A note on countable dense homogeneity

by

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R. Bennett [1], calls a topological space *countable dense homogeneous* provided that X is separable and for each two countable dense subsets A and B of X there is a homeomorphism h from X onto X such that $h(A) = B$. The purpose of this note is to prove the following.

THEOREM. *If X is a connected, locally compact metric space and is countable dense homogeneous, then X is locally connected.*

Proof. Assume the contrary. Then there is a point of X at which X is not connected im kleinen. Bennett has shown that X is homogeneous. Therefore X is not connected im kleinen at any point of X.

Suppose that every open set in X contains an open subset V such that some component of $\overline{V}$ has interior. Let U_1 be an open set of diameter less than 1. Let V_1 be an open subset of U_1 such that some component C_1 of $\overline{V}_1$ contains an open set D_1. Let U_2 be an open set such that $\overline{U}_2$ is a subset of D_1 and $\operatorname{diam}(U_2) < 1/2$. Let V_2 be an open subset of U_2 such that some component C_2 of $\overline{V}_2$ contains an open set D_2. Continue this process. Since X is complete, there is a point x common to $U_1, U_2, U_3, \ldots$ One sees easily that X is connected im kleinen at x, a contradiction.

So there is an open set U such that if C is a component of the closure of an open subset of U, then C has no interior. By the homogeneity of the space X, every point belongs to such an open set U. Since X is separable metric, there exists a countable basis $G = \{U_1, U_2, U_3, \ldots\}$ such that if C is a component of any $\overline{U}_i$, then C has no interior. It follows from the Baire Category Theorem that if M is a point set which is the union of countably many point sets C, each of which is a component of $\overline{U}_i$ for some i, then M contains no open set.

Let x_1 be a point of U_1. Let x_2 be a point of U_2 not in any component of any U_i containing x_1. Let x_3 be a point of U_3 not in any component of any U_i containing x_1 or x_2. This process may be continued, yielding a countable dense subset A of X such that if p is a point then there is an open set U_n containing p such that if C is a component of U_n intersecting A, then C contains only one point of A.

Now, let Q be some definite point of X. By the local compactness of X, there exists a sequence $M_1, M_2, M_3, \ldots$ of nondegenerate connected point sets containing Q such that diam $M_n \leqslant 1/n$, for each $n = 1, 2, 3, \ldots$ For each n, let B_n denote a countable dense subset of M_n, and let $B = A+B_1+B_2+B_3+\ldots$ Then B is a countable dense subset of X such that, for some point Q of X, every open set containing Q has a component which contains infinitely many points of B. Clearly, there is no homeomorphism from X onto X that takes A onto B.

Reference

[1] R. B. Bennett, *Countable dense homogeneous spaces*, Fund. Math. 74 (1972), pp. 189–194.

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Images of Borel sets and k-analytic sets (*)

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Section 1. Introduction and preliminaries. Throughout this paper, all given spaces are assumed to be metrizable. Our aim is to consider the preservation of Borel and "analytic" properties of spaces under images and inverse images by certain maps, generalizing and extending a number of known results to wider classes of mappings and spaces; the familiar results will be mentioned throughout the paper as terminology is introduced. The basic properties of Borel and k-analytic sets are discussed in Kuratowski [9] and Stone [13]. We gather together here some of the basic definitions and establish some notation.

$G_0(X)$ is the family of open sets of X. For each ordinal $a < \omega_1$, $G_a(X)$ is the family of all countable intersections (unions) of sets of class $G_\beta(X)$, $\beta < a$, if a is odd (even). $F_0(X)$ is the family of closed sets of X, and $F_a(X)$ is the family of all countable unions (intersections) of sets of class $F_\beta(X)$, $\beta < a$, if a is odd (even). Since X is perfectly normal, $G_a(X) \subseteq F_{a+1}(X)$ and $F_a(X) \subseteq G_{a+1}(X)$ for each $a < \omega_1$. Hence $\bigcup_a F_a(X) = \bigcup_a G_a(X)$, and this the family of Borel sets of X. A set in $F_a(X) \cap G_a(X)$ is said to be *ambiguous of class a in X*. $G_1, G_2, \ldots$ ($F_1, F_2, \ldots$) sets in X are also called $G_\delta, G_{\delta\sigma}, \ldots$ ($F_\sigma, F_{\sigma\delta}, \ldots$) sets of X. X is called *absolutely* $G_a(F_a)$ if it is a $G_a(F_a)$ set in any (metrizable) space Y in which X is topologically embedded. We denote this by $X \in \mathcal{G}_a(\mathcal{F}_a)$, where $\mathcal{G}_a(\mathcal{F}_a)$ is the property "absolute G_a" ("absolute F_a").

Willard [16] showed the following are equivalent for $a \geqslant 1$: (a) $X \in \mathcal{G}_a$, (b) X is a G_a in some complete space, (c) X is a G_a in βX, (d) X is a G_a in some compactification of X, (e) X is a G_a in every compactification of X.

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