Now, let Q be some definite point of X. By the local compactness of X, there exists a sequence $M_1, M_2, M_3, \ldots$ of nondegenerate connected point sets containing Q such that diam $M_n \leqslant 1/n$, for each $n = 1, 2, 3, \ldots$ For each n, let B_n denote a countable dense subset of M_n, and let $B = A+B_1+B_2+B_3+\ldots$ Then B is a countable dense subset of X such that, for some point Q of X, every open set containing Q has a component which contains infinitely many points of B. Clearly, there is no homeomorphism from X onto X that takes A onto B.

Reference

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Images of Borel sets and k-analytic sets (*)

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Section 1. Introduction and preliminaries. Throughout this paper, all given spaces are assumed to be metrizable. Our aim is to consider the preservation of Borel and "analytic" properties of spaces under images and inverse images by certain maps, generalizing and extending a number of known results to wider classes of mappings and spaces; the familiar results will be mentioned throughout the paper as terminology is introduced. The basic properties of Borel and k-analytic sets are discussed in Kuratowski [9] and Stone [13]. We gather together here some of the basic definitions and establish some notation.

$G_0(X)$ is the family of open sets of X. For each ordinal $\alpha < \omega_1$, $G_\alpha(X)$ is the family of all countable intersections (unions) of sets of class $G_\beta(X)$, $\beta < \alpha$, if α is odd (even). $F_0(X)$ is the family of closed sets of X, and $F_\alpha(X)$ is the family of all countable unions (intersections) of sets of class $F_\beta(X)$, $\beta < \alpha$, if α is odd (even). Since X is perfectly normal, $G_\alpha(X) \subseteq F_{\alpha+1}(X)$ and $F_\alpha(X) \subseteq G_{\alpha+1}(X)$ for each $\alpha < \omega_1$. Hence $\bigcup_\alpha F_\alpha(X) = \bigcup_\alpha G_\alpha(X)$, and this the family of Borel sets of X. A set in $F_\alpha(X) \cap G_\alpha(X)$ is said to be *ambiguous of class α in X*. $G_1, G_2, \ldots$ $(F_1, F_2, \ldots)$ sets in X are also called $G_\delta, G_{\delta\sigma}, \ldots$ $(F_\sigma, F_{\sigma\delta}, \ldots)$ sets of X. X is called *absolutely* $G_\alpha(F_\alpha)$ if it is a $G_\alpha(F_\alpha)$ set in any (metrizable) space Y in which X is topologically embedded. We denote this by $X \in \mathcal{G}_\alpha(\mathcal{F}_\alpha)$, where $\mathcal{G}_\alpha(\mathcal{F}_\alpha)$ is the property "absolute G_α" ("absolute F_α").

Willard [16] showed the following are equivalent for $\alpha \geqslant 1$: (a) $X \in \mathcal{G}_\alpha$, (b) X is a G_α in some complete space, (c) X is a G_α in βX, (d) X is a G_α in some compactification of X, (e) X is a G_α in every compactification of X.

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The equivalence of the analogous statements for $\mathcal{F}_\alpha$ sets ($\alpha \geqslant 2$) was recently established by R. Hansell [6]: (a) $X \in \mathcal{F}_\alpha$, (b) X is an F_α in some complete space, (c) X is an $F_\alpha \cap G_\delta$ set in βX, (d) X is an $F_\alpha \cap G_\delta$ set in some compactification of X, (e) X is an $F_\alpha \cap G_\delta$ set in every compactification of X.

A map (not necessarily continuous) $f\colon X \to Y$ is Borel measurable if for O open in Y, $f^{-1}(O)$ is Borel in X; f is Borel measurable of class α if $f^{-1}(O) \in G_\alpha(X)$, α even, or $F_\alpha(X)$, α odd; f is perfect if it is continuous, closed, and each set $f^{-1}(y)$ is compact. If f is one-to-one and onto, and both f and f^{-1} are Borel measurable, then f is called a *Borel isomorphism*; if f and f^{-1} are Borel measurable of class α and β respectively, f is called a *generalized homeomorphism of class* (α, β).

If $X \subseteq Y$, and k is an infinite cardinal, we say X is *k-analytic in* Y if one can write $X = \bigcup_{t \in B(k)} \bigcap_{n=1}^{\infty} F(t_1, \ldots, t_n)$, where $F(t_1, \ldots, t_n)$ is a zero set of Y and where $B(k)$ is the product of countably many discrete spaces of cardinal k. For separable metrizable spaces and $k = \aleph_0$, this defines the classical analytic or Souslin sets. The notion of k-analytic sets was introduced, and their basic properties were studied, in Stone [13]. We say X is *absolutely k-analytic*, and write $X \in \mathcal{A}_k$, if X is k-analytic in every (metrizable) space Y in which X is topologically embedded. This is equivalent to X being k-analytic in some complete space (Stone, [13]).

Section 2. Inverse images. Throughout this section, f is a function from X onto Y. We are concerned with the following sort of question: if Y is absolutely Borel (absolutely k-analytic), when is X absolutely Borel (absolutely k-analytic)?

The statements "Y is absolutely open", "Y is absolutely closed", "Y is an absolute $F - F'$, i.e., is absolutely the difference of two closed sets", and "Y is an absolute F_σ", are respectively equivalent to "Y is empty", "Y is compact", "Y is locally compact" and "Y is σ-locally compact". (The first two equivalences are routine, and the third and fourth are in Stone, [12].) Hence, if Y is absolutely open, so is X, for any f; if Y is absolutely closed or absolutely $F - F'$, then so is X, provided f is perfect. It is easy to show that the inverse image of a σ-locally compact space by a perfect map is again σ-locally compact. Thus the inverse image of an absolute F_σ by a perfect map is an absolute F_σ.

Vainstein [15] showed that perfect maps also preserve absolute Borel sets of higher class, as well as absolute $\aleph_0$-analytic sets, under inverse images. Our first aim is to extend this theorem to a wider class of spaces and maps. For this purpose, we first use a technique of R. W. Hansell [6] to investigate the embedding of absolute k-analytic sets in their Stone–Čech compactifications.

PROPOSITION 2.1. *The following statements are equivalent:*

(1) *X is absolutely k-analytic,*

(2) *X is the intersection of a k-analytic set and a G_δ set in some compactification BX,*

(3) *X is the intersection of a k-analytic set and a G_δ set in βX,*

(4) *X is k-analytic in some complete metric space,*

(5) *X is k-analytic in every metrizable space Y in which X is topologically embedded.*

Proof. (4) implies (5) implies (1) is in Stone, [13].

(1) implies (2). X is k-analytic in its completion (with respect to any chosen metric), $\tilde{X}$. $\tilde{X}$ is in turn a G_δ in $\beta\tilde{X}$. Since each zero set of $\tilde{X}$ is the restriction of a zero set of $\beta\tilde{X}$, it follows that X is the intersection of a k-analytic set with a G_δ in its compactification $\beta\tilde{X}$.

(2) implies (3). If X is the intersection of a k-analytic set K and a G_δ set G in BX, and if $\tilde{i}\colon \beta X \to BX$ is the Stone extension of the identity map i on X, then $\tilde{i}^{-1}(X) = X$ and $X = \tilde{i}^{-1}(K) \cap \tilde{i}^{-1}(G)$, which are respectively k-analytic and G_δ in βX.

(3) implies (4). Pick a metric d for X and let $(\tilde{X}, \tilde{d}) = \tilde{X}$ be its completion. Let $i\colon X \to \tilde{X}$ be the injection map and let $\tilde{i}\colon \beta X \to \beta\tilde{X}$ be its Stone extension. Put $X^* = \tilde{i}^{-1}(\tilde{X})$ and, for x and y in X^*, let $s(x, y) = \tilde{d}(\tilde{i}(x), \tilde{i}(y))$. It is easy to verify that $\tilde{i}\colon (X^*, s) \to (\tilde{X}, \tilde{d})$ is perfect and that (X^*, s) is a complete pseudo-metric space with a topology coarser than the βX-induced topology. Let X^* denote the space (X^*, s). One can verify that:

(a) if U is open in βX and $V = \operatorname{Int}_{X^*}(U \cap X^*)$, then $U \cap X = V \cap X$,

(b) if Z is a zero set in βX, then there is a G_δ set, G, in X^* such that $G \supseteq X$ and $\operatorname{cl}_G(Z \cap X) \subseteq Z$.

Now suppose $X = \bigcup_{t \in B(k)} \bigcap_{n=1}^{\infty} F(t_1, t_2, \ldots, t_n) \cap H$, where each $F(t_1, \ldots, t_n)$ is a zero set of βX and where $H = \bigcap_{n=1}^{\infty} H_n$, with H_n open in βX. Let $E = \bigcup_{t \in B(k)} \bigcap_{n=1}^{\infty} F(t_1, \ldots, t_n)$. For each $(t_1, \ldots, t_n)$, $n = 1, 2, \ldots$, pick, using (b), a G_δ set $G(t_1, \ldots, t_n)$ in X^*, containing X, and such that

$$\operatorname{cl}_{G(t_1, \ldots, t_n)}\big(F(t_1, \ldots, t_n) \cap X\big) \subseteq F(t_1, \ldots, t_n).$$

Let

$$K = \bigcup_{t \in B(k)} \bigcap_{n=1}^{\infty} \big[\operatorname{cl}_{G(t_1, \ldots, t_n)}\big(F(t_1, \ldots, t_n) \cap X\big)\big];$$

K is k-analytic in X^* and $K \cap X = E \cap X$.

Finally, for each $n = 1, 2, \ldots$, let $O_n = \mathrm{Int}_{X^*}(H_n \cap X^*)$. It follows, using (a), that $X = \bigcap_{n=1}^{\infty} (O_n \cap K)$. Hence X is the intersection of a k-analytic and a G_δ set in X^*. In particular, X is k-analytic in X^*, so we can write $X = \bigcup_{t \in B(k)} \bigcap_{n=1}^{\infty} L(t_1, \ldots, t_n)$, where each $L(t_1, \ldots, t_n)$ is closed in X^*. It follows from the definitions of s and $\tilde{i}$, and the fact that the L's are closed, that

$$\tilde{i}\Big(\bigcap_{n=1}^{\infty} L(t_1, \ldots, t_n)\Big) = \bigcap_{n=1}^{\infty} \tilde{i}\big(L(t_1, \ldots, t_n)\big).$$

Hence

$$\tilde{i}(X) = \bigcup_{t \in B(k)} \bigcap_{n=1}^{\infty} L(t_1, \ldots, t_n) \subseteq \tilde{X},$$

and since $\tilde{i}$ is closed, it follows that $\tilde{i}(X) = X$ is k-analytic in $\tilde{X}$.

DEFINITION 2.2 (Krolovec, [8]). Let g be a continuous map from the T_1 space X onto the T_1 space Y. f is *locally perfect* if each x in X has a neighborhood U_x such that $f(\mathrm{cl}\, U_x)$ is closed in Y and $f|\mathrm{cl}\, U_x$ is perfect.

Every perfect map between T_1 spaces and every continuous map of a locally compact T_2 space onto a T_1 space is locally perfect. The projection map of $[0,1] \times [0,1)$ onto $[0,1]$ is locally perfect but not closed. By mapping any non-locally compact metrizable space onto a point, one sees that a continuous closed map between metrizable spaces need not be locally perfect.

THEOREM 2.3. *Let f be a locally perfect map of X onto Y. If $Y \in \mathcal{A}_k$, then $X \in \mathcal{A}_k$. If $Y \in \mathcal{G}_\alpha$ $(1 \leqslant \alpha < \omega_1)$, then $X \in \mathcal{G}_\alpha$. If $Y \in \mathcal{F}_\alpha$ $(2 \leqslant \alpha < \omega_1)$, then $X \in \mathcal{F}_\alpha$.*

Proof. There is a Tychonoff space $\tilde{X}$ which contains X as a dense open subset, and an extension of f to a perfect map $\tilde{f}$ of $\tilde{X}$ onto Y (Krolovec, [8]). Let $\tilde{f}^*$ denote the Stone extension of $\tilde{f}$ to a map $\tilde{f}^*\colon \beta\tilde{X} \to \beta Y$. Since $\tilde{f}$ is perfect, $(\tilde{f}^*)^{-1}(Y) = X$. If $Y \in \mathcal{A}_k$, then $Y = K \cap H$, where K and H are respectively k-analytic and G_δ in βY. Then $\tilde{X} = (\tilde{f}^*)^{-1}(K) \cap (\tilde{f}^*)^{-1}(H)$ is the intersection of a k-analytic and a G_δ set in $\beta\tilde{X}$, and hence, since X is open in $\tilde{X}$, X is also such an intersection in its compactification $\beta\tilde{X}$. Therefore $X \in \mathcal{A}_k$ by Proposition 2.1.

If $Y \in \mathcal{G}_\alpha$, then Y is a G_α in βY, so $(\tilde{f}^*)^{-1}(Y) = X$ is a G_α in $\beta\tilde{X}$; therefore X is G_α in $\beta\tilde{X}$, and so $X \in \mathcal{G}_\alpha$. By Hansell's theorem, $Y \in \mathcal{F}_\alpha$ if and only if Y is the intersection of an F_α and a G_δ in βY; from this, as above, it follows that X is an $F_\alpha \cap G_\delta$ in its compactification $\beta\tilde{X}$; hence, again by Hansell's result, $X \in \mathcal{F}_\alpha$.

Our next result extends to k-analytic sets a similar result of Vainstein, using entirely different methods. We need the following lemma.

LEMMA 2.4. *Let $\mathcal{A}$ be a σ-discrete family of sets in the space X. Then X admits a metric s with respect to which $\mathcal{A}$ is σ-metrically discrete, i.e., $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$, where, for each n, there is a positive ε_n such that any two sets in $\mathcal{A}_n$ are at s-distance $\geqslant \varepsilon_n$.*

Proof. Without loss, we may assume the sets in $\mathcal{A}$ are closed. Let $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$, where each family $\mathcal{A}_n$ is discrete. Pick any metric, d, for X satisfying $d \leqslant 1$. The set $F_n = \bigcup \{A\colon A \in \mathcal{A}_n\}$ is closed; define a metric d_n' on X by extending to all of X (Hausdorff, [7]) the following equivalent metric on F_n: $d_n'(x, y) = d(x, y)$ if both x and y are in some set $A \in \mathcal{A}_n$, and $d_n'(x, y) = 1$ otherwise. Now put $d_n''(x, y) = \min\{d_n'(x, y), 1\}$. Then $d_n'' \leqslant 1$, d_n'' is equivalent to d_n' on X, and, if B and C are in $\mathcal{A}_n$, $B \neq C$, then $d_n''(B, C) = 1$. Thus $\mathcal{A}_n$ is metrically discrete with respect to d_n''. Now let $s(x, y) = \sum_{n=1}^{\infty} d_n''(x, y)/2^n$. s is equivalent to d and, with respect to s, $\mathcal{A}$ is σ-metrically discrete.

THEOREM 2.5. *Let f be a continuous closed map of X onto Y. If $Y \in \mathcal{A}_k$, then $X \in \mathcal{A}_k$, provided each set $f^{-1}(y) \in \mathcal{A}_k$. If $Y \in \mathcal{G}_\alpha$ and each set $f^{-1}(y) \in \mathcal{G}_\alpha$ $(1 \leqslant \alpha < \omega_1)$, then $X \in \mathcal{G}_{\alpha+1}$. If $Y \in \mathcal{F}_\alpha$ and each set $f^{-1}(y) \in \mathcal{F}_\alpha$ $(2 \leqslant \alpha < \omega_1)$, then $X \in \mathcal{F}_{\alpha+1}$.*

Proof. Let $Y_1 = \{y \in Y\colon f^{-1}(y)$ is compact$\}$, $Y_2 = Y - Y_1$, $\mathcal{C}_1 = \{f^{-1}(y)\colon y \in Y_1\}$, and $\mathcal{C}_2 = \{f^{-1}(y)\colon y \in Y_2\}$. Put $X_1 = \bigcup \mathcal{C}_1$, $X_2 = \bigcup \mathcal{C}_2 = X - X_1$.

$\mathcal{C}_2$ is σ-discrete (Lašnev [10]) and since f is continuous and closed it follows that Y_2 is a σ-discrete set of points. Hence Y_1 is a G_δ in Y and therefore is in $\mathcal{A}_k$, $\mathcal{G}_\alpha$ or $\mathcal{F}_\alpha$ whenever Y is in $\mathcal{A}_k$, $\mathcal{G}_\alpha$ or $\mathcal{F}_\alpha$ respectively. Since $f^{-1}(Y_1) = X_1$ and $f|X_1$ is perfect, it follows from Theorem 2.3 that X_1 is respectively in $\mathcal{A}_k$, $\mathcal{G}_\alpha$ or $\mathcal{F}_\alpha$ whenever Y is.

Since $\mathcal{C}_2$ is σ-discrete we may, using Lemma 2.4, assume that X has a metric, d, with respect to which $\mathcal{C}_2$ is σ-metrically discrete. Hence the family of sets $\mathcal{C}_2$ is also σ-metrically discrete in the completion $(\tilde{X}, \tilde{d})$. If each set of $\mathcal{C}_2$ is in $\mathcal{A}_k$, $\mathcal{G}_\alpha$ or $\mathcal{F}_\alpha$, then X_2, as a union of a σ-discrete family of such sets in $\tilde{X}$, is k-analytic, $G_{\alpha+1}$, or $F_{\alpha+1}$ in $\tilde{X}$, and hence is in $\mathcal{A}_k$, $\mathcal{G}_{\alpha+1}$ or $\mathcal{F}_{\alpha+1}$.

Thus if each set $f^{-1}(y)$ is in $\mathcal{A}_k$, $\mathcal{G}_\alpha$, or $\mathcal{F}_\alpha$, then $X = X_1 \cup X_2$ is respectively k-analytic, $G_{\alpha+1}$, or $F_{\alpha+1}$ in the complete space $\tilde{X}$, and hence $X \in \mathcal{A}_k$, $\mathcal{G}_{\alpha+1}$, or $\mathcal{F}_{\alpha+1}$.

It turns out that in certain special cases the assumption that f is continuous may be weakened to the assumption that f is Borel measurable of some fixed class $\beta < \omega_1$. This is discussed in Section 3, Corollary 3.10.

In any event, some assumption on the sets $f^{-1}(y)$ similar to that in 2.5 seems necessary as the following example shows.

EXAMPLE 2.6. For each $a < \omega_1$, let R_a be the real line and let G_a be a subset of R which is of additive class a but not of additive class β for any $\beta < a$. Let A be the disjoint topological sum of the spaces R_a, $a < \omega_1$, and let $X = \bigcup\{G_a\colon a < \omega_1\}$. Put $Y = \{a\colon a < \omega_1\}$, with the discrete topology. Y is an absolute G_δ. Define $f\colon X \to Y$ by $f(x) = a$ if and only if $x \in G_a$. Then f is continuous, open, closed and each set $f^{-1}(a)$ is absolutely Borel, since it is Borel in R_a. Yet X is not absolutely G_ξ for any ξ, else its open subset $G_{\xi+1}$ would be absolutely of additive class ξ, contrary to construction. X is, however, absolutely $\aleph_0$-analytic, by Theorem 2.5.

That the pathology of this example lies precisely in the fact that no countable bound exists on the Borel classes of the sets $f^{-1}(a)$ is shown in the preceding theorem.

Section 3. Forward images. In Section 3 we take the opposite point of view. If f is a function from X onto Y and X is absolutely Borel (absolutely k-analytic), when is Y absolutely Borel (absolutely k-analytic)? For X an absolutely open, closed, $F-F'$, or F_σ set, this comes down to finding conditions on f which preserve the topologically equivalent properties mentioned in Section 2. For example, continuous closed maps f preserve absolute $F-F'$ sets and absolute F_σ sets.

Čoban [1] recently announced that continuous closed maps preserve absolute Borel sets, as do continuous open maps, provided each set $f^{-1}(y)$ is complete. Our chief aim in Section 3 is to show that these results also hold whenever the hypothesis of continuity is replaced by Borel measurability of some fixed class, and that they apply, in this case, to absolutely k-analytic sets as well.

As a first step, we include a considerably simplified proof of a special case of a result of Vainstein [15] on the extension of perfect maps between metric spaces.

THEOREM 3.1. *Let f be a perfect map of X onto Y. Then there exist (metrizable) absolute G_δ sets M and N, containing X and Y respectively, such that f extends to a perfect map f^* of M onto N.*

Proof. Let $\tilde{X}$ and $\tilde{Y}$ denote the completions of X and Y with respect to any metrics. Then f extends to a continuous map $\tilde{f}\colon B \to A = \tilde{f}(B)$, where B is a G_δ in $\tilde{X}$, $B \supset X$, and where $\tilde{Y} \supset A \supset Y$ (Kuratowski, [9]). Since B is an absolute G_δ, it is completely metrizable. By remetrizing X, we may assume $\tilde{X} = B$, so that we have a continuous extension of f, $\tilde{f}\colon \tilde{X} \to A = \tilde{f}(\tilde{X}) \subseteq \tilde{Y}$. Let $\tilde{f}^*\colon \beta X \to \beta A$ be the Stone extension of f. Since f is perfect, $\tilde{f}^*(\beta\tilde{X} - X) \subseteq \beta A - Y$, so $\tilde{f}^*(\beta\tilde{X} - \tilde{X}) \subseteq \beta A - Y$.

Put $M = \beta\tilde{X} - (\tilde{f}^*)^{-1}\overline{(\tilde{f}^*)(\beta\tilde{X} - \tilde{X})}$. We have $\tilde{X} \supset M \supset X$, so M is metrizable. Since $\tilde{X}$ is a G_δ in $\beta\tilde{X}$, $\beta\tilde{X} - \tilde{X}$ is an F_σ in $\beta\tilde{X}$, and thus, since $\tilde{f}^*$ is closed, M is a G_δ in $\beta\tilde{X}$ and therefore also in $\tilde{X}$ itself. Hence M is an absolute G_δ. Define $f^* = \tilde{f}^*|M$. Since M is an inverse image set of $\tilde{f}^*$, f^* is a perfect map of M onto $N = f^*(M)$. Since $N \subseteq A \subseteq \tilde{Y}$, N is metrizable, and, as a perfect image of an absolute G_δ is an absolute G_δ (Frolík, [4]).

We shall also need the following simple characterization of closed maps.

PROPOSITION 3.2. *Let f map X onto Y and let $\Gamma = \{(x, f(x)) \in X \times Y\colon x \in X\}$ be its graph. Let π_y denote the projection map of $X \times Y \to Y$. Then f is closed (not necessarily continuous) if and only if $\pi_y|\Gamma$ is a closed map of Γ onto Y.*

Proof. The proof is routine using the following characterization of closed maps from X onto Y (Kuratowski, [9]): f is closed if and only if, given $y \in Y$, a sequence $\{y_n\colon n = 1, 2, \ldots\}$ in Y, with $y_n \neq y$ for each n, and, for each n, a point $x_n \in f^{-1}(y_n)$, then there exists a point $x \in f^{-1}(y)$ to which some subsequence of $\{x_n\colon n = 1, 2, \ldots\}$ converges.

An analogous result for open maps is in Proposition 3.7.

We can now state and prove the following theorem.

THEOREM 3.3. *Let f be a closed map of X onto Y that is Borel measurable of class a, $a < \omega_1$. If $X \in \mathcal{A}_k$, then $Y \in \mathcal{A}_k$. If X is absolutely Borel, Y is absolutely Borel. More specifically, if $X \in \mathcal{G}_\xi$, $1 \leqslant \xi < \omega_1$ (or $\mathcal{F}_\xi$, $2 \leqslant \xi < \omega_1$), then*

(1) $Y \in \mathcal{G}_\xi$ $(\mathcal{F}_\xi)$ *if* $a+1 \leqslant \xi$ *and* $\xi \geqslant \omega_0$,

(2) $Y \in \mathcal{F}_{\xi+1}$ $(\mathcal{G}_{\xi+1})$ *if* $a+1 \leqslant \xi$ *and* $\xi < \omega_0$,

(3) $Y \in \mathcal{G}_{a+1}$ $(\mathcal{F}_{a+1})$ *if* $a+1 > \xi$ *and* $\xi \geqslant \omega_0$,

(4) $Y \in \mathcal{F}_{a+2}$ $(\mathcal{G}_{a+2})$ *if* $a+1 > \xi$ *and* $\xi < \omega_0$.

Proof. We first assume $a = 0$, i.e., that f is continuous. We may assume, without further loss of generality, then f is perfect. Indeed, if f is continuous and closed, then there is a closed set $A \subseteq X$ such that $f(A) = Y$ and $f|A$ is perfect. Since $A \in \mathcal{A}_k$ if $X \in \mathcal{A}_k$, and $A \in \mathcal{G}_\xi$ $(\mathcal{F}_\xi)$ if $X \in \mathcal{G}_\xi$ $(\mathcal{F}_\xi)$, no generality is lost. For convenience we also assume $\xi \geqslant 2$, since the case $\xi = 1$ is well known anyway (Frolík, [4]).

Let M, N, and f^* be chosen as in Theorem 3.1. Remetrizing X and Y if necessary, and replacing M and N by $\mathrm{cl}_M X$ and $\mathrm{cl}_N Y$, we may assume that M and N are respectively $\tilde{X}$ and $\tilde{Y}$, the metric completions of X and Y. Since f^* is perfect, $(f^*)^{-1}(Y) = X$.

Let $F\colon \tilde{Y} \to 2^{\tilde{X}}$, the space of closed subsets of $\tilde{X}$, by $F(y) = (f^*)^{-1}(y)$. Since f^* is continuous and closed, F is upper semi-continuous, and, since each $F(y)$ is complete, there is a $g\colon \tilde{Y} \to \tilde{X}$ satisfying $g(y) \in F(y)$ for each y and $g^{-1}(O)$ is an F_σ in $\tilde{Y}$ whenever O is open in $\tilde{X}$, i.e., g is Borel measurable of class 1 (Engelking, [2]).

Let $\Gamma = \{(x, f^*(x)) : x \in \tilde{X}\}$, $\Gamma' = \{(g(y), y) : y \in \tilde{Y}\}$, and $\Gamma'' = \{(x, f^*(x)) : x \in X\}$. Since g is Borel measurable of class 1, Γ' is a G_δ in $\tilde{X} \times \tilde{Y}$ and therefore an absolute G_δ (Kuratowski, [9]). Γ'' is homeomorphic to X. Hence Γ'' and $\Gamma'' \cap \Gamma'$ are in $\mathcal{A}_k$ if $X \in \mathcal{A}_k$; and Γ'' and $\Gamma'' \cap \Gamma'$ are in $\mathcal{G}_\xi$ $(\mathcal{F}_\xi)$ if $X \in \mathcal{G}_\xi$ $(\mathcal{F}_\xi)$.

Let π be the projection from $\tilde{X} \times \tilde{Y}$ to $\tilde{X}$. Since $(f^*)^{-1}(Y) = X$, $g(Y) = \pi(\Gamma'' \cap \Gamma')$. But $\pi|\Gamma$ is a homeomorphism, and $\Gamma \supseteq \Gamma'' \cap \Gamma'$. Hence if $X \in \mathcal{A}_k$, then $g(Y) \in \mathcal{A}_k$; and if $X \in \mathcal{G}_\xi$ $(\mathcal{F}_\xi)$, so is $g(Y)$.

Now put $g(\tilde{Y}) = B$, $g(Y) = A$, and let $h = f^*|B$. h and g are inverse functions between B and $\tilde{Y}$. Since g is Borel measurable of class 1, $h \colon B \to \tilde{Y}$ is a generalized homeomorphism of class $(0, 1)$ and $h(A) = Y$.

If $X \in \mathcal{A}_k$, then $A \in \mathcal{A}_k$, so we can write $A = \bigcup_{t \in B(k)} \bigcap_{n=1}^{\infty} F(t_1, \ldots, t_n)$, where each $F(t_1, \ldots, t_n)$ is a zero set of B. Then

$$Y = h(A) = \bigcup_{t \in B(k)} \bigcap_{n=1}^{\infty} hF(t_1, \ldots, t_n),$$

and each $hF(t_1, \ldots, t_n)$ is a G_δ set in Y. Hence Y is k-analytic in $\tilde{Y}$, so $Y \in \mathcal{A}_k$, by Proposition 2.1.

If $X \in \mathcal{G}_\xi$ $(\mathcal{F}_\xi)$, then $A \in \mathcal{G}_\xi$ $(\mathcal{F}_\xi)$. If $\xi < \omega_0$, then, since f carries open (closed) sets of B to $F_\sigma(G_\delta)$ sets of $\tilde{Y}$, it follows by a simple induction that $Y = h(A)$ is an $F_{\xi+1}$ $(G_{\xi+1})$ in $\tilde{Y}$, so $Y \in \mathcal{F}_{\xi+1}$ $(\mathcal{G}_{\xi+1})$. If $\xi = \omega_0$, then $A = \bigcup_{n=1}^{\infty} B_n$ or $\bigcap_{n=1}^{\infty} B_n$, where each B_n is Borel of a finite class in X. Then Y is a union or intersection, respectively, of Borel sets of finite class in $\tilde{Y}$, so $Y \in \mathcal{G}_{\omega_0}$ $(\mathcal{F}_{\omega_0})$. Finally, if $\xi > \omega_0$, a simple transfinite induction, beginning this time at ω_0, shows that Y is a G_ξ $(\mathcal{F}_\xi)$ in $\tilde{Y}$, so $Y \in \mathcal{G}_\xi$ $(\mathcal{F}_\xi)$, completing the proof of the theorem for continuous closed maps f.

For the general case, suppose now that $f \colon X \to Y$ is closed and Borel measurable of class α. Then Γ, the graph of f, is a Borel set of multiplicative class α in $X \times \tilde{Y}$, where now, $\tilde{Y}$ is any metric completion of Y. If $X \in \mathcal{A}_k$, then $X \times \tilde{Y}$ is in $\mathcal{A}_k$, so $\Gamma \in \mathcal{A}_k$; if $X \in \mathcal{G}_\xi$ $(\mathcal{F}_\xi)$, then $X \times \tilde{Y} \in \mathcal{G}_\xi$ $(\mathcal{F}_\xi)$ and

(a) $\Gamma \in \mathcal{G}_\xi$ $(\mathcal{F}_\xi)$ if $\alpha + 1 \leqslant \xi$,

(b) $\Gamma \in \mathcal{G}_{\alpha+1}$ $(\mathcal{F}_{\alpha+1})$ if $\alpha + 1 > \xi$.

Let $\pi' \colon X \times Y \to Y$ be the projection map. By Proposition 3.2, $\pi'|\Gamma$ is closed. Thus $X \in \mathcal{A}_k$, $Y = \pi'(\Gamma) \in \mathcal{A}_k$ by the first part of the proof. And if $X \in \mathcal{G}_\xi$ $(\mathcal{F}_\xi)$, then (a) and (b) hold, $Y = \pi'(\Gamma)$, and using the computations from the first part of the proof, conditions (1)–(4) of the theorem follow immediately.

Letting Y be the topological sum of a non-Borel subset of $\boldsymbol{R}$, the real numbers, with its complement, and mapping $\boldsymbol{R}$ to Y by the identity, we see that a one-to-one closed open image of $\boldsymbol{R}$ needn't be absolutely Borel, so the assumption of Borel measurability in Theorem 3.3 is not superfluous. It might be interesting to know if the hypothesis "f is Borel measurable of class α" could be replaced by "f is Borel measurable"; but it is not known if this class of mappings is really more extensive anyway.

COROLLARY 3.4. *Let $f \colon X \to Y$ be a generalized homeomorphism of class $(\alpha, 0)$. If $X \in \mathcal{A}_k$, then $Y \in \mathcal{A}_k$; if X is absolutely Borel, so is Y, and* (1)–(4) *of Theorem* 3.3 *still hold.*

The preceding corollary slightly sharpens a result achieved by R. Hansell [5] using entirely different techniques. It can obviously also be viewed as an "inverse image" theorem for generalized homeomorphisms of class $(0, \alpha)$. The following corollary extends to k-analytic sets a result announced by Čoban [1].

COROLLARY 3.5. *Let f be a continuous open map of X onto Y such that, for some fixed metric on X, each set $f^{-1}(y)$ is complete. If $X \in \mathcal{A}_k$, then $Y \in \mathcal{A}_k$. If $X \in \mathcal{G}_\xi$, $1 \leqslant \xi < \omega_1$ $(\mathcal{F}_\xi$, $2 \leqslant \xi < \omega_1)$, then $Y \in \mathcal{F}_{\xi+1}$ $(\mathcal{G}_{\xi+1})$ if $\xi < \omega_0$ and $Y \in \mathcal{G}_\xi$ $(\mathcal{F}_\xi)$ if $\xi \geqslant \omega_0$. In particular, the result holds if f is continuous, open and finite-to-one.*

Proof. There is a closed subset A of X such that $f(A) = Y$ and $f|A$ is perfect (Michael, [11]). If $X \in \mathcal{A}_k$, then $A \in \mathcal{A}_k$; and if $X \in \mathcal{G}_\xi$ $(\mathcal{F}_\xi)$, then $A \in \mathcal{G}_\xi$ $(\mathcal{F}_\xi)$. Then the result follows from Theorem 3.3 with $\alpha = 0$.

For open maps, the hypothesis on the sets $f^{-1}(y)$ cannot be improved very much, if at all, as the following example shows.

EXAMPLE 3.6. In [14], Taimonov gives a construction to show that any Borel set in $[0, 1]$, say B_λ, is the continuous image of a $G_{\delta\sigma}$ in $[0, 1] \times [0, 1]$, say C_λ, by a continuous, open, countable-to-one map, f_λ. Let this be done for each $\lambda < \omega_1$, always choosing B_λ to be a Borel set of exact additive class λ, i.e., B_λ is of additive class λ, but not of any lower class (Kuratowski, [9]). Let X be the disjoint topological sum of the C_λ's, and Y the disjoint topological sum of the B_λ's. Then $f = \bigcup_\lambda f_\lambda$ is continuous, open, and countable-to-one. In particular, each set $f^{-1}(y)$ is an absolute F_σ. Also $X \in \mathcal{G}_{\delta\sigma}$, but Y is not absolutely Borel. For if it were, say, absolutely of additive class α, then its open subset $B_{\alpha+1}$ would also be absolutely of additive class α, contrary to construction.

As in the case of continuous closed maps, however, the assumption "f is continuous" can be weakened to "f is Borel measurable of some class α". To do this we need the following proposition, all but the last statement of which is due to R. Hansell. It is the open analogue of Proposition 3.2.

PROPOSITION 3.7. *Let f be a Borel measurable mapping of class α, $\alpha < \omega_1$, of X onto Y. Let X' be the space (X, T'), where T' is the topology generated by the original topology on X together with all sets of form $f^{-1}(V)$, where V is open in Y. Then the map $g(x) = (x, f(x))$ is a homeomorphism of X' onto Γ, the graph of f. The projection $\pi_x|\Gamma\colon \Gamma \to X$ is a generalized homeomorphism of class $(0, \alpha)$. If f carries open sets in X to sets of additive class β in Y, then the projection map $\pi_y|\Gamma$ carries open sets in Γ to sets additive class β in Y. (In particular, if f is open, so is $\pi_y|\Gamma$; and if f is one-to-one, $\pi_y|\Gamma$ is a generalized homeomorphism of class $(0, \beta)$.)*

Proof. g is one-to-one and onto. It is continuous because $\pi_x \circ g =$ identity map from X' to X, and $\pi_y \circ g\colon X' \to Y$ is continuous by construction. If O is open in X and V is open in Y, then $O \cap f^{-1}(V)$ is a typical basic open set in X', and $g(O \cap f^{-1}(V)) = \Gamma \cap (O \times V)$; hence g is open.

That $\pi_x|\Gamma$ is a $(0, \alpha)$ homeomorphism is in Hansell [5].

Suppose that $\{V_{\alpha,n}\colon \alpha \in A,\ n = 1, 2, \ldots\}$ is a σ-discrete open basis for Y, with each family $\{V_{\alpha,n}\colon \alpha \in A\}$ being discrete. Then a basis for X' is given by $\{U \cap f^{-1}(V_{\alpha,n})\colon \alpha \in A,\ U \text{ open in } X,\ n = 1, 2, \ldots\}$. Fixing n, let P_n be of form $\bigcup_\alpha (U_\alpha \cap f^{-1}(V_{\alpha,n}))$. But then $f(P_n)$ is the union of a discrete collection of sets of additive class β in Y, and therefore is itself of additive class β. Since each open set of X' is a union of countably many P_n's, it follows that f carries open sets in X' to sets of additive class β in Y. Thus if O is open in Γ, $\pi_y|\Gamma(O) = f(g^{-1}(O))$ is of additive class β in Y.

THEOREM 3.8. *Let f be an open map of X onto Y that is Borel measurable of class α, $\alpha < \omega_1$, and suppose that each set $f^{-1}(y)$ is complete in some fixed metric on X. Then if $X \in \mathcal{A}_k$, $Y \in \mathcal{A}_k$. If $X \in \mathcal{G}_\xi$, $1 \leqslant \xi < \omega_1$ ($\mathcal{F}_\xi$, $2 \leqslant \xi < \omega_1$), and $\lambda = \max(\alpha, \xi)$, then Y is absolutely ambiguous of class $\lambda + 2$, and absolutely ambiguous of class $\lambda+1$ if $\lambda \geqslant \omega_0$.*

Proof. Let d be a metric on X making each set $f^{-1}(y)$ complete and let s be any metric on Y. Γ, the graph of f, is of multiplicative class α in $X \times \widetilde{Y}$, where $\widetilde{Y}$ is the completion of (Y, s) (Kuratowski, [9]). Hence $\Gamma \in \mathcal{A}_k$ if $X \in \mathcal{A}_k$. Since $X \times Y \in \mathcal{G}_\xi$ ($\mathcal{F}_\xi$) if $X \in \mathcal{G}_\xi$ ($\mathcal{F}_\xi$), in this case Γ is absolutely ambiguous of class $\lambda+1$.

Metrize $X \times Y$ by $r((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + s(y_1, y_2)$. Let $g = \pi_y|\Gamma$. Then for $y \in Y$, $g^{-1}(y) = f^{-1}(y) \times \{y\}$, an r-complete subset of Γ. It follows from Proposition 3.7 that g is open. Hence Corollary 3.5 applies to $g\colon \Gamma \to Y$ and the conclusion follows immediately.

We remark that the numerical calculations can be slightly sharpened by distinguishing the cases of ξ even and ξ odd.

With the necessary tools now at hand, we conclude with the following inverse image theorem alluded to in Section 2; its Corollary 3.10 strengthens Theorem 2.5 in certain cases.

THEOREM 3.9. *Let f be a closed map of X onto Y that is Borel measurable of class β, $\beta < \omega_1$, and let Γ be its graph. If each set $f^{-1}(y) \in \mathcal{A}_k$, then $\Gamma \in \mathcal{A}_k$; and if each set $f^{-1}(y) \in \mathcal{G}_\xi$ (for fixed ξ), then Γ is absolutely Borel.*

Proof. By Proposition 3.7, $(\pi_x|\Gamma)^{-1}\colon X \to \Gamma$ is a generalized homeomorphism of class $(\beta, 0)$. Let $g = \pi_y|\Gamma\colon \Gamma \to Y$. By Proposition 3.2 g is closed, and, of course, continuous. Since $g^{-1}(y) = h(f^{-1}(y))$, it follows from Corollary 3.4 that each $g^{-1}(y)$ is absolutely Borel of some fixed class if each $f^{-1}(y)$ is, and each $g^{-1}(y) \in \mathcal{A}_k$ if each $f^{-1}(y) \in \mathcal{A}_k$. It now follows from Theorem 2.5 that if Y is absolutely Borel or k-analytic, then Γ is absolutely Borel or absolutely k-analytic respectively. (Clearly, in the Borel case, one could use the previous results to get a bound on the class of Γ if desired).

COROLLARY 3.10. *Let f be a closed map of X onto Y that is Borel measurable of class β, $\beta < \omega_1$. Suppose Y and each set $f^{-1}(y)$ are in $\mathcal{G}_\xi$ for some fixed ξ. Then X is absolutely Borel if either X is absolutely $\aleph_0$-analytic or if both X and Y are separable.*

Proof. $g = \pi_x|\Gamma\colon \Gamma \to X$ is a generalized homeomorphism of class $(0, \beta)$ by Proposition 3.7. If $X = g(\Gamma)$ is absolutely $\aleph_0$-analytic, it is absolutely Borel (Hansell, [5], p. 79). If X and Y are separable, the result follows from a theorem of Stone ([13], p. 32).

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Real functions having graphs connected and dense in the plane

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Introduction. In this paper a theorem proved by Jack Brown in [1] is utilized to prove theorems concerning the class of all real functions having graphs connected and dense in the plane. Only real functions will be considered here and the word graph will refer to the graph of a real function.

Definitions and notation. If f is a point set in the plane, then the X-projection of f is the set of all abscissas of points of f and will be denoted by f_x. The statement that the number set M is c-dense in the number set N means that if I is an open interval containing an element of N, then the cardinality of $I \cap (M \cap N)$ is that of the continuum. The cardinality of the continuum will be denoted by c. The set of all real numbers will be denoted by E.

LEMMA 1. *If the graph f has connected X-projection and intersects every lower semi-continuous graph with X-projection a subinterval of the X-projection of f, then f is connected.*

This lemma follows easily from the theorem that Jack Brown states and proves in [1].

THEOREM 1. *If C_1 is a subset of E such that each of C_1 and $E-C_1$ is c-dense in E, then there is a totally disconnected graph g with X-projection C_1, such that if M is a point set containing g and having X-projection E, then M is connected and dense in the plane.*

Proof of Theorem 1. Suppose C_1 is a subset of E such that each of C_1 and $E-C_1$ is c-dense in E.

Let W denote the collection to which w belongs if and only if w is a lower semi-continuous graph with X-projection an interval. The collection W has cardinality c. Let Q be a meaning of precedes such that (1) W is well ordered with respect to Q and (2) if w is an element of the collection W, then the set of all elements of W that precede w has cardinality less than W.