A note on countable dense homogeneity

by

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R. Bennett [1], calls a topological space countable dense homogeneous provided that $X$ is separable and for each two countable dense subsets $A$ and $B$ of $X$ there is a homeomorphism $h$ from $X$ onto $X$ such that $h(A) = B$. The purpose of this note is to prove the following.

**Theorem.** If $X$ is a connected, locally compact metric space and is countable dense homogeneous, then $X$ is locally connected.

**Proof.** Assume the contrary. Then there is a point of $X$ at which $X$ is not connected im kleinen. Bennett has shown that $X$ is homogeneous. Therefore $X$ is not connected im kleinen at any point of $X$.

Suppose that every open set in $X$ contains an open subset $V$ such that some component of $V$ has interior. Let $U_1$ be an open set of diameter less than 1. Let $V_1$ be an open subset of $U_1$ such that some component $C_1$ of $V_1$ contains an open set $D_1$. Let $U_2$ be an open set such that $U_2$ is a subset of $D_1$ and diam$(U_2) < 1/2$. Let $V_2$ be an open subset of $U_2$ such that some component $C_2$ of $V_2$ contains an open set $D_2$. Continue this process. Since $X$ is complete, there is a point $x$ common to $U_1, U_2, U_3, ...$. One sees easily that $X$ is connected im kleinen at $x$, a contradiction.

So there is an open set $U$ such that if $C$ is a component of the closure of an open subset of $U$, then $C$ has no interior. By the homogeneity of the space $X$, every point belongs to such an open set $U$. Since $X$ is separable metric, there exists a countable basis $\mathcal{B} = \{U_1, U_2, U_3, ...\}$ such that if $C$ is a component of any $U_i$, then $C$ has no interior. It follows from the Baire Category Theorem that if $M$ is a point set which is the union of countably many point sets $C$, each of which is a component of $U_i$, for some $i$, then $M$ contains no open set.

Let $x_1$ be a point of $U_1$. Let $x_2$ be a point of $U_2$ not in any component of any $U_i$ containing $x_1$. Let $x_3$ be a point of $U_3$ not in any component of any $U_i$ containing $x_2$ or $x_1$. This process may be continued, yielding a countable dense subset $A$ of $X$ such that if $p$ is a point then there is an open set $U_n$ containing $p$ such that if $C$ is a component of $U_n$ intersecting $A$, then $C$ contains only one point of $A$.

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Now, let \( Q \) be some definite point of \( X \). By the local compactness of \( X \), there exists a sequence \( M_1, M_2, M_3, \ldots \) of nondegenerate connected point sets containing \( Q \) such that \( \text{diam } M_n < 1/n \), for each \( n = 1, 2, 3, \ldots \). For each \( n \), let \( B_n \) denote a countable dense subset of \( M_n \), and let \( B = A + B_1 + B_2 + B_3 + \ldots \). Then \( B \) is a countable dense subset of \( X \) such that, for some point \( Q \) of \( X \), every open set containing \( Q \) has a component which contains infinitely many points of \( B \). Clearly, there is no homeomorphism from \( X \) onto \( X \) that takes \( A \) onto \( B \).

Reference


Images of Borel sets and \( \mathcal{B} \)-analytic sets (*)

by

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Section 1. Introduction and preliminaries. Throughout this paper, all given spaces are assumed to be metrizable. Our aim is to consider the preservation of Borel and "analytic" properties of spaces under images and inverse images by certain maps, generalizing and extending a number of known results to wider classes of mappings and spaces; the familiar results will be mentioned throughout the paper as terminology is introduced. The basic properties of Borel and \( \mathcal{B} \)-analytic sets are discussed in Kuratowski [9] and Stone [13]. We gather together here some of the basic definitions and establish some notation.

\( G_\alpha(X) \) is the family of open sets of \( X \). For each ordinal \( \alpha < \varepsilon_0 \), \( G_\beta(X) \) is the family of all countable intersections (unions) of sets of class \( G_\beta(X) \), \( \beta < \alpha \), if \( \alpha \) is odd (even). \( F_\alpha(X) \) is the family of closed sets of \( X \), and \( F_\alpha(X) \) is the family of all countable unions (intersections) of sets of class \( F_\alpha(X) \), \( \beta < \alpha \), if \( \alpha \) is odd (even). Since \( X \) is perfectly normal, \( G_\alpha(X) \subseteq F_{\alpha+1}(X) \) and \( F_{\alpha}(X) \subseteq G_{\alpha+1}(X) \) for each \( \alpha < \varepsilon_0 \). Hence \( \bigcup F_{\alpha}(X) = \bigcup G_{\alpha}(X) \), and this is the family of Borel sets of \( X \). A set in \( F_{\alpha}(X) \) \( \cap \bigcap G_{\alpha}(X) \) is said to be ambiguous of class \( \alpha \) in \( X \). \( G_1, G_2, \ldots, (F_1, F_2, \ldots) \) sets in \( X \) are also called \( G_1, G_2, \ldots, (F_1, F_2, \ldots) \) sets of \( X \). \( X \) is called absolutely \( G_\alpha(F) \) if it is a \( G_{\alpha}(F) \) set in any (metrizable) space \( Y \) in which \( X \) is topologically embedded. We denote this by \( X \in \mathcal{G}_\alpha(F) \), where \( \mathcal{G}_\alpha(F) \) is the property "absolute \( G_\alpha(F) \)."

Willard [16] showed the following are equivalent for \( \alpha \geq 1 \): (a) \( X \in \mathcal{G}_\alpha(F) \), (b) \( X \) is a \( G_\alpha \) in some complete space, (c) \( X \) is a \( G_\alpha \) in \( \beta X \), (d) \( X \) is a \( G_\alpha \) in some compactification of \( X \), (e) \( X \) is a \( G_\alpha \) in every compactification of \( X \).

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