Lefschetz number for homomorphisms in \( \Lambda \).

(2.1) Definition. An u.s.c. map \( F : X \to X \) of a space in \( J \) is admissible (rel. to \( \Lambda \)) if there are maps \( G_i : Y_i \to Y_{i+1} \), \( i = 0, \ldots, n \) (where \( Y_n = Y_{n+1} = X \)) satisfying

(i) \( F = G_0 \circ \ldots \circ G_n, \)
(ii) \( G_i \) is acyclic and u.s.c. for each \( i = 0, \ldots, n \), and
(iii) \( Y_i \) is in \( \Lambda \) for \( i = 1, \ldots, n \).

Each such sequence is called an admissible sequence for \( F \).

(2.2) Definition. An admissible map \( F : X \to X \) is an \( MA \)-Lefschetz map (rel. to \( \Lambda \)) if

(i) for each admissible sequence \( G_0, \ldots, G_n \) for \( F, Q_n \circ \ldots \circ Q_0 \) has finite type and

(ii) whenever \( G_0, \ldots, G_n \) is an admissible sequence with Lefschetz number \( \Lambda(Q_n \circ \ldots \circ Q_0) \neq 0 \), then \( F \) must have a fixed point.

A continuous single-valued map \( f : X \to X \) is a Lefschetz map if \( \mathcal{X}(f) = f \) has finite type and whenever \( \Lambda(f) \neq 0 \), then \( f \) has a fixed point.

(2.3) Definition. A space \( X \) is an \( MA \)-space (rel. to \( \Lambda \)) if each compact admissible map \( F : X \to X \) (rel. to \( \Lambda \)) is an \( MA \)-Lefschetz map (rel. to \( \Lambda \)).

A space \( X \) is a \( \Lambda \)-space if each compact continuous (single-valued) map \( f : X \to X \) is a Lefschetz map.

3. Main results. In this section we identify the two topological categories in which we are interested and state the two main theorems.

First consider two maps \( f, g : X \to Y \) and \( u \in \text{Cov} \). Recall that \( f \) and \( g \) are \( \varepsilon \)-near if for each \( x \in X \), there is an element \( U \) of \( A \) containing both \( f(x) \) and \( g(x) \). Similarly, if \( Y \) is a metric space and \( \varepsilon \) is a positive real number, \( f \) and \( g \) are said to be \( \varepsilon \)-near if for each \( x \in X \), \( d(f(x), g(x)) < \varepsilon \).

(3.1) Definition. Let \( \tilde{X} \) be a subspace of \( X \). For \( u \in \text{Cov} \), a continuous map \( r_u : \tilde{X} \to \tilde{X} \) is an \( \alpha \)-retraction if \( r_u \mid \tilde{X} = 1 \) and \( 1_X \) are \( \varepsilon \)-near. \( X \) is an approximative retract of \( Y \) if for each \( u \in \text{Cov} \) there is an \( \alpha \)-retraction \( r_u : \tilde{X} \to \tilde{X} \) of \( X \) if \( X \) is an approximative neighborhood retract of \( Y \) if there is an open set \( U \) in \( Y \) such that \( X \subseteq U \) and \( X \) is an approximative retract of \( U \).

(3.2) Definition. Let \( Y \) be a metric space, \( X \) a subspace and \( \varepsilon \) a positive real number. A continuous map \( r : Y \to X \) is an \( \varepsilon \)-retraction if \( r \mid \tilde{Y} = 1 \tilde{X} \) and \( 1_X \) are \( \varepsilon \)-near. \( X \) is an approximative retract of \( Y \) if for each \( \varepsilon > 0 \) there is an \( \varepsilon \)-retraction \( r : Y \to X \). \( X \) is a weak approximative neighborhood retract of \( Y \) if there is an open set \( U \) in \( Y \) such that \( X \subseteq U \) and \( X \) is a weak approximative retract of \( U \).

(3.3) Definition. A space \( X \) is a (metric) approximative absolute neighborhood retract (A-ANR) if for each homeomorphism \( h : X \to M^k \) with \( M \) a metric space and \( h(X) \) closed in \( M \), the space \( h(X) \) is an approximative neighborhood retract of \( M \).

(3.4) Definition. A space \( X \) is a (metric) weak approximative absolute neighborhood retract (WA-ANR) if for each homeomorphism \( h : X \to M^k \) with \( M \) a metric space and \( h(X) \) closed in \( M \), the space \( h(X) \) is a weak approximative neighborhood retract of \( M \).

It is clear that the category of WA-ANR's contains the category of A-ANR's and that restricting to compact spaces the two categories coincide.

(3.5) Remark. Using the Kuratowski–Wojdyslawski embedding theorem ([1], p. 79) any metric space \( X \) can be embedded as a closed subset of a convex set \( C \) in a Banach space. If \( X \) is an A-ANR [or a WA-ANR], there is an open set \( U \) in \( C \) such that \( X \) is an approximative retract [or weak approximative retract] of \( U \). The notation \( C \) and \( U \) will be used throughout.

It is necessary to consider an additional restriction on the WA-ANR's. Its significance is explained in Remark (5.3).

(3.6) Condition \( \mathcal{K} \). A WA-ANR \( X \) satisfies condition \( \mathcal{K} \) if for each compact subset \( K \) of \( X \) it is possible to choose \( \varepsilon \)-retractions \( r \mid K \to X \) for each \( \varepsilon > 0 \) such that \( \bigsqcup_{\varepsilon \in \mathcal{K}} r \mid K \) is compact.

**Theorem 1.** Every A-ANR is a \( \Lambda \)-space.

**Theorem 2.** Every WA-ANR which satisfies condition \( \mathcal{K} \) is an \( MA \)-space (rel. to \( \Lambda \)) and hence also a \( \Lambda \)-space. (3\( \Lambda \) denotes the category of metric spaces.)
4. Proof of Theorem 1. Recall that we are using the notation of (3.5).

(4.1) LEmMA. Let $X$ be an ANR and let $j: X \to U$ denote the inclusion map. Then for any continuous compact map $f: X \to X$, $\ker_j \subseteq \ker_f$, where $\ker_f: \mathcal{K}(X) \to \mathcal{K}(U)$ and $\ker_j: \mathcal{K}(X) \to \mathcal{K}(U)$.

Proof. Take $a \in \ker_f$. Since $\mathcal{K}$ has compact carriers and agrees with $\mathcal{S}$ on compact spaces, there is a compact subset $K$ of $X$ and $x \in \mathcal{S}(K)$ such that $a(x) = a$ (where $i: K \to X$ is the inclusion). We may assume that $f(x) \subseteq K$. Let $f(x): K \to K$ be defined by $f$. To show that $f \in \ker_j$, we need to show that $f \in \ker_j$.

Now since $K$ is compact and $U$ is open in $C$, there is an $\varepsilon > 0$ such that $B_K(K; \varepsilon) = \{y \in U : d(x, y) < \varepsilon\} \subseteq U$. Take $n_0$ such that $1/n_0 < \varepsilon$. For $x, y \in K$ with $d(x, y) < 1/n_0$, we have $P_{x, y} \subseteq B_K(K; \varepsilon) \subseteq U$, where $P_{x, y}$ is the segment in $C$ connecting $x$ and $y$. Let $a_{n_0}$ be the union of all segments in $\{P_{x, y} : x, y \in K, d(x, y) < 1/n_0\}$. Then $(a_{n_0})_{n_0 \leq n}$ is a nested family of compact sets and $K = \bigcap_{n_0 \leq n} a_{n_0}$.

By the continuity of $\mathcal{S}$, $\mathcal{S}(K)$ is the limit of $(a_{n_0})_{n_0 \leq n}$. Let $f_n: K \to V_n$ be the composition of $f(x)$ with the inclusion $K \to V_n$. Then to show that $f \in \ker_j$, it suffices to show $f_n(x) = 0$ for each $n \geq n_0$.

For $n \geq n_0$, let $a_n$ be the covering of $X$ by balls of radius $1/2n$, $a_n = \{B_n(x; 1/2n) : x \in X\}$ and let $f = f \circ a_{n_0}$ be the $f$-retraction of $X$. Since $X$ is an ANR, we have the $f$-retraction $r_f: U \to X$. Consider the diagram, where $j$ is the inclusion and $f'$ is defined by $j$.

$$
\begin{array}{ccc}
K & \xrightarrow{j} & V_n \\
\downarrow & & \uparrow \\
U & \xrightarrow{r_f} & X
\end{array}
$$

Since $r_f(X)$ and $1_X$ are $f$-near, for each $x \in X$ there is some $q_f^{-1}B(x; 1/2n)$ in $\mathcal{S}$ containing $x$ and $r_f(x)$. Thus $d(f(x), f') = d(f(x), f \circ r_f) = \varepsilon$ for $f \in \ker_j$. Hence the diagram is homeomorphic commutative, and $f_n = f_n \circ r_f = f_n \circ r_f$. Finally, $f_n(x) = 0$ implies $f_n(x) = 0$ and hence $f_n(x) = 0$. Q.E.D.

(4.2) LEmMA. Let $X$ be an ANR and let $j: X \to U$ denote the inclusion. There is an $\alpha_n$ in $\mathcal{C}(X)$ such that for each refinement $\alpha_0$ of $\alpha_n$ and each $f$-retraction $r_f: U \to X$, $j \circ r_f = j \circ r_f$.

Proof. Since $U$ is open in $C$ and $X \subseteq U$, for each $x \in X$ there is an $\alpha_n$ such that $B(x; \varepsilon) \subseteq U$. Moreover, the ball is convex since $C$ is convex. Let $\alpha_n = \{B(x; \varepsilon) : x \in X\}$ be the convex.

Now take $\alpha_0$ and an $f$-retraction $r_f: U \to X$. For each $x \in X$ there is an element $0$ of a containing $x$ and $r_f(x)$. Thus for some $\varepsilon \in X$ we have $x, r_f(x) \in 0 \subseteq B(x; \varepsilon) \subseteq B(x; \varepsilon) \subseteq U$, and hence the segment $P_{x, r_f(x)} \subseteq U$ is contained in $U$. Finally we conclude that $j$ is homotopic to $j \circ r_f$. Q.E.D.

Theorem 1 is now an easy consequence of the following generating theorem. (See [4], (3.4.1).)

(3.4) THeOREM. Let $X$ be a $T_1$-space and $D$ be a cofinal subset of $\mathcal{C}(X)$. Let $f: U \to X$ be a continuous compact map. Suppose that for each $a \in D$ there is a $\alpha_\infty \subseteq \mathcal{C}(X)$ and a continuous map $g_a: X \to X$ satisfying

(a) $g_a$ is an a-compact,
(b) $g_a \circ g_a = f_a$, and
(c) $g_a \circ g_a$ and $g_a$ are a-near.

Then $g_a$ is a Lefschetz map.

THEOREM 1. Each ANR is a $\alpha$-space.

Proof. Let $f: X \to U$ be a continuous compact map of an ANR.

We use (4.3) to show that $f$ is a Lefschetz map. Let $a_0$ be as in (4.2); then $D = \{a \in \mathcal{C}(X) : a \circ a_0 \subseteq \mathcal{C}(X)\}$.

For $a$ in $D$ let $f_a: X \to U$, an ANR, and hence a $\alpha$-space. Let $g_a = f_a \circ r_{a_0}$, where $r_{a_0}$ is a $\alpha$-retraction and $a_\infty = a_0$. Thus $f = f \circ r_{a_0}$ is compact since $f$ is compact and $f \circ r_{a_0} = f_a$. Finally, it follows from (4.1) and (4.2) that $a \circ a_\infty = a_\infty = a_\infty$, and (3.4) applies. Q.E.D.

5. Proof of Theorem 2. The proof of Theorem 2 relies heavily on Lemma (5.1); the proof of the Lemma is postponed to the end of the section.

(5.1) LEmMA. Let $X$ be a WA-ANR satisfying condition $\mathcal{X}$. Then $j: \mathcal{S}(X) \to \mathcal{S}(U)$ is a monomorphism, where $j$ is the inclusion.

The next Lemma corresponds to (4.2) of the previous section.

(5.2) LEmMA. Let $X$ be a WA-ANR and let $j: X \to U$ denote the inclusion.

Then given a compact subset $K$ of $X$, there is a positive real number $\varepsilon_K$ such that for each positive $\varepsilon \leq \varepsilon_K$ and for each $\varepsilon$-retraction $r_\varepsilon: \mathcal{S}(K) \to \mathcal{S}(U)$,

$$
\begin{array}{ccc}
\mathcal{S}(X) & \xrightarrow{j} & \mathcal{S}(U) \\
\downarrow & & \uparrow \\
\mathcal{S}(K) & \xrightarrow{j \circ r_\varepsilon} & \mathcal{S}(K)
\end{array}
$$

where $i: K \to X$ is the inclusion.

$J_i$ in addition, $X$ satisfies condition $\mathcal{X}$, then $r_\varepsilon \circ j \circ i = j \circ i$. Q.E.D.

Proof. Since $K$ is a compact subset of $U$, there is an $\varepsilon_K > 0$ such that $B(x; \varepsilon_K) \subseteq U$. Take a positive $\varepsilon \leq \varepsilon_K$ and an $\varepsilon$-retraction $r_\varepsilon: \mathcal{S}(K) \to \mathcal{S}(U)$.

Then for $x \in K$, $d_r(x, r_\varepsilon(x)) = d_r(x, r_\varepsilon(x)) < \varepsilon$ and hence the segment $P_{x, r_\varepsilon(x)}$ is contained in $U$. Thus $j \circ i$ is homotopic to $j \circ r_\varepsilon \circ i$ and the induced homomorphisms are equal.
When $X$ satisfies condition $X_1$, (5.1) implies that $r_\sigma + j_\sigma + r_\nu = r_\kappa$. Q.E.D.

There is no generating theorem in [7] which applies directly to give the proof of Theorem 2. However, the method of proof is analogous to that used in [7].

**Theorem 2.** Each WA-ANR satisfying condition $X_1$ is an MA-space (rel. to $3\lambda_2$) and hence also a $\Delta$-space.

**Proof.** Let $X$ be a WA-ANR satisfying condition $X_1$ and let $F: X \to X$ be a compact admissible map (rel. to $3\lambda_2$). Pick a compact set $K$ in $X$ containing $F(X)$. Let $G_1, \ldots, G_n$ be an admissible sequence for $F$. We have the commutative diagram, where the $i_k$'s are inclusion maps and each $G'_k$ is defined by $G_k$.

\[ \begin{array}{ccc}
K & \xrightarrow{i_0} & K \\
& \searrow & \downarrow i_k \\
G_k & \xrightarrow{i_{k-1}} & G_{k-1} \ldots G_1 X \xrightarrow{G_1} X \\
\end{array} \]

Now $\varepsilon_K$ is determined by (5.3) and for each $\varepsilon$-retraction $r_k: U \to X$ with $\varepsilon \leq \varepsilon_K, j = i \circ G'_k \circ \ldots \circ G'_1 \circ r_k: U \to U$ is a compact admissible map. Since $U$ is an ANR, it is an MA-space (see (5.5) in [7]) and hence $f_\sigma + j_\sigma \circ G'_k \circ \ldots \circ G'_1 \circ r_k$ has finite type. Then by Lemma (2.5) of [7], $G'_k \circ \ldots \circ G'_1 \circ r_k \circ f_\sigma + j_\sigma \circ G'_k \circ \ldots \circ G'_1 \circ r_k$ also has finite type and their Lebesgue numbers are equal. By (5.3) $G'_k \circ \ldots \circ G'_1 \circ r_k \circ f_\sigma + j_\sigma \circ G'_k \circ \ldots \circ G'_1 \circ r_k$ has finite type and $A(G'_k \circ \ldots \circ G'_1 \circ r_k) = A(A(\varepsilon_K + i_k \circ G'_k \circ \ldots \circ G'_1 \circ r_k))$. Applying (5.4) again, $i_k \circ G'_k \circ \ldots \circ G'_1 \circ r_k \circ f_\sigma + j_\sigma \circ G'_k \circ \ldots \circ G'_1 \circ r_k$ has finite type and $A(G'_k \circ \ldots \circ G'_1 \circ r_k) = A(j_\sigma \circ G'_k \circ \ldots \circ G'_1 \circ r_k)$.

Suppose that $G_1, \ldots, G_n$ is an admissible sequence for $F$ and that $A(G'_k \circ \ldots \circ G'_1 \circ r_k)$ is not 0. We show that $F$ must have a fixed point. For each integer $n$ with $1/n \leq \varepsilon_K$, let $r_n: U \to X$ be a 1/n-retraction. Then $A(j_\sigma \circ G'_k \circ \ldots \circ G'_1 \circ r_n)$ is not 0. Let $y_n$ be a fixed point for $j \circ i \circ G'_k \circ \ldots \circ G'_1 \circ r_n$ and let $x_n = r_n(y_n)$. Then $y_n, j \circ i \circ G'_k \circ \ldots \circ G'_1 \circ r_n(y_n) = j \circ f \circ r_n(y_n) \subseteq F(x_n)$ in $K$. Moreover, $d(x_n, y_n) = d(y_n(y_n), y_n) < 1/n$. Thus $y_n(x_n)$ is a sequence in $K$ and has a subsequence converging to $y_n$ in $K$. The corresponding subsequence of $(x_n, y_n)$ also converges to $y_n$. Hence we have a sequence of terms $(x_n, y_n)$ in the graph of $F$ converging to $(y_n, y_n)$.

But since $F$ is u.a.e. the graph of $F$ is closed and hence contains $(y_n, y_n)$; thus $y_n \in F(y_n)$. Q.E.D.

**Proof of (5.1).** Suppose that $j_\sigma(\epsilon) = 0, j_\sigma: \tilde{H}(X) \to \tilde{H}(U)$. Then there are compact sets $K \subseteq X$ and $K \subseteq U$ with $K \subseteq K_1$ and $\nu \in \tilde{H}(K_1)$ such that

\[ \begin{array}{ccc}
X & \xrightarrow{f} & U \\
\downarrow i_0 & & \downarrow i_1 \\
K & \xrightarrow{j_\sigma} & K_1 \\
\end{array} \]

For each $\varepsilon > 0$ and $K \subseteq X$ and $K \subseteq U$ with $K \subseteq K_1$ and $\nu \in \tilde{H}(K_1)$ such that

\[ i_0(\varepsilon) = \nu \text{ and } j_\sigma(\varepsilon) = 0. \]

(The construction of $\tilde{H}$ as a direct limit assures the existence of $K_1$.)

Applying condition $X_1$, there are $\varepsilon$-retractions $r_k: U \to X$ for each $\varepsilon > 0$ such that $K = K_1 \cup (\bigcup K_2)$ is a compact subset of $X$. The remainder of the proof parallels (4.1). We show that $x$ projects to 0 in $\tilde{H}(K)$ under the homomorphism induced by the inclusion $K_1 \to K_1$. Then $x$ projects to $0$ in $\tilde{H}(X)$. Take $\varepsilon > 0$ such that $\nu_\varepsilon(K_1; \varepsilon) \subseteq U$. Take a positive integer $n_\varepsilon$ with $1/n_\varepsilon < \varepsilon$ and for each $n \geq n_\varepsilon$ form the compact set $V_n$ as in the proof of (4.1). Then $K = \bigcap V_n$ and $\tilde{H}(K) = \lim_{n \to \infty} (\tilde{H}(V_n))_{\varepsilon > 0}$.

It suffices to show that $i_0^*(\varepsilon) = 0$ for each $n \geq n_\varepsilon$, where $i_0: K_1 \to V_n$ is the inclusion.

We have the $1/n$-retraction $r_n: U \to X$ and $i_0(\varepsilon) \subseteq K \subseteq V_n$. Let $r_\varepsilon: K \to V_n$ be defined by $r_\varepsilon$. Then for $x \in K_1, d(i_0(\varepsilon), r_\varepsilon(x)) < 1/n$ and $P_{x_0}(\varepsilon) \subseteq V_n$. This implies that $i_0^*(\varepsilon) = r_\varepsilon^* \circ i_0^*(\varepsilon) = 0$. Q.E.D.

(5.3) Remark. Several unanswered questions remain. A crucial point is whether $\tilde{H}(X) \to \tilde{H}(U)$ is a monomorphism. To prove that $j_\sigma$ is a monomorphism in (5.1) it was necessary to add condition $X_1$; but if doing so the results could be extended from the category of A-ANRs to that of WA-ANRs. Can (5.1) be proved if condition $X_1$ is weakened or perhaps omitted entirely? Lemma (4.1) draws a weaker conclusion than (5.1), but it is sufficient for our purposes. Can (4.1) be used to show that in fact ker$j_\sigma = 0$? Can (4.1) be extended to include multi-valued, admissible, compact maps $F: X \to Y$? An affirmative answer to either question will assure that A-ANRs are $\Delta$-spaces.

Finally, we remark that if $X$ is a compact A-ANR, then condition $X_1$ is automatic and Theorem 2 implies that $X$ is an $\cdot$-Lefschetz space (Def. (3.3), [7]). J. Jaworowski proved this result in [3] when considering only acyclic maps.
Epireflections in the category of $T_{0}$-spaces*

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A $T_{0}$-space with the property that every non-empty irreducible closed set is a point closure will be called a $pc$-space. It is shown that the $pc$-spaces form an epireflective subcategory of the category $\mathcal{C}_{0}$ of all $T_{0}$-spaces, generated in $\mathcal{C}_{0}$ by any $pc$-space which contains a copy of the Alexandroff dyad.

A larger simply generated, epireflective subcategory of $\mathcal{C}_{0}$—the $fe$-spaces—is introduced and it is shown that an $fe$-space is an invariant of its lattice of real-valued lower semi-continuous functions. As a preliminary it is shown that equalizers in $\mathcal{C}_{0}$ correspond to "front-closed" subspaces.

1. Preliminaries. The set of continuous maps from $X$ to $Y$ will be denoted by $(X,Y)$. The closure of $A$ in $X$ will be written $cl_{F}A$ (or $cl A$ when no confusion is possible) and for $x \in X$, $cl_{F}x$ means $cl(x)$. $R$ will always denote the $T_{0}$-space obtained by endowing the real line with its lower topology (i.e. the topology having as non-trivial open sets those of the form $\{x \in R: x > a\} \cup \{x \in R\}$).

Let $X$ be any $T_{0}$-space. One can define a second topology on $X$—the front topology — by specifying the front-closure operator $fel$ as follows: $x \in fel A$ means that for any neighbourhood $N$ of $x$, $N \cap cl_{F}A \neq \emptyset$. The name is motivated by the fact that for $A \subseteq R$, $fel A$ is obtained by adjoining to $A$ those points in $cl_{F}A$ which lie "in front" of some points of $A$. It is easy to verify that $fel$ is a Kuratowski closure operator. This topology is the same as the $b$-topology of [6].

We note in passing that the front topology on $X$ is discrete iff $X$ is a $T_{0}$-space (see [7]) and that if $X$ is a non-discrete $T_{0}$-space then the front topology is strictly larger than the original topology.

2. Equalizers and extremal subobjects in the category of $T_{0}$-spaces. The category of all $T_{0}$-spaces with continuous maps will be denoted by $\mathcal{C}_{0}$.

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