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Fixed point theorems for non-compact approximative ANR's*

by

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1. Introduction. Compact approximative ANR's were first introduced by H. Noguchi and in 1968 A. Granas proved that they were Lefschetz spaces. More recently J. Jaworowski proved the Lefschetz fixed point theorem for upper semi-continuous, acyclic multi-valued maps of these compact approximative ANR's.

On the other hand there is much recent interest in Lefschetz fixed point theory for compact maps of non-compact spaces. A space X is a Λ-space if for every compact continuous map $f\colon X\to X$, the Lefschetz number $\Lambda(f)$ exists and f has a fixed point whenever $\Lambda(f)\neq 0$. For example, ANR's are Λ-spaces ([2], [4]). The corresponding concepts, M-Lefschetz space and $M\Lambda$-space, for certain multi-valued maps have been studied. (The maps used need not be acyclic; it suffices to require that they be compositions of acyclic maps.) ANR's are known to be $M\Lambda$-spaces. (See [7].)

In this note, it is shown that (non-compact) approximative ANR's are Λ-spaces. It is also proved that a second related class of spaces are $M\Lambda$-spaces.

2. Preliminary definitions. In this section we recall the pertinent facts about multi-valued maps, establish the homology theories under which we will be working, and recall the definitions of Λ-space and $M\Lambda$-space. The reader is referred to [7] for the details of this section.

A map is said to be *compact* if its image is contained in a compact set. A multi-valued map $F\colon X\to Y$ is *upper semi-continuous* (u.s.c.) if

(i) $F(x)$ is compact for each x in X and

(ii) for each x in X and each open set V containing $F(x)$, there is an open neighborhood U of x such that $F(U)\subseteq V$.

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If $F\colon X\to Y$ and $G\colon Y\to Z$ are multi-valued maps, the *composition* of F and G is denoted $G\circ F\colon X\to Z$ and is defined by $G\circ F(x) = \bigsqcup_{y\in F(x)} G(y)$. A point x is a *fixed point* for $F\colon X\to X$ if $x\in F(x)$.

Next with regard to the homology theories, let $\mathfrak{J}$ denote the category of Hausdorff spaces and continuous maps and $\mathcal{A}$ the category of graded vector spaces and homomorphisms of degree zero. $\mathcal{H}\colon \mathfrak{J}\to\mathcal{A}$ can be any covarient functor which has compact support, satisfies the homotopy axiom, and agrees with the Čech homology functor $\check{H}$ on the full subcategory of compact spaces. In particular, when working with multi-valued maps we will use the functor $\vec{H}$. ($\vec{H}(X)$ is the direct limit of groups $\check{H}(K)$ taken over the compact subsets K of X. See [6].) Let $F\colon X\to Y$ be an u.s.c. multi-valued map of spaces in $\mathfrak{J}$. F is *acyclic* (w.r.t. $\vec{H}$) if for each x in X, $F(x)$ is an acyclic subset of Y. When F is acyclic the induced homomorphism $\vec{H}(F) = F_*$ is defined ((2.1) [7]).

Finally, we recall the definitions of Λ-space and $M\Lambda$-space. $\mathfrak{J}_0$ will denote a subcategory of $\mathfrak{J}$. Refer to § 2 of [7] for the definitions of generalized trace, finite type, and Lefschetz number for homomorphisms in $\mathcal{A}$.

(2.1) Definition. An u.s.c. map $F\colon X\to X$ of a space in $\mathfrak{J}$ is *admissible* (rel. to $\mathfrak{J}_0$) if there are maps $G_i\colon Y_i\to Y_{i+1}$, $i = 0, \dots, n$ (where $Y_0 = Y_{n+1} = X$) satisfying

(i) $F = G_n\circ\dots\circ G_0$,

(ii) G_i is acyclic and u.s.c. for each $i = 0, \dots, n$, and

(iii) Y_i is in $\mathfrak{J}_0$ for $i = 1, \dots, n$.

Each such sequence is called an *admissible sequence* for F.

(2.2) Definition. An admissible map $F\colon X\to X$ is an *M-Lefschetz map* (rel. to $\mathfrak{J}_0$) if

(i) for each admissible sequence $G_0, \dots, G_n$ for F, $G_{n*}\circ\dots\circ G_{0*}$ has finite type and

(ii) whenever $G_0, \dots, G_n$ is an admissible sequence with Lefschetz number $\Lambda(G_{n*}\circ\dots\circ G_{0*})\neq 0$, then F must have a fixed point.

A continuous single-valued map $f\colon X\to X$ is a *Lefschetz map* if $\mathcal{H}(f) = f_*$ has finite type and whenever $\Lambda(f_*)\neq 0$, then f has a fixed point.

(2.3). Definition. A space X is an *$M\Lambda$-space* (rel. to $\mathfrak{J}_0$) if each compact admissible map $F\colon X\to X$ (rel. to $\mathfrak{J}_0$) is an M-Lefschetz map (rel. to $\mathfrak{J}_0$).

A space X is a Λ-space if each compact continuous (single-valued) map $f\colon X\to X$ is a Lefschetz map.

3. Main results. In this section we identify the two topological categories in which we are interested and state the two main theorems.

First consider two maps $f, g\colon X\to Y$ and $\alpha\in \operatorname{Cov} Y$. Recall that f and g are *α-near* if for each x in X, there is an element U of α containing both $f(x)$ and $g(x)$. Similarly, if Y is a metric space and ε is a positive real number, f and g are said to be *ε-near* if for each x in X, $d(f(x), g(x)) < \varepsilon$.

(3.1) Definition. Let X be a subspace of Y. For $\alpha\in\operatorname{Cov} X$, a continuous map $r_\alpha\colon Y\to X$ is an α-retraction if $r_\alpha|_X$ and 1_X are α-near. X is an *approximative retract* of Y if for each $\alpha\in\operatorname{Cov} X$ there is an α-retraction $r_\alpha\colon Y\to X$. X is an *approximative neighborhood retract* of Y if there is an open set U in Y such that $X\subseteq U$ and X is an approximative retract of U.

(3.2) Definition. Let Y be a metric space, X a subspace and ε a positive real number. A continuous map $r_\varepsilon\colon Y\to X$ is an *ε-retraction* if $r_\varepsilon|_X$ and 1_X are ε-near. X is a *weak approximative retract* of Y if for each $\varepsilon > 0$ there is an ε-retraction $r_\varepsilon\colon Y\to X$. X is a *weak approximative neighborhood retract* of Y if there is an open set U in Y such that $X\subseteq U$ and X is a weak approximative retract of U.

(3.3) Definition. X is a (metric) *approximative absolute neighborhood retract* (A-ANR) if for each homeomorphism $h\colon X\to M$ with M a metric space and $h(X)$ closed in M, the space $h(X)$ is an approximative neighborhood retract of M.

(3.4) Definition. X is a (metric) *weak approximative absolute neighborhood retract* (WA-ANR) if for each homeomorphism $h\colon X\to M$ with M a metric space and $h(X)$ closed in M, the space $h(X)$ is a weak approximative neighborhood retract of M.

It is clear that the category of WA-ANR's contains the category of A-ANR's and that restricting to compact spaces the two categories coincide.

(3.5) Remark. Using the Kuratowski–Wojdyslawski embedding theorem ([1], p. 79) any metric space X can be embedded as a closed subset of a convex set C in a Banach space. If X is an A-ANR [or a WA-ANR], there is an open set U in C such that X is an approximative retract [or weak approximative retract] of U. The notation C and U will be used throughout.

It is necessary to consider an additional restriction on the WA-ANR's. Its significance is explained in Remark (5.3).

(3.6) Condition $\mathcal{K}$. *A WA-ANR X satisfies condition $\mathcal{K}$ if for each compact subset K of U it is possible to choose ε-retractions $r_\varepsilon\colon U\to X$ for each $\varepsilon > 0$ such that $\overline{\bigsqcup_{\varepsilon>0} r_\varepsilon(K)}$ is compact.*

Theorem 1. *Every A-ANR is a Λ-space.*

Theorem 2. *Every WA-ANR which satisfies condition $\mathcal{K}$ is an $M\Lambda$-space (rel. to $\mathfrak{J}_M$) and hence also a Λ-space. ($\mathfrak{J}_M$ denotes the category of metric spaces.)*

4. Proof of Theorem 1. Recall that we are using the notation of (3.5).

(4.1) LEMMA. *Let X be an A-ANR and let $j\colon X\to U$ denote the inclusion map. Then for any continuous compact map $f\colon X\to X$, $\ker j_*\subseteq \ker f_*$, where $j_*\colon \mathcal{H}(X)\to\mathcal{H}(U)$ and $f_*\colon \mathcal{H}(X)\to\mathcal{H}(X)$.*

Proof. Take σ in $\ker j_*$. Since $\mathcal{H}$ has compact carriers and agrees with $\check{H}$ on compact spaces, there is a compact subset K of X and $\varkappa$ in $\check{H}(K)$ such that $i_*(\varkappa)=\sigma$ (where $i\colon K\to X$ is the inclusion). We may assume that $f(X)\subseteq K$. Let $f_K\colon K\to K$ be defined by f. To show that $f_*(\sigma)=0$ it will suffice to show $(f_K)_*(\varkappa)=0$, $(f_K)_*\colon \check{H}(K)\to\check{H}(K)$.

Now since K is compact and U is open in C there is an $\varepsilon>0$ such that $B_C(K;\varepsilon)=\{y\in C|\ d(K,y)<\varepsilon\}\subseteq U$. Take n_0 such that $1/n_0<\varepsilon$ and consider $n\geqslant n_0$. For x,y in K with $d(x,y)\leqslant 1/n$ we have $P_{x,y}\subseteq B_C(K;\varepsilon)\subseteq U$, where $P_{x,y}$ is the segment in C connecting x and y. Let V_n be the union of all segments in $\{P_{x,y}|\ x,y\in K,\ d(x,y)\leqslant 1/n\}$. Then $\{V_n\}_{n\geqslant n_0}$ is a nested family of compact sets and $K=\bigcap_{n\geqslant n_0}V_n$.

By the continuity of $\check{H}$, $\check{H}(K)=\varprojlim\{\check{H}(V_n)\}_{n\geqslant n_0}$. Let $f_n\colon K\to V_n$ be the composition of f_K with the inclusion $K\to V_n$. Then to show that $(f_K)_*(\varkappa)=0$ it suffices to show $f_{n*}(\varkappa)=0$ for each $n\geqslant n_0$.

For $n\geqslant n_0$ let a_n be the covering of X by balls of radius $1/2n$, $a_n=\{B_X(x;1/2n)|\ x\in X\}$ and let $\beta=f^{-1}a_n\in \operatorname{Cov}X$. Since X is an A-ANR we have the β-retraction $r_\beta\colon U\to X$. Consider the diagram, where j' is the inclusion and f' is defined by f.

$$\begin{array}{ccc} K & \xrightarrow{f_n} & V_n \\ {\scriptstyle j'}\downarrow & & \uparrow{\scriptstyle f'} \\ U & \xrightarrow[r_\beta]{} & X \end{array}$$

Since $r_\beta|_X$ and 1_X are β-near, for each x in K there is some $f^{-1}(B_X(\bar{x};1/2n))$ in β containing x and $r_\beta(x)$. Thus $d(f_n(x),f'\circ r_\beta\circ j'(x))=d(f(x),f\circ r_\beta(x))<1/n$ and $P_{f_n(x),f'\circ r_\beta\circ j'(x)}\subseteq V_n$. Hence the diagram is homotopy commutative and $f_{n*}=f'_*\circ r_{\beta*}\circ j'_*$. Finally $j_*(\sigma)=0$ implies $j'_*(\varkappa)=0$ and hence $f_{n*}(\varkappa)=0$. Q.E.D.

(4.2) LEMMA. *Let X be an A-ANR and let $j\colon X\to U$ denote the inclusion. There is an a_0 in $\operatorname{Cov}X$ such that for each refinement a of a_0 and each a-retraction $r_a\colon U\to X$, $j_*=j_*\circ r_{a*}\circ j_*$.*

Proof. Since U is open in C and $X\subseteq U$, for each x in X there is an $\varepsilon_x>0$ such that $B_C(x;\varepsilon_x)\subseteq U$. Moreover, the ball is convex since C is convex. Let $a_0=\{B_X(x;\varepsilon_x)|\ x\in X\}\in\operatorname{Cov}X$.

Now take $a>a_0$ and an a-retraction $r_a\colon U\to X$. For each x in X there

is an element 0 of a containing x and $r_a(x)$. Thus for some $\bar{x}$ in X we have $x,r_a(x)$ in $0\subseteq B_X(\bar{x};\varepsilon_{\bar{x}})\subseteq B_C(\bar{x};\varepsilon_{\bar{x}})\subseteq U$ and hence the segment $P_{x,r_a(x)}$ is contained in U. Finally we conclude that j is homotopic to $j\circ r_a\circ j$. Q.E.D.

Theorem 1 is now an easy consequence of the following generating theorem. (See [4], (4.1).)

(4.3) THEOREM. *Let X be a T_3-space and D a cofinal subset of $\operatorname{Cov}X$. Let $f\colon X\to X$ be a continuous compact map. Suppose that for each a in D there is a Λ-space Y_a and continuous maps $g_a\colon X\to Y_a$ and $h_a\colon Y_a\to X$ satisfying*

(a) *h_a is compact,*

(b) *$h_{a*}\circ g_{a*}=f_*$, and*

(c) *$h_a\circ g_a$ and f are a-near.*

Then f is a Lefschetz map.

THEOREM 1. *Each A-ANR is a Λ-space.*

Proof. Let $f\colon X\to X$ be a continuous compact map of an A-ANR. We use (4.3) to show that f is a Lefschetz map. Let a_0 be as in (4.2); then $D=\{a\in\operatorname{Cov}X|\ a>a_0\}$ is cofinal in $\operatorname{Cov}X$.

For a in D let $Y_a=U$, an ANR and hence a Λ-space. Also let $g_a=j\colon X\to U$ and $h_a=f\circ r_{a'}\colon U\to X$, where $r_{a'}$ is an a'-retraction and $a'=f^{-1}a$. Then $f\circ r_{a'}$ is compact since f is compact and $f\circ r_{a'}\circ j$ and f are a-near since $r_{a'}\circ j$ and 1_X are a'-near. Finally, it follows from (4.1) and (4.2) that $f_*\circ r_{a'*}\circ j_*=f_*$ and (4.3) applies. Q.E.D.

5. Proof of Theorem 2. The proof of Theorem 2 relies heavily on Lemma (5.1); the proof of the Lemma is postponed to the end of the section.

(5.1) LEMMA. *Let X be a WA-ANR satisfying condition $\mathcal{K}$. Then $j_*\colon \vec{H}(X)\to\vec{H}(U)$ is a monomorphism, where j is the inclusion.*

The next Lemma corresponds to (4.2) of the previous section.

(5.2) LEMMA. *Let X be a WA-ANR and let $j\colon X\to U$ denote the inclusion. Then given a compact subset K of X, there is a positive real number ε_K such that for each positive $\varepsilon\leqslant\varepsilon_K$ and for each ε-retraction $r_\varepsilon\colon U\to X$*

$$j_*\circ r_{\varepsilon*}\circ j_*\circ i_*=j_*\circ i_*\colon \mathcal{H}(K)\to\mathcal{H}(U),$$

where $i\colon K\to X$ is the inclusion.

If, in addition, X satisfies condition $\mathcal{K}$, then $r_{\varepsilon*}\circ j_*\circ i_*=i_*$.

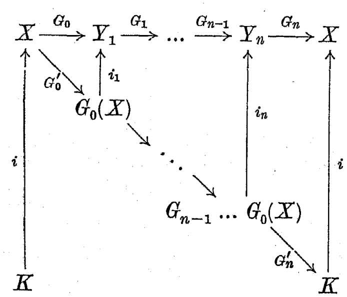
Proof. Since K is a compact subset of U, there is an $\varepsilon_K>0$ such that $B_C(K;\varepsilon_K)\subseteq U$. Take a positive $\varepsilon\leqslant\varepsilon_K$ and an ε-retraction $r_\varepsilon\colon U\to X$. Then for $x\in K$, $d(j\circ r_\varepsilon\circ j\circ i(x),j\circ i(x))=d(r_\varepsilon(x),x)<\varepsilon$ and hence the segment $P_{x,r_\varepsilon(x)}$ is contained in U. Thus $j\circ i$ is homotopic to $j\circ r_\varepsilon\circ j\circ i$ and the induced homomorphisms are equal.

When X satisfies condition $\mathcal{K}$, (5.1) implies that $r_{\varepsilon *} \circ j_* \circ i_* = i_*$. Q.E.D.

There is no generating theorem in [7] which applies directly to give the proof of Theorem 2. However, the method of proof is analogous to that used in [7].

THEOREM 2. *Each* WA-ANR *satisfying condition* $\mathcal{K}$ *is an* $M\Lambda$-*space* (*rel. to* $\mathfrak{I}_M$) *and hence also a* Λ-*space.*

Proof. Let X be a WA-ANR satisfying condition $\mathcal{K}$ and let $F: X \to X$ be a compact admissible map (rel. to $\mathfrak{I}_M$). Pick a compact set K in X containing $F(X)$. Let $G_0, \ldots, G_n$ be an admissible sequence for F. We have the commutative diagram, where the i_k's are inclusion maps and each G'_k is defined by G_k.

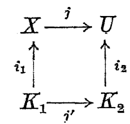


Now ε_K is determined by (5.2) and for each ε-retraction r_ε: $U \to X$ with $\varepsilon \leqslant \varepsilon_K$, $j \circ i \circ G'_n \circ \ldots \circ G'_0 \circ r_\varepsilon$: $U \to U$ is a compact admissible map. Since U is an ANR, it is an $AM\Lambda$-space (see (5.5) in [7]) and hence $j_* \circ i_* \circ G'_{n*} \circ \ldots \circ G'_{0*} \circ r_{\varepsilon *}$ has finite type. Then by Lemma (2.5) of [7], $G'_{n*} \circ \ldots \circ G'_{0*} \circ r_{\varepsilon *} \circ j_* \circ i_*$ also has finite type and their Lefschetz numbers are equal. By (5.2) $G'_{n*} \circ \ldots \circ G'_{0*} \circ r_{\varepsilon *} \circ j_* \circ i_* = G'_{n*} \circ \ldots \circ G'_{0*} \circ i_*$. Applying (5.2), [7] again, $i_* \circ G'_{n*} \circ \ldots \circ G'_{0*} = G_{n*} \circ \ldots \circ G_{0*}$ has finite type and $\Lambda(G_{n*} \circ \ldots \circ G_{0*}) = \Lambda(j_* \circ i_* \circ G'_{n*} \circ \ldots \circ G'_{0*} \circ r_{\varepsilon *})$.

Suppose that $G_0, \ldots, G_n$ is an admissible sequence for F and that $\Lambda(G_{n*} \circ \ldots \circ G_{0*}) \neq 0$. We show that F must have a fixed point. For each integer n with $1/n \leqslant \varepsilon_K$, let r_n: $U \to X$ be a $1/n$-retraction. Then $\Lambda(j_* \circ i_* \circ G'_{n*} \circ \ldots \circ G'_{0*} \circ r_{n*}) \neq 0$. Let y_n be a fixed point for $j \circ i \circ G'_n \circ \ldots \circ G'_0 \circ r_n$ and let $x_n = r_n(y_n)$. Then $y_n \in j \circ i \circ G'_n \circ \ldots \circ G'_0 \circ r_n(y_n) = j \circ F \circ r_n(y_n) = F(x_n) \subseteq K$. Moreover, $d(x_n, y_n) = d(r_n(y_n), y_n) < 1/n$. Thus $\{y_n\}$ is a sequence in K and has a subsequence converging to y_0 in K. The corresponding subsequence of $\{x_n\}$ also converges to y_0. Hence we have a sequence of terms (x_n, y_n) in the graph of F converging to (y_0, y_0).

But since F is u.s.c. the graph of F is closed and hence contains (y_0, y_0); thus $y_0 \in F(y_0)$. Q.E.D.

Proof of (5.1). Suppose that $j_*(\sigma) = 0$, j_*: $\vec{H}(X) \to \vec{H}(U)$. Then there are compact sets $K_1 \subseteq X$ and $K_2 \subseteq U$ with $K_1 \subseteq K_2$ and $\varkappa \in \check{H}(K_1)$ such that



$i_{1*}(\varkappa) = \sigma$ and $j'_*(\varkappa) = 0$. (The construction of $\vec{H}$ as a direct limit assures the existence of K_2.)

Applying condition $\mathcal{K}$, there are ε-retractions r_ε: $U \to X$ for each $\varepsilon > 0$ such that $K = K_1 \sqcup \big(\bigsqcup_{\varepsilon > 0} r_\varepsilon(K_2)\big)$ is a compact subset of X. The remainder of the proof parallels (4.1). We show that $\varkappa$ projects to 0 in $\check{H}(K)$ under the homomorphism induced by the inclusion $K_1 \to K$; then $\varkappa$ projects to $\sigma = 0$ in $\vec{H}(X)$. Take $\varepsilon > 0$ such that $B_C(K; \varepsilon) \subseteq U$. Take a positive integer n_0 with $1/n_0 < \varepsilon$ and for each $n \geqslant n_0$ form the compact set V_n as in the proof of (4.1). Then $K = \bigcap_{n \geqslant n_0} V_n$ and $\check{H}(K) = \varprojlim \{\check{H}(V_n)\}_{n \geqslant n_0}$. It suffices to show that $i_{n*}(\varkappa) = 0$ for each $n \geqslant n_0$, where i_n: $K_1 \to V_n$ is the inclusion.

We have the $1/n$ retraction $r_n = U \to X$ and $r_n(K_2) \subseteq K \subseteq V_n$. Let r'_n: $K_2 \to V_n$ be defined by r_n. Then for x in K_1, $d(i_n(x), r'_n \circ j'(x)) = d(x, r_n(x)) < 1/n$ and $P_{x, r_n(x)} \subseteq V_n$. This implies that i_n is homotopic to $r'_n \circ j'$ and hence $i_{n*}(\varkappa) = r'_{n*} \circ j'_*(\varkappa) = 0$. Q.E.D.

(5.3) Remark. Several unanswered questions remain. A crucial point is whether j_*: $\mathcal{H}(X) \to \mathcal{H}(U)$ is a monomorphism. To prove that j_* is a monomorphism in (5.1) it was necessary to add condition $\mathcal{K}$; but in doing so the results could be extended from the category of A-ANR's to that of WA-ANR's. Can (5.1) be proved if condition $\mathcal{K}$ is weakened or perhaps omitted entirely? Lemma (4.1) draws a weaker conclusion than (5.1), but it is sufficient for our purposes. Can (4.1) be used to show that in fact $\ker j_* = \{0\}$? Can (4.1) be extended to include multi-valued, admissible, compact maps F: $X \to X$? An affirmative answer to either question will assure that A-ANR's are $M\Lambda$-spaces.

Finally, we remark that if X is a compact A-ANR, then condition $\mathcal{K}$ is automatic and Theorem 2 implies that X is an M-Lefschetz space (Def. (3.3), [7]). J. Jaworowski proved this result in [3] when considering only acyclic maps.

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Epireflections in the category of T_0-spaces*

by

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A T_0-space with the property that every non-empty irreducible closed set is a point closure will be called a pc-*space*. It is shown that the pc-spaces form an epireflective subcategory of the category $\mathcal{C}_0$ of all T_0-spaces, generated in $\mathcal{C}_0$ by any pc-space which contains a copy of the Alexandroff dyad.

A larger simply generated, epireflective subcategory of $\mathcal{C}_0$—the fc-spaces—is introduced and it is shown that an fc-space is an invariant of its lattice of real-valued lower semi-continuous functions. As a preliminary it is shown that equalizers in $\mathcal{C}_0$ correspond to "front-closed" subspaces.

1. Preliminaries. The set of continuous maps from X to Y will be denoted by (X, Y). The closure of A in X will be written $\mathrm{cl}_X A$ (or $\mathrm{cl}A$ when no confusion is possible) and for $x \in X$, $\mathrm{cl}x$ means $\mathrm{cl}\{x\}$. R will always denote the T_0-space obtained by endowing the real line with its lower topology (i.e. the topology having as non-trivial open sets those of the form $\{x \in R: x > a\}$ $a \in R$).

Let X be any T_0-space. One can define a second topology on X—the *front topology* — by specifying the front-closure operator fcl, as follows: $x \in \mathrm{fcl}A$ means that for any neighbourhood N of x, $N \cap \mathrm{cl}x \cap A \neq \emptyset$. The name is motivated by the fact that for $A \subset R$, $\mathrm{fcl}A$ is obtained by adjoining to A those points in $\mathrm{cl}A$ which lie "in front" of some points of A. It is easy to verify that fcl is a Kuratowski closure operator. This topology is the same as the b-topology of [6].

We note in passing that the front topology on X is discrete iff X is a T_D-space (see [7]) and that if X is a non-discrete T_0-space then the front topology is strictly larger than the original topology.

2. Equalizers and extremal subobjects in the category of T_0-spaces. The category of all T_0-spaces with continuous maps will be denoted by $\mathcal{C}_0$.

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