On decompositions of smooth continua (1)

by

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Introduction. In [2] Charatonik proved that for every λ-dendroid M there exists a decomposition $\mathfrak{D}$ of M such that

(i) $\mathfrak{D}$ is upper semicontinuous,

(ii) the elements of $\mathfrak{D}$ are continua,

(iii) the decomposition space of $\mathfrak{D}$ is a dendroid, and

(iv) $\mathfrak{D}$ is the finest decomposition among all decompositions satisfying (i), (ii), and (iii).

It is the main purpose of this paper to prove a similar decomposition theorem for a class of compact Hausdorff continua which will be called smooth. Although for certain λ-dendroids (called monostratiform) the decomposition defined by Charatonik consists of a single element [3], the decomposition obtained for smooth continua is never degenerate; in fact, each element of the decomposition has void interior.

In order to obtain the above-mentioned result it is necessary to extend the notion of smoothness (e.g., [4]) to a more general class of continua than dendroids. Mohler [9] has observed that the definition of smoothness is valid for the class of hereditarily unicoherent continua. However, it is easily seen that the definition is applicable more generally to any continuum M satisfying

(*) There exists a point $p \in M$ such that for each point $x \in M$ there exists a unique subcontinuum which is irreducible between p and x.

A second purpose of this paper is to discuss some of the basic properties of continua which satisfy (*) and the smoothness condition. Some of the theorems obtained are generalizations of known results concerning smooth dendroids and generalized trees.

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1. Definitions and preliminary remarks. A *continuum* is a compact connected Hausdorff space. An *arc* is a continuum (not necessarily metrizable) which is irreducibly connected between some pair of points. The continuum M is *unicoherent* provided that the intersection of any two subcontinua whose union is M is connected. The continuum M is *hereditarily unicoherent* in case every subcontinuum of M is unicoherent.

In order to generalize the notion of hereditary unicoherence we make the following definition:

The continuum M is *hereditarily unicoherent at* p in case the intersection of any two subcontinua, each of which contains p, is connected.

The following theorems are immediate consequences of the above definitions. Note that Theorem 1.3 states that (∗) is equivalent to hereditary unicoherence at a point.

THEOREM 1.1. *The continuum M is hereditarily unicoherent if and only if M is hereditarily unicoherent at each point.*

THEOREM 1.2. *The continuum M is hereditarily unicoherent if and only if given any two points of M there exists a unique subcontinuum which is irreducible between them.*

THEOREM 1.3. *The continuum M is hereditarily unicoherent at p if and only if given any point $x \in M$ there exists a unique subcontinuum which is irreducible between p and x.*

Notation. If the continuum M is hereditarily unicoherent at p and $q \in M-\{p\}$ then pq will denote the unique subcontinuum which is irreducible between p and q.

If $\{A_n, n \in D\}$ is a net of subsets of a topological space X then $\mathrm{Li}\, A_n = \{x \in X;$ every neighborhood of x intersects A_n for almost all $n\}$ and $\mathrm{Ls}\, A_n = \{x \in X;$ every neighborhood of x intersects A_n for arbitrarily large $n\}$. A net of subsets $\{A_n, n \in D\}$ is said to *converge to a set* A (denoted by $\mathrm{Lim}\, A_n = A$) in case $\mathrm{Li}\, A_n = A = \mathrm{Ls}\, A_n$. The reader is referred to [7] for a general discussion of nets, and to [10] for properties of nets of subsets.

The following definition is a natural generalization of the definition of smooth dendroid given in [4].

A continuum M is said to be *smooth at the point* p in case M is hereditarily unicoherent at p and for each convergent net of points $\{a_n, n \in D\}$ in M the condition

(i) $\lim a_n = a$

implies that

(ii) $\{pa_n, n \in D\}$ is convergent, and

(iii) $\mathrm{Lim}\, pa_n = pa$.

For an arbitrary continuum M the set $\{p \in M;\ M$ is smooth at $p\}$ is called the *initial set* of M and will be denoted by $I(M)$. If $I(M) \neq \emptyset$, then M is said to be *smooth*.

A tree [12], Theorem 9, page 803, is a hereditarily unicoherent and locally connected continuum. A *generalized tree* is a hereditarily unicoherent, arcwise connected continuum which is smooth. According to a theorem of Koch and Krule [8], the above definition of generalized tree is equivalent to the definition originally given by Ward [12].

2. Some properties of nets of sets. The first two theorems of this section are generalizations of known results concerning convergence of sequences of subsets.

THEOREM 2.1. *Let $\{A_n, n \in D\}$ be a net of connected sets in a compact Hausdorff space S. If $\mathrm{Li}\, A_n \neq \emptyset$ then $\mathrm{Ls}\, A_n$ is a continuum.*

Proof. The proof is obtained by generalizing the argument of [6], Theorem 2–101, page 101, so that it applies to nets. The details will be omitted.

THEOREM 2.2. *Let the continuum M be hereditarily unicoherent at p. If $\{A_n, n \in D\}$ is a net of connected sets each of which contains p, then $\mathrm{Li}\, A_n$ is a continuum.*

Proof. The argument contained in [1], Lemma 1, page 6, is easily generalized to nets.

LEMMA 2.1. *Let $\{A_n, n \in D\}$ be a net of subsets in a topological space S. If $a \in \mathrm{Li}\, A_n$ and $b \in \mathrm{Ls}\, A_n$ then there exist nets $\{A_m, m \in E\}$, $\{a_m, m \in E\}$, and $\{b_m, m \in E\}$ such that*

(i) *$\{A_m, m \in E\}$ is a subnet of $\{A_n, n \in D\}$,*

(ii) *$\{a_m, m \in E\}$ converges to a,*

(iii) *$\{b_m, m \in E\}$ converges to b, and*

(iv) *for each $m \in E$, $a_m \in A_m$ and $b_m \in A_m$.*

Proof. The proof is obtained by applying standard techniques involving nets (see, for example, [7], Lemma 5, page 70).

THEOREM 2.3. *Let the continuum M be hereditarily unicoherent at p. If M is not smooth at p then there exist nets $\{a_m, m \in E\}$ and $\{b_m, m \in E\}$ such that*

(i) *$\{a_m, m \in E\}$ converges to a,*

(ii) *$\{b_m, m \in E\}$ converges to b,*

(iii) *$b_m \in pa_m$ for each $m \in E$, and*

(iv) *$b \in M-pa$.*

Proof. Since M is not smooth at p there exists a convergent net $\{a_n, n \in D\}$ such that $\lim a_n = a$ and either

(a) $\{pa_n, n \in D\}$ is not convergent,

or

(b) $\{pa_n, n \in D\}$ is convergent and $\operatorname{Lim} pa_n \neq pa$.

Since $\operatorname{Li} A_n$ is a continuum containing p and a we have

$$pa \subset \operatorname{Li} pa_n \subset \operatorname{Ls} pa_n .$$

Thus for either case (a) or (b) it follows that

$$pa \neq \operatorname{Ls} pa_n .$$

Choose $b \in \operatorname{Ls} pa_n - pa$. Now the conclusion of the theorem follows immediately from Lemma 2.1.

3. The initial set. This section contains a number of theorems concerning the initial set $I(M)$ of an arbitrary continuum M. In particular we obtain generalizations of each of the following results:

(a) A generalized tree is locally connected at each point of $I(M)$ [8].

(b) A dendroid M is a dendrite if and only if $I(M) = M$ [4].

THEOREM 3.1. *Let M be a smooth continuum and let N be a subcontinuum of M such that $N \cap I(M) \neq \emptyset$. Then for each open set V which contains N there exists a connected open set U such that $N \subset U \subset V$.*

Proof. Let $p \in N \cap I(M)$ and define

$$U = \{x \in M;\ px \cap (M - V) = \emptyset\} .$$

Clearly $N \subset U \subset V$ and U is connected. It remains to be shown that U is open.

Let $\{x_n, n \in D\}$ be a net in $M - U$ which converges to x. Since

$$px_n \cap (M - V) \neq \emptyset ,$$

it follows that

$$\operatorname{Lim} px_n \cap (M - V) \neq \emptyset .$$

Since M is smooth at p,

$$px \cap (M - V) \neq \emptyset .$$

Consequently $x \in M - U$. Thus $M - U$ is closed, i.e., U is open.

COROLLARY 3.1. *A smooth continuum M is locally connected at each point of $I(M)$.*

THEOREM 3.2. *The continuum M is a tree if and only if $I(M) = M$.*

Proof. Suppose that M is a tree and $p \in M$. That $p \in I(M)$ follows immediately from the fact that M is locally connected.

If $I(M) = M$ then by definition M is hereditarily unicoherent at each point. According to Lemma 1.1, M is hereditarily unicoherent; by Corollary 3.1, M is locally connected. Consequently M is a tree.

THEOREM 3.3. *Let M be a smooth continuum and suppose that N is a subcontinuum of M such that $N \cap I(M) \neq \emptyset$. Then, for each subcontinuum H of M, $N \cap H$ is a continuum.*

Proof. Suppose, on the other hand, that $N \cap H = K_1 \cup K_2$ (separation). Let J denote a subcontinuum of H which is irreducible between K_1 and K_2. For $i \in \{1, 2\}$, let $k_i \in K_i \cap J$. Let $\{a_n, n \in D\}$ be a net of points in the k_1-composant of J which converges to k_2. (The existence of such a net follows from [6], Theorem 3–44, page 140.)

Let U be a relatively open subset of $J \cup N$ such that $\operatorname{cl}(U) \subset J - N$. Since J is irreducible between k_1 and k_2, it follows that $\operatorname{cl}(U)$ separates J into two relatively open sets, one containing $K_1 \cap J$, the other containing $K_2 \cap J$.

Choose $p \in I(M) \cap N$. Now there exists an $n_0 \in D$ such that for $n \geqslant n_0$,

$$pa_n \cap \operatorname{cl}(U) \neq \emptyset .$$

Consequently

$$\operatorname{Lim} pa_n \cap \operatorname{cl}(U) \neq \emptyset ,$$

that is,

$$pk_2 \cap \operatorname{cl}(U) \neq \emptyset .$$

But

$$pk_2 \subset N \subset M - \operatorname{cl}(U) ,$$

which is a contradiction.

COROLLARY 3.2. *Every smooth continuum is unicoherent.*

COROLLARY 3.3. *Every indecomposable subcontinuum of a smooth continuum has void interior.*

Proof. Suppose that M is a smooth continuum and N is an indecomposable subcontinuum with non-void interior. Let $p \in I(M)$. If $p \in M - N$, let C_p denote the closure of the component of $M - N$ which contains p; if $p \in N$, let $C_p = \{p\}$. Choose $q \in N - C_p$, and let V be an open set such that

$$C_p \subset V \subset \operatorname{cl}(V) \subset M - \{q\} .$$

According to Theorem 3.1, we may assume that V is connected. By Theorem 3.3, the set $\operatorname{cl}(V) \cap N$ is a continuum. But $\operatorname{cl}(V) \cap N$ has non-void interior relative to N. This contradicts the fact that N is indecomposable [6], Theorem 3–41, page 139.

It is perhaps worth noting that there exists a metric continuum M which is smooth with respect to a point p and such that $M-\{p\}$ is the union of a pairwise disjoint collection of nondegenerate indecomposable continua. An example of such a continuum may be obtained as follows. Let N denote a continuous arc of pseudo-arcs as described by Thomas in [11], page 59. The continuum N is an irreducible plane continuum and there exists a monotone map $f\colon N\to[0,1]$ such that

(i) $f^{-1}(t)$ is a pseudo-arc for each $t\in[0,1]$, and

(ii) $f^{-1}(t)^0=\emptyset$ for each $t\in[0,1]$.

Now define $M=N/f^{-1}(0)$. The continuum M has the desired properties.

4. Monotone mappings on smooth continua. In this section we study monotone mappings on continua which are hereditarily unicoherent at some point and prove that smoothness is an invariant of such mappings.

THEOREM 4.1. *Let M denote a continuum which is hereditarily unicoherent at a point p. Let $f\colon M\to H$ be a monotone map from M onto a Hausdorff space H. Then*

(i) *H is hereditarily unicoherent at $f(p)$,*

(ii) *$f(px)=f(p)f(x)$ for each $x\in M$, and*

(iii) *$f(I(M))\subset I(H)$.*

Proof. (i) Suppose that H is not hereditarily unicoherent at $f(p)$. Then there exist subcontinua R and S such that $f(p)\in R\cap S$ and $R\cap S$ is not connected. Since f is monotone, $f^{-1}(R)$ and $f^{-1}(S)$ are continua containing p. But $f^{-1}(S)\cap f^{-1}(R)$ is not connected which is a contradiction.

(ii) Clearly $f(p)f(x)\subset f(px)$. Suppose that there exists a point $y\in f(px)-f(p)f(x)$, and choose $z\in px$ such that $f(z)=y$. Then $f^{-1}(f(p)f(x))$ is a continuum containing p and x but missing z. Since M is hereditarily unicoherent at p, $f^{-1}(f(p)f(x))\cap px$ is a continuum containing p and x which misses z. This is a contradiction.

(iii) Suppose that M is smooth at p. According to (i), H is hereditarily unicoherent at $f(p)$. Now suppose that H is not smooth at $f(p)$. By Theorem 2.3 there exist nets $\{a_m, m\in E\}$ and $\{b_m, m\in E\}$ such that $\lim a_m=a$, $\lim b_m=b$, $b_m\in pa_m$ for each $m\in E$, and $b\in H-f(p)a$.

By using standard techniques involving nets it is possible to find convergent nets $\{c_n, n\in F\}$ and $\{d_n, n\in F\}$ in M such that

(a) if $\lim c_n=c$ then $f(c)=a$,

(b) $\{f(c_n), n\in F\}$ is a subnet of $\{a_m, m\in E\}$,
$\{f(d_n), n\in F\}$ is a subnet of $\{b_m, m\in E\}$, and

(c) for each $n\in F$, $d_n\in pc_n$.

From (a) and (ii) it follows that $f(pc)=f(p)a$. Since M is smooth at p it follows from (c) that

$$\text{if } \lim d_n=d \text{ then } d\in pc\,.$$

Consequently

$$f(d)\in f(pc)=f(p)a\,.$$

Applying (b) we have

$$f(d)=\lim f(d_n)=\lim b_m=b\,.$$

Thus $b\in f(p)a$; but b was chosen in $H-f(p)a$. This is a contradiction. It follows that H is smooth at $f(p)$.

5. The canonical decomposition of smooth continua. The following result summarizes several theorems from [5].

THEOREM 5.1. *Let M denote a continuum which is irreducible between wo points. If each indecomposable subcontinuum of M has void interior hen there exists a decomposition $\mathcal{E}$ of M such that*

(i) *$\mathcal{E}$ is upper semicontinuous,*

(ii) *the elements of $\mathcal{E}$ are continua,*

(iii) *the decomposition space of $\mathcal{E}$ is an arc, and*

(iv) *if $\mathcal{F}$ is a decomposition satisfying* (i), (ii), *and* (iii) *then $\mathcal{E}\leqslant\mathcal{F}$ (i.e., refines $\mathcal{F}$).*

Furthermore, each element of $\mathcal{E}$ has void interior.

In this section we will apply Theorem 5.1 to obtain an analogous decomposition for smooth continua. For smooth continua the decomposition space is a generalized tree.

Throughout this section let M denote a smooth continuum and let p be a fixed point in $I(M)$. We define an equivalence relation ϱ by the condition

$$(x,y)\in\varrho \text{ if and only if } px=py\,.$$

Let $\varphi\colon M\to M/\varrho$ denote the natural map and let

$$\mathcal{D}=\{\varphi^{-1}(t);\ t\in M/\varrho\}\,.$$

LEMMA 5.1. *Let M be an irreducible continuum. Then the decomposition $\mathcal{D}$ described above coincides with the decomposition $\mathcal{E}$ of Theorem* 5.1.

Proof. For each $x\in M$, let $D(x)$ (resp. $E(x)$) denote the element of $\mathcal{D}$ (resp. $\mathcal{E}$) which contains x. According to Corollary 3.1, M is locally connected at p. Consequently $D(p)=\{p\}=E(p)$.

Let $f\colon M\to[a,b]$ ($[a,b]$ denotes an arc) be the monotone map associated with the decomposition $\mathcal{E}$, i.e.,

$$\mathcal{E}=\{f^{-1}(t);\ a\leqslant t\leqslant b\}.$$

Suppose that $x\in M-\{p\}$ and that $f(p)<f(x)$. We may assume without loss of generality that

$$x\in \operatorname{cl}\{f^{-1}([a,f(x)))\}\cap \operatorname{cl}\{f^{-1}((f(x),b])\}$$

since $E(x)^0=\emptyset$.

It will now be shown that $E(x)\subset D(x)$, i.e., for each $y\in E(x)$, $py=px$. Suppose, without loss of generality, that $y\notin px$. Let $\{x_n,n\in D\}$ be a net in $f^{-1}((f(x),b])$ which converges to x. According to [5], Theorem 2.3, it follows that $E(x)\subset px_n$ for each $n\in D$. Consequently

$$y\in \operatorname{Lim} px_n=px.$$

This is a contradiction; thus $E(x)\subset D(x)$.

Suppose that $w\in D(x)-E(x)$. Without loss of generality we may assume that $f(p)<f(w)<f(x)$. Then $f^{-1}([f(p),f(w)])$ is a continuum containing p and w which misses x. This contradicts the definition of $D(x)$. Consequently $D(x)=E(x)$.

We are now ready to prove that the decomposition $\mathcal{D}$ (called the canonical decomposition of M) has the desired properties. Although $\mathcal{D}$ was defined in terms of a particular point $p\in I(M)$ it will be seen that $\mathcal{D}$ is actually independent of the choice of p.

THEOREM 5.2. *If M is a smooth continuum then there exists a decomposition $\mathcal{D}$ of M such that*

(i) *$\mathcal{D}$ is upper semicontinuous,*

(ii) *the elements of $\mathcal{D}$ are continua,*

(iii) *the decomposition space of $\mathcal{D}$ is arcwise connected, and*

(iv) *if $\mathcal{E}$ is a decomposition satisfying* (i), (ii), *and* (iii) *then $\mathcal{D}\leqslant\mathcal{E}$ (i.e., $\mathcal{D}$ refines $\mathcal{E}$).*

Moreover, the decomposition space of $\mathcal{D}$ is a generalized tree and each element of $\mathcal{D}$ has void interior.

Proof. Let $\mathcal{D}$ denote the decomposition of M described above. (i) In order to prove that $\mathcal{D}$ is upper semicontinuous it suffices to show that ϱ is a closed subset of $M\times M$. Let $\{(x_n,y_n),n\in D\}$ be a net in ϱ which converges to (x,y). By smoothness

$$px=\operatorname{Lim} px_n=\operatorname{Lim} py_n=py.$$

Thus $(x,y)\in\varrho$ and ϱ is closed.

(ii) That the elements of $\mathcal{D}$ are continua, indeed continua with void interior, follows immediately from Lemma 5.1 and Theorem 5.1.

(iii) Let $\varphi(x)$ denote an arbitrary point in $M/\varrho-\{\varphi(p)\}$. Applying Lemma 5.1 to the continuum px it follows that $\varphi(px)$ is an arc containing $\varphi(p)$ and $\varphi(x)$. Thus M/ϱ is arcwise connected.

(iv). Suppose that there exists a mapping $\psi\colon M\to H$ such that the decomposition $\mathcal{E}=\{\psi^{-1}(h);\ h\in H\}$ satisfies (i), (ii), and (iii). If $\mathcal{D}$ does not refine $\mathcal{E}$ then there exists an element $D\in\mathcal{D}$ and elements E_1 and E_2 of $\mathcal{E}$ such that

$$E_i\cap D\neq\emptyset,\quad i\in\{1,2\}.$$

Since H is arcwise connected we may assume that there exists an arc A in H which contains the points $\psi(p)$ and $\psi(E_1)$ but misses $\psi(E_2)$. Now $\psi^{-1}(A)$ is a continuum which contains p and intersects D properly. This contradicts the definition of D; consequently $\mathcal{D}\leqslant\mathcal{E}$.

In order to prove that M/ϱ is a generalized tree we must show that M/ϱ is hereditarily unicoherent. If M/ϱ is not hereditarily unicoherent then there exist subcontinua H and K of M/ϱ such that $H\cap K$ is not connected. Let A denote an arc from $\varphi(p)$ to $H\cup K$ such that

$$A\cap(H\cup K)=\{z\}.$$

Assuming that $z\in H$, define

$$H'=A\cup H.$$

Then $H'\cap K=H\cap K$; hence $H'\cap K$ is not connected. Since φ is monotone $\varphi^{-1}(H')$ and $\varphi^{-1}(K)$ are subcontinua of M such that $\varphi^{-1}(H')\cap\varphi^{-1}(K)$ is not connected. However $\varphi^{-1}(H')\cap I(M)\neq\emptyset$ which contradicts Theorem 3.3.

Thus M/ϱ is hereditarily unicoherent and arcwise connected (iii). By Theorem 4.1, M/ϱ is smooth; hence M/ϱ is a generalized tree.

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Fixed point theorems for non-compact approximative ANR's*

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1. Introduction. Compact approximative ANR's were first introduced by H. Noguchi and in 1968 A. Granas proved that they were Lefschetz spaces. More recently J. Jaworowski proved the Lefschetz fixed point theorem for upper semi-continuous, acyclic multi-valued maps of these compact approximative ANR's.

On the other hand there is much recent interest in Lefschetz fixed point theory for compact maps of non-compact spaces. A space X is a Λ-space if for every compact continuous map $f\colon X\to X$, the Lefschetz number $\Lambda(f)$ exists and f has a fixed point whenever $\Lambda(f)\neq 0$. For example, ANR's are Λ-spaces ([2], [4]). The corresponding concepts, M-Lefschetz space and $M\Lambda$-space, for certain multi-valued maps have been studied. (The maps used need not be acyclic; it suffices to require that they be compositions of acyclic maps.) ANR's are known to be $M\Lambda$-spaces. (See [7].)

In this note, it is shown that (non-compact) approximative ANR's are Λ-spaces. It is also proved that a second related class of spaces are $M\Lambda$-spaces.

2. Preliminary definitions. In this section we recall the pertinent facts about multi-valued maps, establish the homology theories under which we will be working, and recall the definitions of Λ-space and $M\Lambda$-space. The reader is referred to [7] for the details of this section.

A map is said to be *compact* if its image is contained in a compact set. A multi-valued map $F\colon X\to Y$ is *upper semi-continuous* (u.s.c.) if

(i) $F(x)$ is compact for each x in X and

(ii) for each x in X and each open set V containing $F(x)$, there is an open neighborhood U of x such that $F(U)\subseteq V$.

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