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Menger's Theorem for topological spaces

by

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§ 1. Introduction. Menger's Theorem [1] for graphs has been generalized by Nöbeling [3] to locally connected compact metric spaces. In this paper we generalize Menger's Theorem to Hausdorff topological spaces with no other global conditions on the space, but with local conditions on the two subsets involved.

THEOREM 1.1. *Let A and B be disjoint open subsets of a Hausdorff topological space X. Suppose that the maximal number of disjoint arcs from A to B is finite. Then this number is equal to the minimal number of points that have to be removed from X to separate A and B into different arc components.*

When we restrict X to be a graph, our proof of Theorem 1.1 reduces essentially to Ore's proof of Menger's Theorem [4], Chapter 12.

COROLLARY 1.2 (Menger's Theorem). *Let A and B be disjoint sets of vertices of a finite or infinite graph X. Suppose that there is no edge with one vertex in A and the other in B. Then the maximal number of disjoint arcs from A to B is equal to the minimal number of vertices that have to be removed from X to separate A and B into different components.*

Sections 2 and 3 are devoted to proving Theorem 1.1. In section 4 we show, by example, that some of the conditions of Theorem 1.1 and Nöbeling's result cannot be weakened. Finally in section 5 we discuss the case where the maximal number of arcs is infinite.

The author wishes to thank Michael Mather for his helpful comments and Mary Ellen Rudin for Example 5.1.

§ 2. Definitions. Let A and B be subsets of a topological space X. Let $I = [0, 1]$ be the closed unit interval and $\mathring{I} = (0, 1)$ be the open unit interval. An *arc* λ from A to B in X is an injective map λ: $I \to X$ such that $\lambda(0) \in A$ and $\lambda(1) \in B$. The family of arcs $\{\lambda_q\}$ is said to be *disjoint* if

$$\lambda_q(\mathring{I}) \cap \lambda_r(\mathring{I}) = \emptyset$$

for all arcs λ_q, λ_r in the family with $q \neq r$.

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Lemma 2.1. *Let λ and μ be two arcs in a Hausdorff space X and let*

$$w = \sup\{t \in I \mid \lambda(t) \in \mu(I)\}.$$

Then $\lambda(w) \in \mu(I)$.

Proof. The spaces $\lambda(I)$ and $\mu(I)$ are compact subsets of a Hausdorff space X and hence are closed. Now

$$\lambda(w) \in \overline{\lambda(I) \cap \mu(I)} = \lambda(I) \cap \mu(I).$$

Hence $\lambda(w) \in \mu(I)$.

Let $\lambda_1, \ldots, \lambda_n$ be a family of arcs in X. An arc μ from $\mu(0)$ to $\mu(1)$ is called a *cross arc* with respect to $\lambda_1, \ldots, \lambda_n$ if for $1 \leqslant i \leqslant n$, the intersection $\mu(I) \cap \lambda_i(I)$ is a finite number of disjoint closed arcs of the form $\mu[s, s'] = \lambda_i[t, t']$ where $\mu(s) = \lambda_i(t') \neq \mu(s') = \lambda_i(t)$.

Proposition 2.2. *Let μ be a cross arc from A to B with respect to the n disjoint arcs $\lambda_1, \ldots, \lambda_n$ from A to B. Then there exist $n+1$ disjoint arcs from A to B.*

Proof. It can be proved in a similar way to [4], Theorem 12.1.1 that the symmetric difference of $\mu(I)$ and $\lambda_1(I) \cup \ldots \cup \lambda_n(I)$ consists of $n+1$ disjoint arcs from A to B together with a finite number of closed circuits.

Denote by $\Lambda(A, B)$ the maximal number of disjoint arcs from A to B in X and by $\Gamma(A, B)$ the minimal number of points that have to be removed from $X-(A \cup B)$ to separate A and B into different arc components.

§ 3. Menger's Theorem.

Proof of Theorem 1.1. Let $\Lambda(A, B) = n$ and $\lambda_1, \ldots, \lambda_n$ be a maximal set of disjoint arcs from A to B in X. It is clear that $\Lambda(A, B) \leqslant \Gamma(A, B)$ because at least one point must be removed from each of the arcs in order to separate A and B.

For each i, let t_i be the supremum of t in I such that, $\lambda_i(t)$ is in A or that there exists a cross arc from A to $\lambda_i(t)$ with respect to $\lambda_1, \ldots, \lambda_n$. Since A is open, $\lambda_i(t_i)$ is not in A. By Proposition 2.2 there is no cross arc from A to B and, since B is open, $\lambda_i(t_i)$ is not in B.

We will now prove that

$$Y = X - \bigcup_{i=1}^{n} \lambda_i(t_i)$$

separates A and B into different arc components. Suppose that there is an arc from A to B in Y. Then by Proposition 2.2 there is an arc ν in Y from

$$A \cup \bigcup_{i=1}^{n} \lambda_i[0, t_i) \quad \text{to} \quad B \cup \bigcup_{i=1}^{n} \lambda_i(t_i, 1]$$

which is disjoint from $\lambda_1, \ldots, \lambda_n$.

From the definitions of t_i, it follows that there is no cross arc in X from A to $\nu(1)$; hence $\nu(0)$ is not in A. Let $\nu(0) = \lambda_c(r)$ where $1 \leqslant c \leqslant n$ and $0 < r < t_c$. Again from the definition of t_c there is a cross arc μ in X from A to $\lambda_c(s)$ with respect to $\lambda_1, \ldots, \lambda_n$, where $0 < r < s \leqslant t_c$.

If μ and ν are disjoint then, the arc μ followed by the arc $\lambda_c(t)$, $s \geqslant t \geqslant r$ and then followed by the arc ν, is a cross arc from A to $\nu(1)$.

If μ and ν are not disjoint let

$$w = \sup\{t \in I \mid \nu(t) \in \mu(I)\}.$$

By Lemma 2.1 $\nu(w)$ is in $\mu(I)$. Hence the arc μ from A to $\nu(w)$ followed by ν from $\nu(w)$ to $\nu(1)$ is a cross arc from A to $\nu(1)$. This contradicts the definition of the t_i, so that Y separates A and B into different arc components. Hence $\Lambda(A, B) \geqslant \Gamma(A, B)$ which completes the proof of Theorem 1.1.

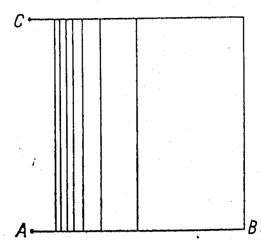
Proof of Corollary 1.2. Let X be any finite or infinite graph. Give all the edges the same length and put the weak topology on X. For the set of vertices A define A^* to be the union of A with all the open edges with one vertex in A. Define B^* similarly. Then A^* and B^* are disjoint open subsets of X. Corollary 1.2 now follows by applying Theorem 1.1 to A^* and B^* and noting that in this case the given construction for finding the points of X to be removed from each arc will always lead to a vertex of the graph.

§ 4. Counterexamples. We show that in Theorem 1.1 the condition that X be Hausdorff cannot be dropped and that in Nöbeling's case [3], when A and B are closed, neither the condition of locally connectedness nor the condition of compactness can be dropped.

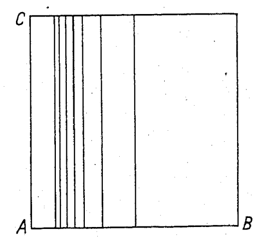
Example 4.1. Let X consist of two copies of $[-1, 1]$ identified at all points except 0. Let $A = [-1, 0)$ and $B = (0, 1]$. Then X is T_1 but not Hausdorff and $\Lambda(A, B) = 1$ while $\Gamma(A, B) = 2$.

Example 4.2. Let

$$X = I \times \{0\} \cup I \times \{1\} \cup \bigcup_{n \geqslant 1} \left\{\frac{1}{n}\right\} \times I$$



Example 4.2



Example 4.3

be a subset of the plane $\boldsymbol{R}^2$ and let $A=(0,0)$, $B=(1,0)$ and $C=(0,1)$. Then X is not compact and $\Lambda(A,B)=1$ while $\Gamma(A,B)=2$. Also $\Lambda(A,C)=1$ while $\Gamma(A,C)=\aleph_0$.

EXAMPLE 4.3. Add the line $\{0\}\times I$ to Example 4.2. This is now compact but not locally connected and $\Lambda(A,C)=2$ while $\Gamma(A,C)=\aleph_0$.

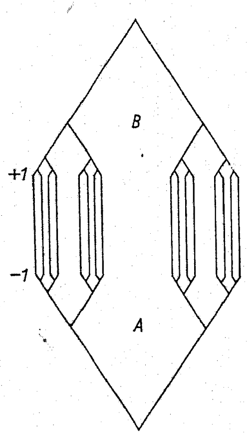
§ 5. Infinite number of arcs. It is always true that $\Lambda(A,B)\leqslant\Gamma(A,B)$. If X is a graph and $\Lambda(A,B)$ is an infinite cardinal we can find a separating set by removing all the vertices in $X-(A\cup B)$ of a maximal disjoint set of arcs from A to B. Since there are only a finite number of vertices on each arc $\Lambda(A,B)=\Gamma(A,B)$.

If X is a topological space and $\Lambda(A,B)$ is at least the cardinality of the continuum, a separating set can be obtained by removing all the points in $X-(A\cup B)$ of a maximal disjoint set of arcs from A to B. Then $\Lambda(A,B)=\Gamma(A,B)$.

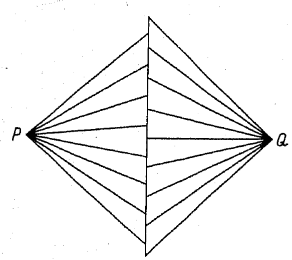
When $\Lambda(A,B)=\aleph_0$ Menger's Theorem for topological spaces is false as the following example due to Mary Ellen Rudin shows.

EXAMPLE 5.1. Let X be the subset of the plane $\boldsymbol{R}^2$ shown in the diagram which consists of the product of the Cantor set with the closed unit interval $[-1,1]$ together with the open set A below the ordinate -1 and the open set B above the ordinate $+1$. Then $\Lambda(A,B)=\aleph_0$ while $\Gamma(A,B)$ has the cardinality of the continuum.

This follows from the fact that there exists an arc from A to B along the interval corresponding to any given point of the Cantor set. Any point of the Cantor set has a triadic expansion that uses only 0's and 2's and an arc can be constructed starting at the lowest vertex of A and



Example 5.1



Example 5.2

turning left or right at the nth junction depending on whether there is a 0 or 2 in the nth place of the triadic expansion.

This example is not locally connected but can be made so by shrinking each closed interval $[-1,1]$ to a point.

For finite $\Lambda(A,B)$ we have shown that there exists a subset V of X which separates A and B, and a set of disjoint arcs $\mathfrak{W}$ from A to B such that each point of V lies on exactly one of these arcs and each arc contains exactly one point of V. Erdös [2], p. 292 has asked whether this result holds for graphs when $\Lambda(A,B)$ is infinite. The following example shows that the result is false for topological spaces, even when $\Lambda(A,B)=\Gamma(A,B)$.

EXAMPLE 5.2. Let X be constructed as follows. Take a unit interval I and two points P and Q outside I. Join P to all the irrational points of I and Q to all the rational points of I. Let A and B be neighbourhoods of P and Q respectively.

Then $\Lambda(A,B)=\Gamma(A,B)=\aleph_0$. But suppose V is a separating set for A and B and that $\mathfrak{W}$ is a set of disjoint arcs from A to B such that each arc contains exactly one point of V. Then, because the rationals and irrationals are dense in one another any arc of $\mathfrak{W}$ can be removed and replaced by an infinite set of disjoint arcs. Each of these new arcs contain a point of the separating set V so that there exists a point of V which did not lie on any of the original arcs of $\mathfrak{W}$.

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