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Real functions having graphs connected and dense in the plane

by

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Introduction. In this paper a theorem proved by Jack Brown in [1] is utilized to prove theorems concerning the class of all real functions having graphs connected and dense in the plane. Only real functions will be considered here and the word graph will refer to the graph of a real function.

Definitions and notation. If f is a point set in the plane, then the X-projection of f is the set of all abscissas of points of f and will be denoted by f_x. The statement that the number set M is c-dense in the number set N means that if I is an open interval containing an element of N, then the cardinality of $I \cap (M \cap N)$ is that of the continuum. The cardinality of the continuum will be denoted by c. The set of all real numbers will be denoted by E.

LEMMA 1. *If the graph f has connected X-projection and intersects every lower semi-continuous graph with X-projection a subinterval of the X-projection of f, then f is connected.*

This lemma follows easily from the theorem that Jack Brown states and proves in [1].

THEOREM 1. *If C_1 is a subset of E such that each of C_1 and $E-C_1$ is c-dense in E, then there is a totally disconnected graph g with X-projection C_1, such that if M is a point set containing g and having X-projection E, then M is connected and dense in the plane.*

Proof of Theorem 1. Suppose C_1 is a subset of E such that each of C_1 and $E-C_1$ is c-dense in E.

Let W denote the collection to which w belongs if and only if w is a lower semi-continuous graph with X-projection an interval. The collection W has cardinality c. Let Q be a meaning of precedes such that (1) W is well ordered with respect to Q and (2) if w is an element of the collection W, then the set of all elements of W that precede w has cardinality less than W.

If w is an element of W, then $w_x \cap C_1$ has cardinality c. If w_a is an element of W, let P_a be a point of w_a such that (1) the abscissa of P_a is in C_1 and (2) if w_β is in W and w_β precedes w_a, then the X-projection of P_a is not the X-projection of P_β. (This construction is possible because the set of all elements of W preceding w_a has cardinality less than the cardinality of $(w_a)_x \cap C_1$.) Let g' be the point set to which P belongs if and only if there is an element w_a of W such that P is P_a. No two points of g' have the same X-projection; therefore g' is a graph. Furthermore g' intersects every element of the collection W. The X-projection of g' is a subset of C_1. Let g be a graph containing g' and having X-projection C_1. The X-projection of g is totally disconnected. Thus g is totally disconnected.

Suppose M is a point set containing g such that $M_x = E$. Then M contains a graph f such that f contains g and $f_x = E$. Then f contains g' and therefore intersects every element of W. Thus f intersects every lower semi-continuous graph with X-projection an interval and is therefore dense in the plane. From Lemma 1 it follows that f is connected. Thus f is a connected subset of M that is dense in M. Therefore, M is connected. Obviously M is dense in the plane for the same reasons that f is.

THEOREM 2. *If C_1 is a subset of E such that each of C_1 and $E - C_1$ is c-dense in E and f is a graph such that C_1 is a subset of the X-projection of f, then there is a graph g that is connected and dense in the plane such that the X-projection of $f \cap g$ contains C_1.*

Proof of Theorem 2. Suppose C_1 is a subset of E such that each of C_1 and $E - C_1$ is c-dense in E and f is a graph such that C_1 is a subset of the X-projection of f.

Let C_2 be $E - C_1$. Then each of C_2 and $E - C_2$ is c-dense in E. It follows from Theorem 1 that there is a graph g_1 with X-projection C_2 such that if M is a point set containing g_1 and having X-projection E, then M is connected and dense in the plane.

Let g be a graph such that (1) $g(x) = g_1(x)$ if x is in C_2 and (2) $g(x) = f(x)$ if x is in C_1. Obviously, g_x is E and g contains g_1. Thus g is connected and dense in the plane. Also, it is clear that $(g \cap f)_x$ contains C_1.

THEOREM 3. *If f is a graph with X-projection E, then there exist two graphs, h and k, each connected and dense in the plane, such that if x is a number, then $f(x) = h(x) + k(x)$.*

Proof of Theorem 3. Suppose f is a graph with X-projection E. Let C_1 and C_2 be mutually exclusive subsets of E, each c-dense in E, such that $C_1 \cup C_2$ is E. From Theorem 1 it follows that there is a simple graph h_1, such that $(h_1)_x$ is C_1 and if M is a point set with X-projection E and containing h_1, then M is connected and dense in the plane. Similarly, there is a graph k_2 with X-projection C_2 such that if M is a point set having X-projection E and containing k_2, then M is connected and dense in the plane.

Let h be the graph such that: (1) $h(x) = h_1(x)$ if x is in C_1, and (2) $h(x) = f(x) - k_2(x)$ if x is in C_2. Let k be the graph such that: (1) $k(x) = k_2(x)$ if x is in C_2, and (2) $k(x) = f(x) - h_1(x)$ if X is in C_1. If x is in C_1, then $h(x) + k(x) = h_1(x) + [f(x) - h_1(x)] = f(x)$. If x is in C_2, then $h(x) + k(x) = [f(x) - k_2(x)] + k_2(x) = f(x)$. Thus if x is in E, then $h(x) + k(x) = f(x)$.

THEOREM 4. *If f is a graph with X-projection E then f is the point-wise limit of a sequence $f_1, f_2, f_3, \ldots$, each term of which is a connected graph dense in the plane.*

Proof of Theorem 4. Let $\beta = B_1, B_2, B_3, \ldots$, be a sequence of subsets of E such that (1) each term of β is c-dense in E, (2) no two terms of β intersect, and (3) $\bigcup_{p=1}^{\infty} B_p$ is E. Let $a = A_1, A_2, A_3, \ldots$, be a sequence such that for each positive integer n, $A_n = \bigcup_{p=1}^{n} B_p$. Then if n is a positive integer, A_n is a subset of A_{n+1}, and each of A_n and $E - A_n$ is c-dense in E. If x is in E, there is a positive integer n such that if m is an integer greater than n, then A_m contains x.

From Theorem 2 it follows that for each positive integer n there is a graph f_n that is connected and dense in the plane such that $(f \cap f_n)_x$ includes A_n. Then $f_1, f_2, f_3, \ldots$, is a sequence each term of which is a connected graph dense in the plane, such that for each positive integer n, $(f \cap f_n)_x$ contains A_n. Clearly, f is the point-wise limit of the sequence $f_1, f_2, f_3, \ldots$

Comment. Theorems 7 and 8 are generalizations of theorems about Darboux functions, as stated in [2] in the sense that every real function with a connected graph is a Darboux function but the converse is not true.

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