On a problem of Tamano

by

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Introduction. In [1], Tamano asked whether or not a space which is the closure-preserving union of compact sets has to be paracompact. We give a partial answer to this question with the following theorem. Let X be a space, and let $\mathfrak{F} = \{F(\alpha)|\ \alpha \in \Gamma\}$ be a closure-preserving family of compact closed sets whose union is X. Suppose that for each $x \in X$, there is a countable subfamily $\mathfrak{F}(x)$ of $\mathfrak{F}$ such that $x \in \operatorname{int} \bigcup \{F|\ F \in \mathfrak{F}(x)\}$. Then X is the disjoint union of open and closed σ-compact subsets.

LEMMA 1. *Let X be a space, and $\mathfrak{F} = \{F(\alpha)|\ \alpha \in \Gamma\}$ a closure-preserving family of compact closed sets whose union is X. Suppose that, for each $x \in X$, there is a countable subfamily $\mathfrak{F}(x)$ of $\mathfrak{F}$ such that $x \in \operatorname{int} \bigcup\{F|\ F \in \mathfrak{F}(x)\}$. Then for each compact set K there is a countable subfamily, $\mathfrak{F}(K)$, of $\mathfrak{F}$ such that $K \subset \operatorname{int} \bigcup \{F|\ F \in \mathfrak{F}(K)\}$.*

Proof. The family $\{\operatorname{int} \bigcup \{F|\ F \in \mathfrak{F}(x)\}|\ x \in K\}$ is an open cover of K, and hence has a finite subcover, say, $\{\operatorname{int} \bigcup\{F|\ F \in \mathfrak{F}(x_i)\}|\ i = 1, 2, \ldots, n\}$, for some points $x_1, x_2, \ldots, x_n \in K$.

Then $\mathfrak{F}(K) = \bigcup\{\mathfrak{F}(x_i)|\ i = 1, 2, \ldots, n\}$ is a countable subfamily of $\mathfrak{F}$, and $K \subset \operatorname{int} \bigcup \{F|\ F \in \mathfrak{F}(K)\}$.

THEOREM 1. *Let X be a space, and $\mathfrak{F} = \{F(\alpha)|\ \alpha \in \Gamma\}$ a closure-preserving family of compact closed sets whose union is X. If for each $x \in X$ there is a countable subfamily $\mathfrak{F}(x)$ of $\mathfrak{F}$ such that $x \in \operatorname{int} \bigcup\{F|\ F \in \mathfrak{F}(x)\}$, then X is the disjoint union of open and closed σ-compact subsets.*

Proof. For each $\alpha \in \Gamma$, $F(\alpha)$ is compact, whence, by Lemma 1, there is a countable subfamily $\Gamma(\alpha)$ of Γ such that $F(\alpha) \subset \operatorname{int}\bigcup \{F(\beta)|\ \beta \in \Gamma(\alpha)\}$.

Let $\Gamma(0) = \{\alpha\}$.

Let $\Gamma(1) = \{\beta \in \Gamma|\ \beta \in \Gamma(\gamma)$, for some $\gamma \in \Gamma(0)\} = \Gamma(\alpha)$.

Let $\Gamma(2) = \{\beta \in \Gamma|\ \beta \in \Gamma(\gamma)$, for some $\gamma \in \Gamma(1)\}$.

Inductively, let $\Gamma(i+1) = \{\beta \in \Gamma|\ \beta \in \Gamma(\gamma)$, for some $\gamma \in \Gamma(i)\}$.

Let $\hat{\Gamma}(\alpha) = \bigcup\{\Gamma(i)|\ i = 0, 1, 2, \ldots\}$.

Let $G(\alpha) = \bigcup\{F(\beta)|\ \beta \in \hat{\Gamma}(\alpha)\}$.

It is easy to see that $G(\alpha)$ is closed and σ-compact. Also, $G(\alpha)$ is open. To see this, we let $x \in G(\alpha)$, and find an open set about x that lies inside $G(a)$. Now $x \in G(a) = \bigcup\{F(\beta)|\ \beta \in \hat{\Gamma}(a)\}$ means that there is a $\beta(x) \in \hat{\Gamma}(a)$ such that $x \in F(\beta(x))$. Since $\beta(x) \in \hat{\Gamma}(a) = \bigcup\{\Gamma(i)|\ i = 0, 1, \ldots\}$, there is a natural number $i(\beta(x))$ such that $\beta(x) \in \Gamma(i(\beta(x)))$.

Now recall that for an index γ to appear in a set $\Gamma(i+1)$, it is necessary and sufficient that γ belong to $\Gamma(\beta)$ for some $\beta \in \Gamma(i)$. Since $\beta(x) \in \Gamma(i(\beta(x)))$, every index $\gamma \in \Gamma(\beta(x))$ qualifies for membership in $\Gamma(i(\beta(x))+1)$. Thus $\bigcup\{F(\gamma)|\ \gamma \in \Gamma(\beta(x))\} \subset \bigcup\{F(\gamma)|\ \gamma \in \Gamma(i(\beta(x))+1)\}$ and this latter set is in turn a subset of $\bigcup\{F(\gamma)|\ \gamma \in \hat{\Gamma}(a)\}$. But we also know that $x \in F(\beta(x))$, which is a subset of int $\bigcup\{F(\gamma)|\ \gamma \in \Gamma(\beta(x))\}$.

Thus $x \in \operatorname{int}\bigcup\{F(\gamma)|\ \gamma \in \Gamma(\beta(x))\} \subset \bigcup\{F(\gamma)|\ \gamma \in \hat{\Gamma}(a)\} = G(a)$, and $G(a)$ is seen to be open.

Note further that the family $\{G(a)|\ a \in \Gamma\}$ is closure-preserving. This is a straightforward result following from the fact that each set $G(a)$ is the union of members of a closure-preserving family of compact closed sets.

Now suppose the index set Γ to be well-ordered. For each $a \in \Gamma$, let $V(a) = G(a) - \bigcup\{G(\beta)|\ \beta < a\}$. Then the following facts about the family $\{V(a)|\ a \in \Gamma\}$ are easily verified: each set $V(a)$ is open, closed and σ-compact; the members of $\{V(a)|\ a \in \Gamma\}$ are pairwise disjoint.

COROLLARY 1. *If, in addition to the hypotheses of Theorem* 1, X *is required to be* T_3, *then* X *is paracompact.*

Proof. A T_3, σ-compact space is paracompact, whence X is the disjoint union of open paracompact subspaces, whence is itself paracompact.

Note that if X is not required to be T_3, X may fail to be paracompact. To see this, let X be any countable connected T_2 space; $X = \{x(i)|\ i \in Z^+\}$. For each positive integer j, let $X(j) = \{x(i)|\ i \leqslant j\}$. Then the family $\{X(j)\}$ is a countable closure-preserving family of compact sets whose union is X, but X is not paracompact, nor even normal or regular.

COROLLARY 2. *Let* X *be a space, and* $\mathfrak{F} = \{F(a)|\ a \in \Gamma\}$ *a closure-preserving family of compact closed sets whose union is* X. *If the family* $\mathfrak{F}$ *is either point-countable or star-countable, then* X *is the disjoint union of open and closed* σ*-compact subsets.*

Proof. Both cases are special cases of Theorem 1. In the event that the family $\mathfrak{F}$ is star-countable, the result can be obtained without well-ordering the index set.

Various other modification of Theorem 1 are also possible. If in Theorem 1 the members of $\mathfrak{F}$ are required only to be closed and σ-compact, the same result follows. If they are required to be closed and Lindelöf, then X is the pairwise disjoint union of open and closed Lindelöf subspaces.

COROLLARY 3. *Let* $X = \bigcup\{F(a)|\ a \in \Gamma\}$, *where each* $F(a)$ *is open, each* $\overline{F(a)}$ *is compact or* σ*-compact, and the family* $\{F(a)|\ a \in \Gamma\}$ *is closure-preserving. Then* X *is the disjoint union of open and closed* σ*-compact subspaces.*

Proof. The family $\{\overline{F(a)}|\ a \in \Gamma\}$ satisfies the hypotheses of Theorem 1.

COROLLARY 4. *Let* X *be locally compact,* T_2. *If every open cover of* X *has an open closure-preserving refinement, then* X *is the disjoint union of open and closed* σ*-compact subspaces.*

Proof. Cover X with open sets whose closures are compact. Let $\mathfrak{W}$ be a closure-preserving open refinement which covers X. Then the family $\{\overline{W}|\ W \in \mathfrak{W}\}$ satisfies the hypotheses of Corollary 3, and the conclusion follows.

Note that any space X as described in Corollary 3 is locally compact, or locally σ-compact, but not every locally compact space will admit such an open cover. An easy example is the space of countable ordinals with the usual topology.

Reference

[1] H. Tamano, *A characterization of paracompactness*, Fund. Math. 72 (1971), pp. 189–201.

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