The local contractibility of the homeomorphism space of a 2-polyhedron

by

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1. Introduction. It is proved by Černavskiĭ in [3] and by Edwards and Kirby in [4] that the space of homeomorphisms of a compact manifold (with the compact-open topology) is locally contractible. A natural question to ask is whether this result holds for the homeomorphism group of a compact polyhedron. The main result of this paper is

THEOREM 1.1. *The space of homeomorphisms of a compact 2-dimensional polyhedron is locally contractible.*

2. Preliminaries. Let X be a topological space and $A \subset X$. The closure, point set interior, and frontier of A in X will be denoted $\mathrm{cl}[A]$ (sometimes $\bar{A}$), $\mathrm{int}[A]$, and $\mathrm{fr}[A]$, respectively. A set $U \subset X$ is a neighborhood of $A \subset X$ if $A \subset \mathrm{int}[U]$.

By a complex we always mean a finite simplicial complex contained in some Euclidean space. Small Greek letters represent closed simplexes and we write $\alpha < J$ if α is a simplex of the complex J. The notation α^0 denotes the simplex less its boundary. We write $K < J$ to denote a subcomplex K of J. The notation $|J|$ denotes the subset of Euclidean space which carries J. If $K < J$ and $\alpha \cap |K|$ is a simplex of K for each simplex $\alpha < J$, we say K is a full subcomplex of J. The star of σ in J is denoted $\mathrm{St}(\sigma, J)$; thus, $\mathrm{St}(\sigma, J) = \{\alpha < J | \alpha < \beta \text{ and } \sigma < \beta \text{ for some } \beta < J\}$. If S is a set, we let $N(S, J) = \{\sigma < J | \sigma \text{ is a face of some simplex of } J \text{ which meets } S\}$ and $C(S, J) = \{\sigma < J | \sigma \cap S = \emptyset\}$.

Let X be a polyhedron and $x \in X$. The intrinsic dimension of x in X, denoted $d(x; X)$, is the greatest integer t such that there is a triangulation J of X with a t-simplex containing x in its interior. The intrinsic i-skeleton of X, denoted $I^i(X)$, is defined to be the set $\{x \in X | d(x; X) \leqslant i\}$.

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Remarks. If $x \in X$ and h is a homeomorphism of X then $d(x; X) = d(h(x); X)$, i.e., intrinsic dimension is a topological invariant. Akin proved in [1] that $I^i(X) - I^{i-1}(X)$ is a manifold.

Let X be a 2-polyhedron. The thin intrinsic 1-skeleton, denoted $T(X)$, is the subset of $I^1(X)$ consisting of all points contained in the boundary of some 2-simplex of some triangulation of X.

Remarks. The definition of $T(X)$ is independent of the triangulation of X since a point x of $I^1(X)$ is contained in the boundary of a 2-simplex of some triangulation of X if and only if x is contained in the boundary of a 2-simplex of each triangulation of X. Each of the sets $I^0(X)$, $I^1(X)$, and $T(X)$ carries a subcomplex of each triangulation of X. Any homeomorphism of X maps each of $I^0(X)$, $I^1(X)$, and $T(X)$ onto itself.

Suppose X is a compact Hausdorff space. The space of homeomorphisms of X will be denoted $\mathcal{H}(X)$. If $Y \subset X$ then $\mathcal{H}(X, Y)$ denotes the subspace of $\mathcal{H}(X)$ consisting of those homeomorphisms of X which are the identity on Y.

Remarks. Let ϱ be the usual Euclidean metric and

$$M(h, \varepsilon) = \{g \in \mathcal{H}(X) |\ \varrho(g(x), h(x)) < \varepsilon \text{ for each } x \in X\}.$$

The collection $\{M(h, \varepsilon)|\ h \in \mathcal{H}(X), \varepsilon > 0\}$ is a basis for the compact-open topology for $\mathcal{H}(X)$. It is well-known (see [2]) that $\mathcal{H}(X)$ is a topological group and the evaluation map $\mathcal{H}(X) \times X \to X$, defined by $(f, x) \to f(x)$, is continuous.

If M is a manifold and $U \subset M$, a proper imbedding of U into M is an imbedding $h: U \to M$ such that $h^{-1}(\partial M) = U \cap \partial M$. An isotopy of U into M is a family of imbeddings $h_t: U \to M$, $t \in I$ (where I denotes the unit interval), such that the map $h: U \times I \to M$, defined by $h(x, t) = h_t(x)$, is continuous. An isotopy is proper if each imbedding in the isotopy is proper.

If $C \subset U \subset M$, let $I(U, C; M)$ denote the set of proper imbeddings of U into M which are the identity on C. The topology for $I(U, C; M)$ is the compact-open topology.

If $P \subset I(U, C; M)$, $W \subset U$, and $\Phi: P \times I \to I(U, C; M)$ is a deformation (i.e., $\Phi(h, 0) = h$ for each $h \in P$) such that $\Phi(h, t)(w) = w$ for each $w \in W$ and $(h, t) \in P \times I$, we say Φ is modulo W.

Suppose $A_1, A_2, \ldots, A_{n+1} \subset I(U, C; M)$ and $\Phi_i: A_i \times I \to I(U, C; M)$ is a deformation of A_i into A_{i+1} (i.e., $\Phi_i(A_i \times 1) \subset A_{i+1}$) for each i. Inductively we define a deformation

$$\Phi_n * \Phi_{n-1} * \ldots * \Phi_1: A_1 \times I \to I(U, C; M)$$

of A_1 into A_{n+1} by letting, for each $h \in A_1$,

$$\Phi_n * \Phi_{n-1} * \ldots * \Phi_1(h, t) = \begin{cases} \Phi_{n-1} * \ldots * \Phi_1(h, 2t) & \text{if } 0 \leqslant t \leqslant 1/2, \\ \Phi_n(\Phi_{n-1} * \ldots * \Phi_1(h, 1), 2t-1) & \text{if } 1/2 \leqslant t \leqslant 1. \end{cases}$$

LEMMA 2.1. *Let X be a locally compact Hausdorff space and 1_X be the identity map on X. If P is a neighborhood of 1_X in $\mathcal{H}(X)$ and $\Phi: P \times I \to \mathcal{H}(X)$ is a deformation of P into 1_X then*

(1) *we may assume $(1_X \times I) = 1_X$ and*

(2) *if Q is a neighbourhood of 1_X in $\mathcal{H}(X)$ and $\Phi(1_X \times I) = 1_X$, we may choose P so that $\Phi(P \times I) \subset Q$.*

Proof. (1) Let $\Phi'(h, t) = \Phi(1_X, t)^{-1}\Phi(h, t)$ for each (h, t) in $P \times I$. Then $\Phi': P \times I \to \mathcal{H}(X)$ is a deformation of P into 1_X such that $\Phi'(1_X \times I) = 1_X$. Replace Φ by Φ'.

(2) Since $\Phi(1_X \times I) \subset Q$, $\Phi^{-1}(Q)$ is a neighborhood of $1_X \times I$ in $\mathcal{H}(X) \times I$. Let S be a neighborhood of 1_X in $\mathcal{H}(X)$ such that $S \times I \subset \Phi^{-1}(Q)$ and set $\Lambda = \Phi | S \times I$. Then $\Lambda: S \times I \to \mathcal{H}(X)$ is a deformation of S into 1_X such that $\Lambda(S \times I) \subset Q$. q.e.d.

Let X be a space such that $\mathcal{H}(X)$ is a topological group. If there exists a neighborhood P of 1_X in $\mathcal{H}(X)$ and a deformation $\Phi: P \times I \to \mathcal{H}(X)$ of P into 1_X, we say $\mathcal{H}(X)$ is locally contractible.

Remark. To see that this definition of local contractibility is equivalent to the usual one, use lemma 2.1 and the fact that, in a topological group, local properties at the identity are local properties at every point.

3. The proof of Theorem 1.1.

In this section we state two lemmas needed in the proof of Theorem 1.1. Then we prove Theorem 1.1 assuming these lemmas. The remainder of the paper is then devoted to proving the lemmas.

LEMMA 3.1. *Let X be a compact 2-dimensional polyhedron. There exists a neighborhood P of 1_X in $\mathcal{H}(X)$ and a deformation $\Phi: P \times I \to \mathcal{H}(X)$ of P into $\mathcal{H}(X, I^1(X))$ such that $\Phi(1_X \times I) = 1_X$.*

LEMMA 3.2. *Let X be a compact 2-dimensional polyhedron. There exists a neighborhood Q of 1_X in $\mathcal{H}(X, I^1(X))$ and a deformation $\Psi: Q \times I \to \mathcal{H}(X)$ of Q into 1_X.*

Proof of Theorem 1.1. By Lemma 3.1 there is a neighborhood P of 1_X in $\mathcal{H}(X)$ and a deformation $\Phi_1: P \times I \to \mathcal{H}(X)$ of P into $\mathcal{H}(X, I^1(X))$ such that $\Phi(1_X \times I) = 1_X$. By Lemma 3.2 there is a neighborhood Q of 1_X in $\mathcal{H}(X, I^1(X))$ and a deformation $\Phi_2: Q \times I \to \mathcal{H}(X)$ of Q into 1_X. By the continuity of Φ_1, we may assume $\Phi_1(P \times 1) \subset Q$. Let $\Phi = \Phi_2 * \Phi_1$. Then $\Phi: P \times I \to \mathcal{H}(X)$ is a deformation of P into 1_X. q.e.d.

4. Cell-sets. Throughout the remainder of this paper X, Y, and Z will denote a fixed compact 2-polyhedron, $(X-I^1(X))\cup T(X)$, and $\mathrm{cl}[X-Y]\cap Y$, respectively. The set Y is a homogeneously 2-dimensional compact subpolyhedron of X. The homogeneity of Y means Y contains no open sets homeomorphic to a subset of 1-dimensional Euclidean space. This implies each simplex of a triangulation of Y is the face of at least one 2-simplex. The set Z is a finite set of points each of which is in $I^0(X)$. A typical element of Z is a point of intersection of an arc in $\mathrm{cl}[X-Y]$ with Y.

A triangulation J of X is fixed and taken to be the second barycentric subdivision of some triangulation L of X. Let K, L_1, and L_0 denote the subcomplexes of J carried by Y, $I^1(Y)$, and $I^0(Y)$, respectively. Since any triangulation of X contains subcomplexes carried by Y, $I^1(Y)$, and $I^0(Y)$, each of K, L_1, and L_0 is full in J.

We now describe three types of cell-sets related to the vertices of L_1.

Suppose v is a vertex of L_1 and λ is a 1-simplex of L_1 with $v<\lambda$. Let σ_1 be a 2-simplex of K such that $\lambda<\sigma_1$ and λ_1 be the other 1-face of σ_1 such that $v<\lambda_1$. If $\lambda_1<L_1$, we stop the process. Otherwise, there is exactly one other 2-simplex of K, say σ_2, having λ_1 as a face. Let λ_2 be the other 1-face of σ_2 having vertex v. If $\lambda_2<L_1$, we stop the process. Otherwise, we continue as before. Inductively, we get a finite sequence $\sigma_1, \sigma_2, \ldots, \sigma_n$ of distinct 2-simplexes of K and a finite sequence $\lambda_1, \lambda_2, \ldots, \lambda_n$ of distinct 1-simplexes of K such that $\sigma_i\cap\sigma_{i+1}=\lambda_i \not< L_1$ if $i<n$ $v<\lambda_i\cap\sigma_i$ for each i, and $\lambda_n<L_1$, where λ_n is the other 1-face of σ_n having v as a vertex. Let $C=\bigcup_{i=1}^{n}\sigma_i$. If $\lambda\neq\lambda_n$, C is a PL 2-cell such that $C\cap I^1(Y)=\lambda\cup\lambda_n$ and $v=\lambda\cap\lambda_n$ is in the boundary of C. If $\lambda=\lambda_n$, C is a PL 2-cell such that $C\cap I^1(Y)=\lambda$ and $v\in\mathrm{int}[C]$. If $\lambda\neq\lambda_n$, we say C is a *cell-set of type* 1 *at* v. If $\lambda=\lambda_n$, we say C is a *cell-set of type* 2 *at* v.

Suppose v is a vertex of L_1 and σ is a 2-simplex of K such that $v<\sigma$ and σ is not contained in any cell-set of type 1 or 2 at v. Then there is a finite sequence $\sigma_1, \sigma_2, \ldots, \sigma_n$ of distinct 2-simplexes of K such that $\sigma=\sigma_1$, $\sigma_i\cap\sigma_{i+1}$ is a 1-simplex of $K-L_1$, $\sigma_i\cap\sigma_j=v$ if $1<|i-j|<n-1$, and $\sigma_1\cap\sigma_n$ is a 1-simplex of $K-L_1$. Let $C=\bigcup_{i=1}^{n}\sigma_i$ and notice that C is a PL 2-cell containing v in its interior such that $C\cap I^1(Y)=v$. In this case C is called a *cell-set of type* 3 *at* v.

Remarks. If $v<L_1$ then each 2-simplex of K having v as a vertex is contained in exactly one cell-set of one of the above types at v. If $v<L_1-L_0$, there are no cell-sets of type 2 or 3 at v. If C_1 and C_2 are distinct cell-sets at v then $C_1\cap C_2\subset I^1(Y)$.

5. Neighborhoods related to components of $I^1(Y)-I^0(Y)$. The components of the 1-manifold $I^1(Y)-I^0(Y)$ are either open arcs or simple closed curves. If σ is an open arc component then $\bar{\sigma}$ is either a closed polygonal arc with endpoints in L_0 or a polygonal simple closed curve with $\bar{a}-a$ equal to a single point of L_0. We are going to use cell-sets of type 1 to describe some nice simplicial neighborhoods which will be used in the proof of Lemma 3.1.

We first consider the open arc components of $I^1(Y)-I^0(Y)$ with 1-cell closures. Let a be such a component. Then $\bar{a}$ is a polygonal closed arc and there are 1-simplexes $a_1, a_2, \ldots, a_r$ in L_1 such that $a_i\cap a_j=\emptyset$ if $|i-j|\neq 1$ and $\bigcup_{i=1}^{r}a_i=\bar{a}$. Let v_{i-1} and v_i be the vertices of a_i for each i. Then $a_i\cap a_{i+1}=v_i$ and $v_0, v_r<L_0$. Let $\tau_1, \tau_2, \ldots, \tau_t$ be the set of all 2-simplexes of K which have a_1 as a face. The 2-simplex τ_1 generates a cell-set $C(1,1)=\bigcup_{j=1}^{n(1)}\sigma(1,j)$ of type 1 at v_1, where each $\sigma(1,j)$ is a 2-simplex in K, such that $C(1,1)\cap I^1(Y)=a_1\cup a_2$, $\sigma(1,1)=\tau_1$, and $a_2<\sigma(1,n(1))$. Next, the 2-simplex $\sigma(1,n(1))$ generates a cell-set $C(1,2)=\bigcup_{j=1}^{n(2)}\sigma(2,j)$ of type 1 at v_2, where each $\sigma(2,j)$ is a 2-simplex of K, such that $C(1,2)\cap I^1(Y)=a_2\cup a_3$, $\sigma(2,1)=\sigma(1,n(1))$, and $a_3<\sigma(2,n(2))$. Continuing in this manner we get, for each $i<r$, a cell-set $C(1,i)=\bigcup_{j=1}^{n(i)}\sigma(i,j)$ of type 1 at v_i, where each $\sigma(i,j)$ is a 2-simplex of K, such that $C(1,i)\cap C(1,i-1)=\sigma(i,1)=\sigma(i-1,n(i-1))$ and $C(1,i)\cap I^1(Y)=a_i\cup a_{i+1}$ for $i=3,4,\ldots,r-1$. Let $C_1=\bigcup_{i=1}^{r-1}C(1,i)$ and notice that C_1 is a PL 2-cell containing $\bar{a}$ in its boundary and $C_1\cap I^1(Y)=\bar{a}$. In a similar manner each τ_q, $2\leqslant q\leqslant t$, generates a PL 2-cell C_q with $\bar{a}$ in its boundary such that $C_q\cap I^1(Y)=\bar{a}$. In fact, if $1\leqslant q<s\leqslant t$, $C_q\cap C_s=\bar{a}$. It is easy to see that $|N(a,K)|=\bigcup_{q=1}^{t}C_q$ is a neighborhood of a in Y homeomorphic to $\bigcup_{q=1}^{t}B_q\times\bar{a}$, where in polar coordinates $B_q=\{(r,\theta_q)\in R^2|\ 0\leqslant r\leqslant 1$ and $\theta_q=(q-1)\pi/q\}$. Let $u_a\colon \bigcup_{q=1}^{t}B_q\times\bar{a}\to|N(a,K)|$ be a homeomorphism with the property that $u_a(B_q)=C_q$ for each $q=1,2,\ldots,t$ and $u_a((0,0),x)=x$ for each $x\in\bar{a}$. We will refer to the pair $(|N(a,K)|, u_a)$ as a *type* 1 *neighborhood pair of* a.

Now let us turn our attention to the open arc components of $I^1(Y)-I^0(Y)$ with simple closed curve closures. Let β be such a component and $v=\bar{\beta}-\beta$. Let $\beta_1, \beta_2, \ldots, \beta_r$ be the 1-simplexes of L_1 such that β_1 has endpoints v and v_1, and β_i has endpoints v_{i-1} and v_i for $1 < i < r$, β_r has endpoints v_{r-1} and v, and $\bar{\beta} = \bigcup_{i=1}^{r} \beta_i$. Next, let $\tau_1, \tau_2, \ldots, \tau_t$ be the 2-simplexes of K which contain β_1 as a face. By piecing together cell-sets of type 1 at $v_1, v_2, \ldots, v_{r-1}$ as in the preceding paragraph, each τ_q generates a set C_q which is a PL 2-cell with two points on its boundary identified. Viewing C_q as a pinched annulus, there is a homeomorphism $u_q\colon B_q \times \bar{\beta}/B_q \times v \to C_q$ such that $u_q\langle(0,0), y\rangle = y$ for all $y \in \bar{\beta}$, where $B_q \times \bar{\beta}/B_q \times v$ denotes the quotient space obtained by identifying all points of $B_q \times v$ and $\langle x, y\rangle$ denotes the quotient map image of $(x, y) \in B_q \times \bar{\beta}$. Since the sets $C_q - \bar{\beta}$ are pairwise disjoint and $|N(\beta, K)| = \bigcup_{q=1}^{t} C_q$, we can define a homeomorphism $u_\beta\colon \bigcup_{q=1}^{t} B_q \times \bar{\beta}/\bigcup_{q=1}^{t} B_q \times v \to |N(\beta, K)|$ by letting $u_\beta\langle x, y\rangle = u_q\langle x, y\rangle$ if $(x, y) \in B_q \times \bar{\beta}$. The set $|N(\beta, K)|$ is a neighborhood of β in X and can be viewed as $\bigcup_{q=1}^{t} B_q \times S^1$ with $\bigcup_{q=1}^{t} B_q \times w$ pinched to a point for some $w \in S^1$. Notice that for $x \in \bigcup_{q=1}^{t} B_q$, $u_\beta p(x \times \bar{\beta})$ is homeomorphic to $\bar{\beta}$, where p is the quotient map from $\bigcup_{q=1}^{t} B_q \times \bar{\beta}$ onto $\bigcup_{q=1}^{t} B_q \times \bar{\beta}/\bigcup_{q=1}^{t} B_q \times v$. We will refer to the pair $(|N(\beta, K)|, u_\beta)$ as a *type 2 neighborhood pair of* β.

Finally, we consider the simple curve components of $I^1(Y) - I^0(Y)$. Let ζ be one such component and α and β be open arcs in ζ such that $\bar{\alpha}$ and $\bar{\beta}$ are closed polygonal arcs which carry subcomplexes of L and $\alpha \cup \beta = \zeta$. Let L_α and L_β be the subcomplexes of K carried by $\bar{\alpha}$ and $\bar{\beta}$, respectively. Piecing together cell-sets of type 1 at the vertices of L_α which lie in α, we construct a type 1 neighborhood pair $(|N(\alpha, K)|, u_\alpha)$ at α just as we did above. Similarly, we construct a type 1 neighborhood pair $(|N(\beta, K)|, u_\beta)$ at β. It is easy to see that $|N(\alpha, K)| \cup |N(\beta, K)|$ is a neighborhood of ζ and $(|N(\alpha, K)| \cup |N(\beta, K)|) \cap I^1(Y) = \zeta$.

6. Proof of Lemma 3.1. Step 1. Let X_1 be the set of open arc components of $I^1(X) - I^0(X)$ with 1-cell closures. Clearly, there is a neighborhood P_1' of $1_{I^1(X)}$ in $\mathcal{H}(I^1(X), I^0(X))$ each homeomorphism of which maps each component of $I^1(X) - I^0(X)$ onto itself, and a deformation $\Phi_1'\colon P_1' \times I \to \mathcal{H}(I^1(X), I^0(X))$ of P_1' into $\mathcal{H}(I^1(X), I^0(X) \cup X_1)$ such that Φ_1' is modulo $I^1(X) - X_1$, $\Phi_1'(h, t)$ maps each component of $I^1(X) - I^0(X)$ onto itself, and $\Phi_1'(1_{I^1(X)} \times I) = 1_{I^1(X)}$. Let P_1 be a neighborhood of 1_X in $\mathcal{H}(X)$ such that $h|I^1(X) \in P_1'$ for each $h \in P_1$. Since $I^0(X)$ is a finite set of points and each homeomorphism of X maps $I^0(X)$ onto itself, for P_1 sufficiently small, each homeomorphism of P_1 is the identity on $I^0(X)$.

Next, we describe how to construct a deformation $\Phi_1\colon P_1 \times I \to \mathcal{H}(X)$ of P_1 into $\mathcal{H}(X, I^0(X) \cup X_1)$ with the properties $\Phi_1(1_X \times I) = 1_X$ and $\Phi_1(h, t)|I^1(X) = \Phi_1'(h|I^1(X), t)$ for each $(h, t) \in P_1 \times I$.

Let $Y_1 = \{\alpha_1, \ldots, \alpha_j\}$ be the arcs of X_1 contained in $T(X)$. For each $\alpha \in Y_1$ consider $(|N(\alpha, K)|, u_\alpha)$, the type 1 neighborhood pair of α. Note that if α and α' are distinct open arcs in Y_1 then

$$|N(\alpha, K)| \cap |N(\alpha', K)| \subset \bar{\alpha} - \alpha \subset I^0(X).$$

We now define Φ_1 partially by letting

$$\Phi_1(h, t)(z) = \begin{cases} \Phi_1(h|I^1(X), t)(z) & \text{if} \quad z \in I^1(X), \\ h(z) & \text{if} \quad z \in X - \big(I^1(X) \cup \big(\bigcup_{\alpha \in Y_1} |N(\alpha, K)|\big)\big) \end{cases}$$

for each $(h, t) \in P_1 \times I$.

Let $\alpha \in Y_1$ and recall that $u_\alpha\colon \bigcup_{q=1}^{t} B_q \times \bar{\alpha} \to |N(\alpha, K)|$ is a homeomorphism such that $u_\alpha((0, 0), x) = x$ for each $x \in \bar{\alpha}$. Let $\pi_\alpha\colon \bigcup_{q=1}^{t(\alpha)} B_q \times \bar{\alpha} \to (0, 0) \times y$ be the projection so that $\pi_\alpha(x, y) = ((0, 0), y))$ for each pair (x, y). For each $x \in \bigcup_{q=1}^{t(\alpha)} B_q$, let $\pi_{\alpha,x} = (\pi_\alpha|x \times \bar{\alpha})^{-1}$. Notice that $\pi_{\alpha,x}$ is a homeomorphism from $0 \times \bar{\alpha}$ onto $x \times \bar{\alpha}$ and $\pi_{\alpha,x}(0, y) = (x, y)$ for each $y \in \bar{\alpha}$. Now for each $z = u_\alpha(x, y) \in |N(\alpha, K)|$ and $(h, t) \in P_1 \times I$ let $\Phi_1(h, t)(z) = hu_\alpha \pi_{\alpha,x} u_\alpha^{-1} h^{-1} \Phi_1'(h|I^1(X), t(1 - \|x\|)) u_\alpha \pi_\alpha u_\alpha^{-1}(z)$, where $\|x\| = \varrho(x, 0)$. Notice that if $\|x\| = 1$ or $x \in \bar{\alpha} - \alpha$ then $\Phi_1(h, t)(z) = h(z)$. Thus,

$$\Phi_1(h, t)|\mathrm{fr}[|N(\alpha, K)|] = h|\mathrm{fr}[|N(\alpha, K)|]$$

for each $(h, t) \in P_1 \times I$.

If we define Φ_1 on each $|N(\alpha, K)|$, $\alpha \in Y_1$, as above, we get a deformation $\Phi_1\colon P_1 \times I \to \mathcal{H}(X)$ with the desired properties.

Step 2. Let X_2 be the set of open arc components of $I^1(X) - I^0(X)$ with simple closed curve closures. Clearly, there exists a neighborhood P_2' of $1_{I^1(X)}$ in $\mathcal{H}(I^1(X), I^0(X) \cup X_1)$ such that each homeomorphism of P_2' maps each component of $I^1(X) - I^0(X)$ onto itself, and a deformation $\Phi_2'\colon P_2' \times I \to \mathcal{H}(I^1(X), I^0(X) \cup X_1)$ of P_2' into $\mathcal{H}(I^1(X), I^0(X) \cup X_1 \cup X_2)$ with the properties Φ_2' is modulo $I^1(X) - X_2$, $\Phi_2'(h, t)$ maps each component of $I^1(X) - I^0(X)$ onto itself, and $\Phi_2'(1_{I^1(X)} \times I) = 1_{I^1(X)}$. Pick a neighborhood P_2 of 1_X in $\mathcal{H}(X, I^1(X) \cup X_1)$ such that $h|I^1(X) \in P_2'$ for each $h \in P_2$.

We now describe how to define a deformation $\Phi_2\colon P_2 \times I \to \mathcal{H}(X)$ of P_2 into $\mathcal{H}(X, I^0(X) \cup X_1 \cup X_2)$ with the properties $\Phi_2(1_X \times I) = 1_X$ and $\Phi_2(h, t)|I^1(X) = \Phi_2'(h|I^1(X), t)$ for each $(h, t) \in P_2 \times I$.

Let $Y_2 = \{\beta_1, \dots, \beta_n\}$ be the open arcs of X_2 contained in $T(X)$. For each $\beta \in Y_2$, let $v_\beta = \overline{\beta} - \beta$ and consider the type 2 neighborhood pair $(|N(\beta, K)|, u_\beta)$ of β, recalling that

$$u_\beta\colon \bigcup_{q=1}^{t(\beta)} B_q \times \overline{\beta} / \bigcup_{q=1}^{t(\beta)} B_q \times v_\beta \to |N(\beta, K)|$$

is a homeomorphism such that $u_\beta\langle(0,0), x\rangle = x$ for each $x \in \overline{\beta}$. The sets $|N(\beta, K)|$ are such that $|N(\beta, K)| \cap |N(\beta', K)| = \emptyset$ if $v_\beta \neq v_{\beta'}$ and $= v_\beta$ otherwise. Furthermore, $|N(\beta, K)| \cap I^1(X) = \overline{\beta}$ for each $\beta \in Y_2$. For each $\beta \in Y_2$, let $\pi_\beta\colon \bigcup_{q=1}^{t(\beta)} B_q \times \overline{\beta} \to (0,0) \times \overline{\beta}$ be the projection given by $\pi_\beta(x, y) = (0, y)$ for each $(x, y) \in \bigcup_{q=1}^{t(\beta)} B_q \times \overline{\beta}$, and let

$$p_\beta\colon \bigcup_{q=1}^{t(\beta)} B_q \times \overline{\beta} \to \bigcup_{q=1}^{t(\beta)} B_q \times \overline{\beta} / \bigcup_{q=1}^{t(\beta)} B_q \times v_\beta$$

be the quotient map. For each $x \in \bigcup_{q=1}^{t(\beta)} B_q$, let $\pi_{\beta,x} = \big(p_\beta \pi_\beta p_\beta^{-1} | p_\beta(x \times \overline{\beta})\big)^{-1}$ and notice that for each β, the map $\pi_{\beta,x}$ is a homeomorphism from $p_\beta(0 \times \overline{\beta})$ onto $p_\beta(x \times \overline{\beta})$ such that $\pi_{\beta,x}\big(p_\beta(0, y)\big) = p_\beta(x, y)$ for each $y \in \overline{\beta}$.

The deformation Φ_2 is now defined by letting

$$\Phi_2(h, t)(z) = \begin{cases} \Phi_2'\big(h|I^1(X), t\big)(z) & \text{if } z \in I^1(X), \\ h(z) & \text{if } z \in X - \bigcup_{\beta \in Y_2} |N(\beta, K)|, \\ h u_\beta \pi_{\beta,x} u_\beta^{-1} h^{-1} \Phi_2'\big(h|I^1(X), t(1 - \|x\|)\big) u_\beta \pi_\beta u_\beta^{-1}(z) & \\ \quad \text{if } z = u_\beta(x, y) \in |N(\beta, K)|,\ \beta \in Y_2, & \end{cases}$$

for each $(h, t) \in P_2 \times I$. Notice that $\Phi_2(h, t)|\mathrm{fr}[|N(\beta, K)|] = h|\mathrm{fr}[|N(\beta, K)|]$ for each $(h, t) \in P_2 \times I$. It is easy to verify that Φ_2 is the desired deformation.

Step 3. Let $X_3 = \{\zeta_1, \zeta_2, \dots, \zeta_n\}$ be the simple curve components of $I^1(X) - I^0(X)$ and $Y_3 = \{\zeta \in X_3 |\ \zeta \subset T(X)\}$. For each i, $1 \leqslant i \leqslant n$, let $\{\alpha(i,1), \alpha(i,2), \alpha(i,3)\}$ and $\{\beta(i,1), \beta(i,2)\}$ be sets of open arcs satisfying

(1) the closure of each open arc is a polygonal closed arc which carries a subcomplex of L,

(2) $\alpha(i,2) \cup \beta(i,2) = \zeta_i$,

(3) $\mathrm{cl}[\alpha(i,j)] \subset \alpha(i, j-1)$ for $j = 1, 2$, and

(4) $\mathrm{cl}[\beta(i,2)] \subset \beta(i,1)$

for each i. Let $S = \bigcup_{i=1}^{n} \zeta_i$, $A_j = \bigcup_{i=1}^{n} \mathrm{cl}[\alpha(i,j)]$ for $j = 0, 1, 2$, and $B_j = \bigcup_{i=1}^{n} \mathrm{cl}[\beta(i,j)]$ for $j = 1, 2$.

By Theorem 5.1 of [4], there is a neighborhood Q_α of the inclusion $\eta_\alpha\colon A_0 \subset S$ in $I(A_0; S)$ and a deformation $\Lambda_\alpha\colon Q_\alpha \times I \to I(A_0; S)$ of Q_α into $I(A_0, A_1; S)$ such that Λ_α is modulo $\mathrm{fr}_{\mathrm{rel}\,S}[A_0]$ and $\Lambda_\alpha(\eta_\alpha \times I) = \eta_\alpha$.

As described in the preceding section, for each $\zeta_i \in Y_3$ there is a type 1 neighborhood pair $\big(|N(\alpha(i,0), K)|, u_{\alpha(i,0)}\big)$ of $\alpha(i,0)$, where $u_{\alpha(i,0)}$ is a homeomorphism from $\bigcup_{q=1}^{t(i)} B_q \times \mathrm{cl}[\alpha(i,0)]$ onto $|N(\alpha(i,0), K)|$ such that $u_{\alpha(i,0)}\big((0,0), y\big) = y$ for each $y \in \mathrm{cl}[\alpha(i,0)]$. Notice that if $\alpha(i,0)$ and $\alpha(j,0)$ are distinct arcs of Y_3,

$$|N(\alpha(i,0), K)| \cap |N(\alpha(j,0), K)| = \emptyset .$$

Also, $|N(\alpha(i,0), K)| \cap I^1(X) = \mathrm{cl}[\alpha(i,0)]$ for each $\alpha(i,0) \in Y_3$.

Let P_3' be a neighborhood of 1_X in $\mathcal{H}(X, I^0(X) \cup X_1 \cup X_2)$ such that $h|A_0 \in Q_\alpha$ for each $h \in P_3'$. As in step 1, we can use the neighborhood pair $\big(|N(\alpha(i,0), K)|, u_{\alpha(i,0)}\big)$ to smear Λ_α and get a deformation $\Phi_3'\colon P_3' \times I \to \mathcal{H}(X)$ of P_3' into $\mathcal{H}(X, I^0(X) \cup X_1 \cup X_2 \cup A_1)$ with the properties $\Phi_3'(1_X \times I) = 1_X$, $\Phi_3'(h, t)|A_0 = \Lambda_\alpha(h|A_0, t)$ for each $(h, t) \in P_3' \times I$, and Φ_3' is modulo $X - \Big(A_0 \cup \big(\bigcup_{\zeta_i \in Y_3} |N(\alpha(i,0), K)|\big)\Big)$.

Applying Theorem 5.1 of [4] again, we get a neighborhood Q_β of the inclusion $\eta_\beta\colon B_1 \subset S$ in $I(B_1, B_1 \cap A_1; S)$ and a deformation $\Lambda_\beta\colon Q_\beta \times I \to I(B_1, B_1 \cap A_2; S)$ of Q_β into $I\big(B_1, B_1 \cap (A_2 \cup B_2); S\big)$ such that Λ_β is modulo $\mathrm{fr}_{\mathrm{rel}\,S}[B_1]$ and $\Lambda_\beta(\eta_\beta \times I) = \eta_\beta$.

Let P_3'' be a neighborhood of 1_X in $\big(X, I^0(X) \cup Y_1 \cup Y_2 \cup A_1\big)$ such that $h|B_1 \in Q_\beta$ for each $h \in P_3''$. In a manner similar to the above, we can use type 1 neighborhood pairs of the $\beta(i,1)$'s to get a deformation $\Phi_3''\colon P_3'' \times I \to \mathcal{H}(X)$ of P_3'' into $\mathcal{H}\big(X, I^0(X) \cup X_1 \cup X_2 \cup A_2 \cup B_2\big) = \mathcal{H}\big(X, I^1(X)\big)$ with the properties $\Phi_3''(h, t)|B_1 = \Lambda_\beta(h|B_1, t)$ for each $(h, t) \in P_3'' \times I$, Φ_3'' is modulo $X - B_1 \cup \big(\bigcup_{\zeta_i \in Y_3} |N(\beta(i,1), K)|\big)$, and $\Phi_3''(1_X \times I) = 1_X$.

Step 4. By the continuity of Φ_3', Φ_2, and Φ_1, we may assume $\Phi_3'(P_3' \times 1) \subset P_3''$, $\Phi_2(P_2 \times 1) \subset P_3'$, and $\Phi_1(P_1 \times 1) \subset P_2$. Let $P = P_1$ and $\Phi = \Phi_3'' * \Phi_3' * \Phi_2 * \Phi_1$. It easy to verify that Φ and P are as desired. q.e.d.

7. Cuts. A subset A of a topological space X is said to be *thin* if it has empty interior. We say A *nowhere cuts* X if A is a thin subset of X with the property: whenever $x \in A$ and U is a neighborhood of x, $U - A$ does not split into two disjoint open sets each having x in its closure. Michael proved in [5] that if X is a Tychonoff space and A is a thin subset of X then there exists an essentially unique Tychonoff space X_*, a nowhere cutting subset A_*, and a map $p\colon X_* \to X$ which maps $X_* - A_*$ homeomorphically onto $X - A$ and maps A_* (compact, totally disconnected)-to-one onto A. A triple (X_*, A_*, p) satisfying Michael's theorem is called an (X, A)-cut. "Essentially unique" in the above theorem means if (X_*, A_*, p) and (X'_*, A'_*, p') are (X, A)-cuts then there is a homeomorphism $h\colon X_* \to X'_*$ such that $p = p'h$.

Returning to our problem, let M be the disjoint union of the 2-simplexes of K and $\pi\colon M \to K$ be the natural projection. If σ and σ' are 2-simplexes of M with faces a and a', respectively, such that $\pi(a) = \pi(a') < K - L_1$, we write $x \equiv x'$ whenever $x \in a$, $x' \in a'$, and $\pi(x) = \pi(x')$. An equivalence relation R is defined on J by letting xRy whenever there is a sequence $x = x_0, x_1, \ldots, x_n = y$ such that $x_i \equiv x_{i+1}$ for $i = 0, 1, 2, \ldots \ldots, n-1$. Let $W = J/R$ be the quotient space and $q\colon J \to W$ be the quotient map. Since xRy implies $\pi(x) = \pi(y)$, there is a map $p\colon W \to |K|$ such that $\pi = pq$. Now let $K_* = \{q(\sigma)|\ \sigma < M\}$. Since K is the second barycentric subdivision of some triangulation of Y, K_* is a complex and $p\colon K_* \to K$ is a simplicial map of K_* onto K. Notice that $I^1(Y) = |L_1|$ is a thin subset of Y. Michael proved in [5] that $(|K_*|, |L_*|, p)$ is a $(Y, I^1(Y))$-cut, where $L_* = p^{-1}(L_1)$.

Lemma 7.1 *$|K_*|$ is a 2-manifold and $|L_*|$ comprises the boundary of $|K_*|$ together with a finite set of points.*

Proof. Since $|L_*|$ nowhere cuts $|K_*|$ and each 1-simplex of L_* is the face of exactly one 2-simplex of K_*, for each point x of $|L_*|$ which is not a vertex of L_* there is a neighborhood V of x such that the pair (V, x) is homeomorphic to the pair $(R^2_+, (0, 0))$, where

$$\{R^2_+ = \{(x_1, x_2) \in R^2|\ x_2 \geqslant 0\}.$$

Let v be a vertex is L_*. Then $p(v)$ is a vertex in L_1. There are two cases to be considered.

Case 1. Suppose $p(v) \in I^1(Y) - I^0(Y)$. In this case there is a triangulation $\widetilde{K}$ of Y containing a 1-simplex a which contains $p(v)$ in its interior rel $I^1(Y)$. If $\widetilde{L}_1$ denotes the subcomplex of $\widetilde{K}$ carried by $I^1(Y)$ then $a < \widetilde{L}_1$. Let $(|\widetilde{K}_*|, |\widetilde{L}_*|, \widetilde{p})$ be the $(Y, I^1(Y))$-cut determined by $\widetilde{K}$ and $\widetilde{L}_1$. By an argument similar to the one given above, each point w of $\widetilde{p}^{-1}(v)$ has a neighborhood $V(w)$ such that the pair $(V(w), w)$ is homeomorphic to $(R^2_+, (0, 0))$. But, since $(|K_*|, |L_*|, p)$ is essentially unique, v has a neighborhood V in $|K_*|$ such that (V, v) is homeomorphic to $(R^2_+, (0, 0))$.

Case 2. Suppose $p(v) \in I^0(Y)$. Then $|\mathrm{St}(p(v), K)| = \bigcup_{i=1}^{n} C_i$ where the C_i's are the cell-sets at $p(v)$. Since $|L_*|$ nowhere cuts $|K_*|$, $p^{-1}(|\mathrm{St}(p(v), K)|) = \bigcup_{i=1}^{n} D_i$ where the D_i's are pairwise disjoint 2-cells such that $p(D_i) = C_i$ for each i. Let j be such that $v \in D_j$. There are three possibilities:

(i) if C_j is a cell-set of type 1 at $p(v)$ then D_j is a cell-set of type 1 at v and the pair $(\mathrm{int}_{\mathrm{rel}|K_*|} D_j, v)$ is homeomorphic to the pair $(R^2_+, (0, 0))$,

(ii) if C_j is a cell-set of type 2 at $p(v)$ then D_j is a cell-set of type 2 at v and the pair $(\mathrm{int}_{\mathrm{rel}|K_*|} D_j, v)$ is homeomorphic to the pair $(R^2_+, (0, 0))$, and

(iii) if C_j is a cell-set of type 3 at $p(v)$ then D_j is a cell-set of type 3 at v and the pair $(\mathrm{int}_{\mathrm{rel}|K_*|} D_j, v)$ is homeomorphic to the pair $(R^2, (0, 0))$. In (i) and (ii) v is a boundary point of $|K_*|$ and in (iii) v is an interior point of $|K_*|$.

We have shown that each point $x \in |L_*|$ has a neighborhood V in $|K_*|$ such that the pair (V, x) is homeomorphic to one of $(R^2, (0, 0))$ and $(R^2_+, (0, 0))$. Since $p|(|K_*| - |L_*|)$ is a homeomorphism onto $|K| - |L_1| = Y - I^1(Y) = I^2(Y) - I^1(Y)$ which is an open 2-manifold, $|K_*|$ is a 2-manifold. q.e.d.

Remark. The finite set of points in $|L_*|$ which are not boundary points of $|K_*|$ are in one-to-one correspondence with the cell-sets of type 3 at points of $I^0(Y)$.

Lemma 7.2. (a) *There is a $\delta > 0$ such that if h is a δ-homeomorphism of $|K|$ which is the identity on $|L_1| \cup Z$ then $h_* \in \mathcal{H}(|K_*|, |L_*| \cup p^{-1}(Z))$, where $h_* = p^{-1}hp$ on $|K_*| - |L_*|$ and is the identity on $|L_*|$.*

(b) *If $\varepsilon > 0$ is given, δ can be closen so that h_* is an ε-homeomorphism.*

Proof. To prove (a) we choose a $\delta > 0$ so that each δ-homeomorphism $h \in \mathcal{H}(|K|, |L_1| \cup Z)$ has the following properties:

(i) $h(|\mathrm{St}(\hat{a}, K'')|) \subset \mathrm{int}[|N(a^0, K)|]$ for each 1-simplex $a < L_1$, where $\hat{a}$ denotes the barycenter of a and K'' denotes the second barycentric subdivision of K,

(ii) if a is a 1-simplex of L_1, $\delta < \varrho(|C(a, \mathrm{St}(\hat{a}, \sigma''))|, X - \sigma)$ for each 2-simplex σ in K such that $a < \sigma$,

(iii) $h(|\mathrm{St}(v, K'')|) \subset |\mathrm{St}(v, K)|$ for each vertex $v < L_1$, and

(iv) for each vertex $v < L_1$ and each component D of the set $|C(I^1(Y), \mathrm{St}(v, K''))|$, the ϱ distance between D and the complement of the component of $|\mathrm{St}(v, K)| - I^1(Y)$ that contains D is less than δ.

Let h be a fixed δ-homeomorphism of $\mathcal{H}(|K|, |L_1| \cup Z)$ for the remainder of the proof of part (a).

Let a be a 1-simplex of L_1. Conditions (i) and (ii) imply if C'' and C are components of $|\mathrm{St}(\hat{a}, K'')| - a$ and $|\mathrm{St}(\hat{a}, K)| - a$, respectively, such that $C'' \subset C$ then $h(C'') \subset C$. From this it follows that if a_* is a 1-simplex of L_* such that $p(a_*) = a$ and $z \in a^0_*$ then the sequence $\{h_*(z_i)\}_i$ converges to z for each sequence $\{z_i\}_i$ in $|K_*|$ which converges to z.

Let v be a vertex in L_1. Conditions (iii) and (iv) imply if C'' and C are components of $|\mathrm{St}(v, K'')| - I^1(Y)$ and $|\mathrm{St}(v, K)| - I^1(Y)$, respectively, such that $C'' \subset C$ then $h(C'') \subset C$. Consequently, if $z \in p^{-1}(v)$ and $\{z_i\}_i$ is a sequence in $|K_*|$ which converges to z, then $\{h_*(z_i)\}_i$ converges to z.

The above two paragraphs verify that h_* is continuous at points of $|L_*|$. Since $h_*|(|K_*|-|L_*|)$ is a homeomorphism onto $|K_*|-|L_*|$, h_* is continuous on $|K_*|$. Since $|K_*|$ is compact and h_* is one-to-one, continuous, and onto, h_* is a homeomorphism.

Pick $\varepsilon > 0$. To prove (b), we will assign to each $x \in |K|$ a number $\delta(x) > 0$ so that the $\delta(x)$-neighborhood of x in $|K|$, denoted $V(x, \delta(x))$, will have certain properties. Next, we will choose a finite subcover of $|K|$ from $\{V(x, \delta(x)) | x \in |K|\}$ and complete the proof by carefully choosing δ.

Let $x \in |K|$. There are three cases.

Case 1. Suppose x is a vertex of L_1. Let $C_1, C_2, \ldots, C_n$ be the cell-sets at x in $|K|$ and recall that $|\mathrm{St}(x, K)| = \bigcup_{i=1}^{n} C_i$. There are subsets $D_1, D_2, \ldots, D_n$ of $|K_*|$ such that $p^{-1}(|\mathrm{St}(x, K)|) = \bigcup_{i=1}^{n} D_i$, $D_i \cap D_j$ is at most one point of $I^0(Y) - p^{-1}(x)$ for $i \neq j$, and $p(D_i) = C_i$. If $x_i = p^{-1}(x) \cap D_i$ then x_i is a vertex of L_* and $D_i = |\mathrm{St}(x_i, K_*)|$ is the only cell-set at x_i in $|K_*|$. Let $\delta(x) > 0$ be such that $V(x, \delta(x)) \subset |\mathrm{St}(x, K'')|$ and $p^{-1}(V(x, \delta(x))) \cap D_i \subset V(x_i, \varepsilon/2)$.

Case 2. Suppose $x \in \alpha^0$, where α is a 1-simplex of L_1. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the 2-simplexes of K having α as a 1-face and observe that $|N(\alpha^0, K)| = \bigcup_{i=1}^{n} \sigma_i$. Let σ_{i*} be the unique 2-simplex of K_* such that $p(\sigma_{i*}) = \sigma_i$ and let α_{i*} denote the unique 1-simplex of L_* such that $\alpha_{i*} < \sigma_{i*}$ and $p(\alpha_{i*}) = \alpha$. Let $\delta(x) > 0$ be such that $\mathrm{cl}[V(x, \delta(x))] \subset \mathrm{int}[|N(\alpha^0, K|)]$ and $p^{-1}(V(x, \delta(x))) \cap \sigma_{i*} \subset V(x_i, \varepsilon/2)$, where $x_i = p^{-1}(x) \cap \alpha_{i*}$.

Case 3. In case $x \in |K| - |L_1|$, let $\delta(x) > 0$ be such that $V(x, \delta(x)) \cap |L_1| = \emptyset$ and $p^{-1}(V(x, \delta(x))) \subset V(p^{-1}(x), \varepsilon/2)$.

Let $\{y_1, y_2, \ldots, y_r\} \subset X$ such that $\bigcup_{i=1}^{r} V(y_i, \delta(y_i$ [illegible] X and pick $\delta > 0$ to satisfy conditions (i) through (iv) in the proof of (a) plus two more conditions, namely,

(v) δ is a Lebesgue number for the cover $\{V(y_i, \delta(y_i)) | i = 1, 2, \ldots, r\}$ and

(vi) if h is a δ-homeomorphism in $\mathcal{H}(|K|, |L_1| \cup Z)$ and $y_i \in \alpha^0$ for some i and some 1-simplex $\alpha < L_1$ then $\mathrm{cl}[h(V(y_i, \delta(y_i))) \cap \sigma] \subset \sigma^0 \cup \alpha^0$ for each 2-simplex σ in K such that $\alpha < \sigma$.

It is straightforward to check that h is a δ-homeomorphism in $(|K|, |L_1| \cup Z)$ implies h_* is an ε-homeomorphism. q.e.d.

8. Proof of Lemma 2.2. We will show that $\mathcal{H}(Y, T(X))$ is locally contractible. But before doing so, we show how this implies the lemma.

By the local contractibility of $\mathcal{H}(Y, T(X))$ there is a neighborhood S of 1_Y in $\mathcal{H}(Y, T(X))$ and a deformation Λ: $S \times I \to \mathcal{H}(Y, T(X))$ of S into 1_Y. Choose a neighborhood Q of 1_X in $\mathcal{H}(X, I^1(x))$ such that $h|Y \in S$ for each $h \in Q$. The deformation Ψ: $Q \times I \to \mathcal{H}(X)$ of Q into 1_X is defined by letting

$$\Psi(h, t)(x) = \begin{cases} \Lambda(h|Y, t)(x) & \text{if} \quad x \in Y, \\ h(x) = x & \text{if} \quad x \in I(X) \end{cases}$$

for each $(h, t) \in Q \times I$.

To see that $\mathcal{H}(Y, T(X))$ is locally contractible, let (Y_*, W_*, p) be a $(Y, I^1(Y))$-cut as previously described. By lemma 7.1 Y_* is a 2-manifold and $W_* = \partial Y_* \cup$ (a finite set of interior points of Y_*). Let $V_* = W_* \cup p^{-1}(Z)$. Then $V_* = \partial Y_* \cup$ (a finite set of interior points of Y_*). It follows from Edwards and Kirby (see [4]) that $\mathcal{H}(Y_*, V_*)$ is locally contractible. Let S_* be a neighborhood of 1_{Y_*} in $\mathcal{H}(Y_*, V_*)$ and Ψ_*: $S_* \times I \to \mathcal{H}(Y_*, V_*)$ be a deformation of S_* into 1_{Y_*}. By Lemma 7.2 there is a neighborhood S of 1_Y in $\mathcal{H}(Y, T(X))$ such that $h_* \in S_*$ for each $h \in S$. Define a deformation Λ: $S \times I \to \mathcal{H}(Y, T(X))$ by letting

$$\Lambda(h, t)(x) = \begin{cases} p\Psi_*(h_*, t)p^{-1}(x) & \text{if} \quad x \in Y - T(X), \\ h(x) = x & \text{if} \quad x \in T(X) \end{cases}$$

for each $(h, t) \in S \times I$. Λ deforms S into 1_Y. q.e.d.

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