It is easy to determine the distinguished codes of D_n and to prove that they form an $\mathcal{A}$-set. Hence X_0 is an $\mathcal{A}$-base of F_0.

Let $b_0 = \{J(m, n): (m > 0) \ \& \ m \equiv a'_n \ (\mathrm{mod}\ 2^{\overline{n}})\}$. Since b_0 is primitive recursive, it is definable. It remains to show that whenever F is an ultrafilter and $F \supseteq X_0$, then Stsf is arithmetical in $R_F(b_0)$.

Thus assume that each D_n belongs to F. Since

$$q \in R_F(b_0) \equiv \{i > 0: i \equiv a'_q \ (\mathrm{mod}\ 2^{\overline{q}})\} \in F$$

we obtain taking $q = 2^n + e'_n$

$$2^n + e'_n \in R_F(b_0) \equiv D_n \in F$$

whence $2^n + e'_n \in R_F(b_0)$. We now show that $2^n + e'_n$ is a unique element n of $R_F(b_0)$ such that $2^n \leqslant m < 2^{n+1}$. To see this we notice that if $m \in R_F(b_0)$ and $2^n \leqslant m < 2^{n+1}$, then $\overline{m} = n$, $a'_m < 2^n$ and

$$\{i > 0: \ i \equiv a'_m \ (\mathrm{mod}\ 2^n)\} \in F.$$

This set must intersect with D_n since they both belong to F. It follows that $a'_m \equiv e'_n \ (\mathrm{mod}\ 2^n)$ and since both a'_m, e'_n are $< 2^n$ we obtain $a'_m = e'_n$ and $m = 2^n + e'_n$.

The last term ε_{n-1} of e_n where $n > 0$ can therefore be defined as the integral part of $x/2^{n-1}$ where x is a unique integer $< 2^n$ such that $2^n + x \in R_F(b_0)$. Since $n \in \mathrm{Stsf} \equiv \varepsilon_n = 1$ it follows that Stsf is arithmetical in $R_F(b_0)$ and the proof is finished.

In [3] the theorem was proved only for models which are elementarily equivalent to the principal model. It would be interesting to verify whether it holds for ω-models of the system $\overline{A}_2$ resulting from A_2 by omitting the choice axiom.

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Reçu par la Rédaction le 6. 3. 1971

The restricted cancellation law in a Noether lattice

by

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In [2], R. P. Dilworth defined the concept of a Noether lattice. The definition is based on the idea of a principal element. $A \in L$ is a *principal element* if for all $B, C \in L$, $(B \wedge (C\colon A))A = BA \wedge C$ and $(B \vee CA)\colon A = B\colon A \vee C$, thus a principal element is a generalization of the idea of a principal ideal in a Noetherian ring. The ramifications of this concept have been investigated in [2], [5], [6], and [7].

In [3], R. Gilmer considered the restricted cancellation law (RCL) in commutative rings. An element A of a Noether lattice satisfies RCL if for any $B, C \in L$, $AB = AC \neq 0$ implies $B = C$. We show this condition is closely related to the idea of a weak join principal element. $A \in L$ is weak join principal if $BA\colon A = B \vee 0\colon A$.

In section 1, we consider a theorem of Gilmer [3] in which he characterizes a commutative ring in which every ideal satisfies RCL. In a Noether lattice L, we show a similar result holds when RCL is assumed on the prime elements of L. Such lattices are characterized as Dedekind or local with maximal M in which either $M^2 = 0$ or M is principal with $M^k = 0$ for some k.

In section 2, the situation in which (L, M) is a local Noether lattice with maximal M such that M satisfies RCL is investigated. With the aid of the lattice RL_n introduced by Bogart [1], these lattices are characterized. In addition, we show the maximal element M in (L, M) is join principal.

Finally, we consider a local Noether lattice in which the maximal is weak join principal. We investigate the distributive case in which the maximal has a minimal representation as the join of two principals.

The author is indebted to Professor Eugene W. Johnson for his helpful suggestions.

Section 1. In this section we will characterize Noether lattices in which every prime element satisfies the restricted cancellation law.

LEMMA 1.11. *If A satisfies RCL and $AB \leqslant AC \neq 0$ for some $B, C \in L$, then $B \leqslant C$.*

Proof. $A(B\vee C) = AB\vee AC = AC \neq 0$, so by RCL for A, $B\vee C = C$ and $B \leqslant C$. q.e.d.

DEFINITION 1.12. If P is prime, the *dimension of* P is the maximum length of a chain of distinct proper primes greater than P.

DEFINITION 1.13. If P is prime, P *has rank* r if r is the maximum length of a chain of distinct primes less than P.

LEMMA 1.14. *$A \in L$ satisfies RCL if and only if $AB\colon A = B\vee 0\colon A$ for every $B \in L$ and $0\colon A \leqslant B$ whenever $AB \neq 0$.*

Proof. Assume A satisfies RCL. $(AB\colon A)A \leqslant AB$. If $AB = 0$ then $AB\colon A \leqslant B\vee 0\colon A$. If $AB \neq 0$ then by Lemma 1.11, $AB\colon A \leqslant B\vee 0\colon A$. Hence $AB\colon A = B\vee 0\colon A$. Furthermore, if $AB \neq 0$ then $(0\colon A)A \leqslant AB$ and by Lemma 1.11, $0\colon A \leqslant B$.

Conversely, assume $AB\colon A = B\vee 0\colon A$ and $AB \neq 0$ implies $0\colon A \leqslant B$. If $AB = AC \neq 0$, then $AB\colon A = B\vee 0\colon A = B$ and $AC\colon A = C\vee 0\colon A = C$, so $B = C$. q.e.d.

LEMMA 1.15. *If P is a prime in L and $A \in L$ satisfies RCL, then* $[A]$ *satisfies RCL in* L_P.

Proof. Let $[B] \in L_P$. $[A][B]\colon [A] = [AB\colon A] = [B\vee 0\colon A] = [B]\vee$ $\vee[0]\colon [A]$. If $[A][B] \neq [0]$, then $AB \neq 0$ in L and by Lemma 1.14, $0\colon A \leqslant B$, so $[0]\colon [A] \leqslant [B]$. Hence by Lemma 1.14, $[A]$ satisfies RCL in L_P. q.e.d.

LEMMA 1.16. *If (L, M) is a local Noether lattice and M satisfies RCL, then rank $M \leqslant 1$.*

Proof. There exist principals $E_1, \ldots, E_k$ such that $M = E_1\vee E_2\vee\ldots$ $\ldots\vee E_k$ and this is a minimal representation of M as the join of principals. $M^{nk+n} = M^{nk}(E_1^n\vee\ldots\vee E_k^n)$. If $M^{nk+n} = 0$ then M is the only prime of L and rank $M = 0$. If $M^{nk+n} \neq 0$, then $M^n = E_1^n\vee\ldots\vee E_k^n$ by RCL. By [4], there exists a polynomial $p(x)$ with rational coefficients such that the degree of $p(x) =$ rank $M-1$ and $p(n)$ is the number of elements in a minimal base for M^n. $p(n) \leqslant k$ for every n implies $p(x)$ is a constant polynomial and hence rank $M \leqslant 1$. q.e.d.

COROLLARY 1.17. *If P is a prime in L and P satisfies RCL, then rank $P \leqslant 1$.*

Proof. L_P is a local Noether lattice in which the maximal element $[P]$ satisfies RCL. Hence, by Lemma 1.16, rank $[P] \leqslant 1$. Since rank $P =$ rank $[P]$, rank $P \leqslant 1$. q.e.d.

THEOREM 1.18. *If (L, M) is a local Noether lattice with RCL on primes and* $\dim L = 0$, *then L is either special primary or $M^2 = 0$.*

Proof. If $M^2 = 0$ then the theorem holds so assume $M^2 \neq 0$. Let $M = E_1\vee\ldots\vee E_n$ be a minimal representation of M as the join of principals where $E_i \nleqslant M^2$ for each i. Since $M^2 \neq 0$, $ME_j \neq 0$ for some j. Rank $M = 0 = \dim L$ implies M is the only prime of L. Therefore E_j is M-primary and there exists an integer k such that $M^k \leqslant E_j$ and $M^{k-1} \nleqslant E_j$. If $k = 1$ then M is principal and the theorem holds. If $k > 1$, then M^k $= M^k\wedge E_j = (M^k\colon E_j)E_j$. $E_j \nleqslant M^k$ implies $M^k \leqslant ME_j \neq 0$ and by RCL, $M^{k-1} \leqslant E_j$. This contradicts the choice of k. Therefore $E_j = M^k$ and since $E_j \nleqslant M^2$, $k = 1$. q.e.d.

THEOREM 1.19. *If (L, M) is a local Noether lattice with RCL on primes and rank $M = 1$, then L is regular local altitude one.*

Proof. Since rank $M = 1$, M is not a minimal prime of 0. Let $P < M$ be prime, then P is a minimal prime of 0. Let $0 = \bigwedge_{i=1}^{n} Q_i$ be a normal decomposition of 0 as the meet of primary elements, where $\sqrt{Q_i} = P_i$ and $P = P_1$. There exist integers $k(1), \ldots, k(n)$ such that $P_i^{k(i)} \leqslant Q_i$, $P^{k(1)}(\bigwedge_{i=2}^{n} P_i^{k(i)}) \leqslant 0 \leqslant P^{k(1)+1}$. If $P^{k(1)+1} \neq 0$, then $\bigwedge_{i=2}^{n} P_i^{k(i)} \leqslant P$ and $P_j \leqslant P$ for some j, $j \neq 1$. Since rank $M = 1$, $P_j = P$ contradicting the normality of the decomposition. Therefore $P^{k(1)+1} = 0$ and P is the unique minimal prime of 0. Hence L has only two primes, M and P, with $P^n = 0$ for some n. Since $M \nleqslant P$, we may choose a principal X such that $X \leqslant M$, $X \nleqslant M^2$ and $X \nleqslant P$. X is M primary. $XM \neq 0$ since $X \nleqslant P$ and $M \nleqslant P$ so as in the proof of Theorem 1.18, $M = X$. However L is local so every nonzero element of L is a power of $X = M$. Hence $P = 0$ and L is regular local altitude one. q.e.d.

LEMMA 1.20. *If L is a Noether lattice with RCL on primes and $P < P' < I$, P and P' prime, then $0_{P'} = P = 0_P$. Furthermore, if P is an associated prime of 0, then rank $P = 0$.*

Proof. $L_{P'}$ is local with RCL on primes and rank $[P'] = 1$. Therefore $L_{P'}$ is regular local altitude one and $[0]$ is prime in $L_{P'}$. Since $[P] \neq [P']$ and $[P]$ is prime in $L_{P'}$, $[P] = [0]$. Thus $P = 0_{P'}$. However $P \leqslant P'$ implies $0_{P'} \leqslant 0_P \leqslant P$ so $0_{P'} = 0_P = P$. Furthermore, we have shown if rank $P \neq 0$, then P is not an associated prime of 0. q.e.d.

LEMMA 1.21. *If $0_P = P$, then $P = 0$ or $P^2 = 0$. Furthermore if $P^k = 0$ for some prime P and some integer k then $P = 0$ or P is the only prime in L.*

Proof. $0 \leqslant P^2$ so $0_P \leqslant (P^2)_P \leqslant P$. Therefore $(P^2)_P = P$. P is a minimal prime of P^2 so let $P^2 = P\wedge Q_1\wedge\ldots\wedge Q_n$ be normal decomposition of P^2. $P(\bigwedge_{i=1}^{n} Q_i) \leqslant P^2$. If $P^2 = 0$, the result holds. If $P^2 \neq 0$ then $\bigwedge_{i=1}^{n} Q_i \leqslant P$ contradicting the normality of the decomposition.

Now assume $P^k = 0$ for some prime P and some integer k. Since P is the unique minimal prime of 0 and by Lemma 1.20, 0 has no imbedded primes, 0 is P-primary. Suppose P is not maximal and hence the only prime of L, then there exists a prime M such that $P < M$. L_M is regular local altitude one by Theorem 1.19. Hence $[0] = [P]$ and $0_M = P$. Since 0 is P-primary $0_M = 0$. Therefore $P = 0$. Hence either P is maximal and minimal or 0 is prime. q.e.d.

LEMMA 1.22. *If L is a Noether lattice with RCL on primes and if for every nonzero prime P of L, no power of P is 0, then L is Dedekind.*

Proof. If 0 is prime is L, let P be a prime such that $0 < P$. Since rank $P \leqslant 1$, P is maximal. L_P is regular local altitude one by Theorem 1.19 and hence linear. Therefore, by [6], every element of L is a product of primes. 0 is prime, so L is Dedekind.

If 0 is not prime in L, let P be any prime of L. $P < P'$ where P' is prime, implies $[0] = [P]$ in $L_{P'}$ and $0_P = P$ by Lemma 1.20. Therefore $P = 0$ or $P^2 = 0$, both of which are contradictions. Hence P is maximal. Furthermore, by the same reasoning, P is minimal. Let $0 = \bigwedge_{i=1}^{n} Q_i$ be a normal decomposition of 0 as the meet of primaries, where $\sqrt{Q_i} = P_i$. Since any prime of L is maximal and minimal, $\{P_1, \ldots, P_k\}$ is the set of all primes of L. For each i, there exists a positive integer $n(i)$ such that $P_i^{n(i)} \leqslant Q_i$. Therefore, since $P_i \vee P_j = I$, for $i \neq j$, $0 = \bigwedge_{i=1}^{k} P_i^{n(i)} = \prod_{i=1}^{k} P_i^{n(i)}$ and $L \simeq L/P_1^{n(1)} \oplus \ldots \oplus L/P_k^{n(k)}$. $(L/P_i^{n(i)})_{P_i} \simeq L_{P_i}/[P_i]^{n(i)} \simeq L_{P_i}$ since $[P_i]^{n(i)} = [0]$ in L_{P_i}. L_{P_i} is local with RCL on primes and rank $[P_i] = 0$, so by Theorem 1.18, L_{P_i} is special primary or $[P_i]^2 = 0$. Hence L is the direct sum of special primary lattices and local lattices in which the maximal squares to 0. If $k > 1$, then $(P_1, I, \ldots, I)(P_1^{n(1)-1}, P_2, \ldots, P_k) = (0, P_2, \ldots, P_k) = (P_1, I, \ldots, I)(0, P_2, \ldots, P_k) \neq 0$, but $(P_1^{n(1)-1}, P_2, \ldots, P_k) \neq (0, P_2, \ldots, P_k)$. Hence, L does not satisfy RCL on primes which is a contradiction. Therefore $k = 1$ and L is special primary or local with $M^2 = 0$, but this is impossible by hypothesis. Therefore, 0 is prime and L is Dedekind. q.e.d.

THEOREM 1.23. *If L is a Noether lattice with RCL on primes, then L is one of the following*

(i) *local with maximal M and $M^2 = 0$,*

(ii) *special primary,*

(iii) *Dedekind.*

Proof. If $P^k \neq 0$, for every nonzero prime P and every integer k, then by Lemma 1.22 L is Dedekind and 0 is prime. If there exists a prime P and an integer k, such that $P^k = 0$ then by Lemma 1.21, $P = 0$ or P is the only prime in L. If $P = 0$, then L is Dedekind. If $P \neq 0$ then L is local with RCL on primes and by Theorem 1.18, L is special primary or $P^2 = 0$. q.e.d.

Section 2. In this section we consider a local Noether lattice (L, M) in which the maximal element M satisfies RCL. Rank M is less than or equal to one by Lemma 1.16. If rank M is 0, then L is special primary or $M^2 = 0$. If 0 is prime in L then L is Dedekind. Thus, the only remaining case is rank $M = 1$ and 0 is not prime in L. Throughout this section we will consider this situation.

LEMMA 2.11. *If (L, M) is a local Noether lattice in which M satisfies RCL and rank $M = 1$, then M is not a prime of 0.*

Proof. Since rank $M = 1$, $M \cdot M^n \neq 0$ for every integer n. Hence by Lemma 1.14, $0{:}\ M \leqslant M^n$ for every n. By the Intersection Theorem, [2], this implies $0{:}\ M = 0$. q.e.d.

LEMMA 2.12. *If (L, M) is a local Noether lattice in which M satisfies RCL and $M^n \neq 0$, for each n, then if M^k is principal for some k, M is principal.*

Proof. By induction, it suffices to show that M^{k-1} is principal. Let $A, B \in L$. Clearly $A \vee (B{:}\ M^{k-1}) \leqslant (AM^{k-1} \vee B){:}\ M^{k-1}$. $XM^{k-1} \leqslant AM^{k-1} \vee B$ implies $XM^k \leqslant AM^k \vee BM$ and $X \leqslant (AM^k \vee BM){:}\ M^k = A \vee BM{:}\ M^k = A \vee (BM{:}\ M){:}\ M^{k-1} = A \vee (M \vee 0{:}\ M){:}\ M^{k-1} = A \vee B{:}\ M^{k-1}$ since M^k is principal and $0{:}\ M = 0$. Hence, $(AM^{k-1} \vee B){:}\ M^{k-1} = A \vee BM^{k-1}$ and M^{k-1} is join principal. Clearly $(A{:}\ M^{k-1} \wedge B)M^{k-1} \leqslant A \wedge BM^{k-1}$. If $X \leqslant A \wedge BM^{k-1}$, then $XM \leqslant (A \wedge BM^{k-1})M \leqslant AM \wedge BM^k = (AM{:}\ M^k \wedge B)M^k = ((A \vee 0{:}\ M){:}\ M^{k-1} \wedge B)M^k = (A{:}\ M^{k-1} \wedge B)M^k$. If $(A{:}\ M^{k-1} \wedge B)M^k \neq 0$ then by RCL, $X \leqslant (A{:}\ M^{k-1} \wedge B)M^{k-1}$. If $(A{:}\ M^{k-1} \wedge B)M^k = 0$, then $(A{:}\ M^{k-1} \wedge B)M^{k-1} \leqslant 0{:}\ M = 0$ and $(A \wedge BM^{k-1})M = 0$ so $A \wedge BM^{k-1} \leqslant 0{:}\ M = 0$. In either case, $A \wedge BM^{k-1} = (A{:}\ M^{k-1} \wedge B)M^{k-1}$. q.e.d.

LEMMA 2.13. *If (L, M) is a local Noether lattice in which M satisfies RCL and rank $M = 1$, then if there is a principal E such that $E \nleqslant P$, for every prime P which is an associated prime of 0, 0 is prime and L is regular local altitude one.*

Proof. $E \nleqslant P$, for each prime P of 0 implies that E is M-primary since the only primes of L are M and the minimal primes of 0. Choose j so that $M^j \leqslant E$, $M^{j-1} \nleqslant E$, then $M^j = (M^j{:}\ E)E$. If $M^j \neq E$, then $M^j \leqslant ME \neq 0$ and $M^{j-1} \leqslant E$. This is a contradiction. Hence $M^j = E$, and by Lemma 2.12, M is principal. Since rank $M = 1$, $M^k \neq 0$, for every k. Therefore since every nonzero element of L is a power of M, 0 is prime. q.e.d.

COROLLARY 2.14. *If R is a local Noetherian ring in which the maximal ideal M satisfies RCL, then R is*

(i) *a ring with trivial multiplication,*

(ii) *special primary,*

(iii) *regular local altitude one.*

Proof. Since in a local Noether lattice, E is principal if and only if E is join irreducible, every principal element of $L(R)$ is a principal ideal. So, if rank $M=0$, Theorem 1.18 implies $M^2=0$ or R is special primary. If rank $M=1$, then $M \nsubseteq \bigcup_{i=1}^{n} P_i$ where $P_1, \dots, P_n$ are the associated primes of 0 by [8], so there is an element $a \in R$ such that $a \notin P_i$, for each i. Thus by Lemma 2.13, $L(R)$ is regular local altitude one.

In general Corollary 2.14, is not true. A counterexample is the lattice RL_2/XY. If R is a field and X and Y indeterminates over R, then RL_2 is the sublattice of the lattice of ideals of $R[X, Y]$ consisting precisely of the ideals (X), (Y) and all finite joins of power products of these ideals, [1]. For simplicity of notation, we will write X to denote the principal ideal (X). RL_2/XY is the sublattice of RL_2 of elements which are greater than or equal to XY. Every element of this lattice is of the form $X^k \vee Y^n \vee XY$, where $k, n \geqslant 0$. If $k, n \geqslant 1$, then

$$\begin{aligned}(X^k \vee Y^n \vee XY)(X \vee Y)\colon (X \vee Y) &= (X^{k+1} \vee Y^{n+1} \vee XY)\colon (X \vee Y)\\ &= (X^{k+1} \vee Y^{n+1} \vee XY)\colon X \wedge (X^{k+1} \vee Y^{n+1} \vee XY)\colon Y\\ &= (X^k \vee Y) \wedge (Y^n \vee X) = X^k \vee Y^n \vee XY.\end{aligned}$$

If $n=0$, $k \geqslant 1$,

$$\begin{aligned}(X^k \vee XY)(X \vee Y)\colon (X \vee Y) &= (X^{k+1} \vee XY)\colon (X \vee Y)\\ &= (X^k \vee Y) \wedge X = X^k \vee XY.\end{aligned}$$

Hence, $X \vee Y$ satisfies RCL in RL_2/XY. However RL_2/XY is not of one of the types listed in Corollary 2.14. It can be shown in a similar manner that $X_1 \vee \dots \vee X_n$ satisfies RCL in $RL_n/\bigvee_{i \neq j} X_i X_j$. We will show that these are the only Noether lattices, other than those mentioned in Theorem 1.23, in which the maximal element satisfies RCL.

LEMMA 2.15. *If (L, M) is a local Noether lattice where $M = E \vee F$ is a minimal representation of M as the join of principals, $M^2 \neq 0$ and M satisfies RCL, then $L \simeq RL_2/XY$.*

Proof. First, notice that rank $M=1$, for if rank $M=0$, then L is special primary and M is principal by Theorem 1.18 contradicting $E \vee F$ is a minimal representation of M as the join of principals. Also 0 is not prime, for then by Lemma 2.13, M is principal. If $E \leqslant P$, P a prime with $P \neq M$, then $P = P \wedge (E \vee F) = E \vee (P \wedge F) = E \vee (P\colon F)F = E \vee PF$ since $F \nleqslant P$. So $P \leqslant E \vee MP$ and by the Intersection Theorem $P = E$. Similarly, if $F \leqslant P'$, for some prime P', then $P' = F$. Lemma 2.13 implies E and F are prime. $M^3 = (E^2 \vee F^2)M \neq 0$, so $M^2 = E^2 \vee F^2$ and $EF \leqslant E^2 \vee F^2$. Hence $EF\colon F = E \vee 0\colon F \leqslant (E^2 \vee F^2)\colon F = E^2\colon F \vee F$. So

$E \leqslant E^2\colon F \vee F$ and $E = E \wedge (E^2\colon F \vee F) = E^2\colon F \vee E \wedge F = E^2\colon F \vee EF \leqslant E^2\colon F \vee ME$. Hence, $E \leqslant E^2\colon F \leqslant E$. Similarly, $F = F^2\colon E$ and $EF \leqslant E^2 \wedge F^2 = (E^2\colon F^2)F^2 \leqslant EF^2 \leqslant MEF$. So $EF = 0$ and $E \wedge F = (E\colon F)F = EF = 0$ is normal decomposition for 0. Now, if C is principal in L, then by Lemma 2.13, $C \leqslant E$ or $C \leqslant F$. If $C \leqslant E \wedge F$, then $C = 0$. Assume $C \leqslant E$, $C \nleqslant F$. Choose k so that $C \leqslant E^k$, $C \nleqslant E^{k+1}$, then $C = C \wedge E^k = (C\colon E^k)E^k$. If $C\colon E^k = F_1 \vee \dots \vee F_n$, where F_i is principal for each i, then $C = E^k F_1 \vee \dots \vee E^k F_n$, but in a local lattice, principals are join irreducible, so $C = F_j E^k$, for some j. If $F_j \leqslant F$, then $C \leqslant E \wedge F$ and $C = 0$. If $F_j \leqslant E$ then $C \leqslant E^{k+1}$, contrary to the choice of k. Hence $F_j \nleqslant E$ and $F_j \nleqslant F$, so $F_j = I$ by Lemma 2.13 and $C = E^k$. Furthermore, $E^n \neq E^{n+j}$ for $j > 0$ by the Intersection Theorem. Therefore $\{0, I, E, E^2, \dots, F, F^2, \dots\}$ is the complete set of principals of L. If $E^n \vee F^j = E^k \vee F^m$, $k \geqslant n$ and $m \geqslant 1$, then $E^n \leqslant E^k \vee F^m$ and $E^n = E^n \wedge (E^k \vee F^m) = E^k \vee (E^n \wedge F^m) = E^k$, and $n = k$. If $m = 0$ and $j > 0$ then $F^j \leqslant E$ and $F \leqslant E$ which is a contradiction. Hence $j = 0$, $E^n = E^k$ and $n = k$. Hence, every element of L has a unique representation as the join of principals and L is clearly isomorphic to RL_2/XY. q.e.d.

THEOREM 2.16. *If (L, M) is a local Noether lattice in which M satisfies RCL, $M^2 \neq 0$ and $M = E_1 \vee \dots \vee E_k$, $k \geqslant 2$, is a minimal representation of M as the join of principals, then if $P_i = E_1 \vee \dots \vee \hat{E}_i \vee \dots \vee E_k$, where $\wedge$ denotes omission, $\{P_1, \dots, P_k, M\}$ is the set of primes of L, $E_i E_j = 0$ for $i \neq j$. Furthermore, L is a distributive lattice and is isomorphic to $RL_k/\bigvee_{i \neq j} X_i X_j$.*

Proof. Since M is not principal, rank $M = 1$ and 0 is not prime. If $k = 2$, the result follows by Lemma 2.15. Assume the theorem is true for $k \geqslant 2$ and $M = E_1 \vee \dots \vee E_{k+1}$ is a minimal representation of M as the join of principal elements. In L/E_i, the maximal M is the join of k principals and this is a minimal representation for M. If $A, B \in L/E_i$, and $A \circ M = B \circ M \neq E_i$, then $AM \vee E_i = BM \vee E_i$. Hence, $(B \vee A)M \leqslant BM \vee E_i$ and $(B \vee A)M = (B \vee A)M \wedge (BM \vee E_i) = BM \vee (B \vee A)M \wedge E_i$, by modularity, $= BM \vee \big((B \vee A)M\colon E_i\big)E_i \leqslant BM \vee ME_i$, since $E_i \nleqslant M^2$, $E_i \nleqslant (B \vee A)M$. Hence, $(B \vee A)M \leqslant (B \vee E_i)M = BM \neq 0$ and $(B \vee A) \leqslant B$. So $A \leqslant B$. Similary, $B \leqslant A$ and RCL holds in L/E_i. In L/E_1, $P_2, \dots, P_{k+1}$ are prime, $E_1 \leqslant P_i$, for $i = 2, \dots, k+1$, so $P_2, \dots, P_{k+1}$ are prime in L. By considering L/E_2, P_1 is also prime in L. Therefore, $P_1, \dots, P_{k+1}$ are prime in L.

Now in L/E_i, $\bigwedge_{\neq i} P_j = E_i$, so $\bigwedge_{i=1}^{k+1} P_i = P_i \wedge E_i = (P_i\colon E_i)E_i = P_i E_i$. $M^2 = E_1^2 \vee \dots \vee E_{k+1}^2$, as in Lemma 2.15, so $M^2 = P_1^2 \vee E_1^2$ and $P_1 E_1 \leqslant P_1^2 \vee E_1^2$. Hence, $P_1 \leqslant P_1^2\colon E_1 \vee E_1$ and $P_1 = (P_1^2\colon E_1 \vee E_1) \wedge P_1 = P_1^2\colon E_1 \vee P_1 E_1$ and $P_1 = P_1^2\colon E_1$, by the Intersection Theorem. Hence, $P_1 E_1 \leqslant P_1^2$ and $P_1^k E_1 \leqslant P_1^{k+1}$, for $k \geqslant 1$. Suppose Q is P_1-primary. Choose k, such that $P_1^{k+1} \leqslant Q$, $P_1^k \leqslant Q$. If $k \geqslant 1$, then $P_1^k E \leqslant P_1^{k+1} \leqslant Q$ and $P_1^k \leqslant Q$, $E_1 \leqslant P_1$, contradicting Q is P_1-primary. Hence, $k = 0$ and the only element of L which is primary for P_1 is P_1.

If $P_1, \ldots, P_{k+1}$ are the only primes of L which are less than M, then $0 = P_1 \wedge \ldots \wedge P_{k+1}$. Suppose P^* is prime in L, $P^* \neq P_i$, for $i = 1, \ldots, k+1$, and $P^* \neq M$. If $P^* \leqslant P_i$, then rank $M = 1$ implies $P^* = P_i$, contrary to the choice of P^*. Furthermore, if $E_i \leqslant P^*$, for some i, $i = 1, \ldots, k+1$, then P^* is prime in L/E_i. By the induction hypothesis, $P_1, \ldots, P_i, \ldots, P_{k+1}$, M are the only primes of L/E_i and $P^* = P_j$, for some $j \neq i$ or $P^* = M$. This is a contradiction. Hence $E_i \nleqslant P^*$, for $i = 1, \ldots, k+1$. Let $P^* = F_1 \vee \ldots \vee F_m$ be a representation of P^* as the join of principals. Since $P^* \nleqslant P_1$, there is an F_j, such that, $F_j \nleqslant P_1$, say $F_1 \nleqslant P_1$. If $i \geqslant 2$, $F_1 \vee E_i$ is principal in L/E_i which by induction is distributive and isomorphic to $RL_k/\bigvee_{i\neq j} X_i X_j$. Hence, $F_1 \vee E_i = E_{j(i)}^{n(i)} \vee E_i$, where $n(i)$ and $j(i)$ are integers, since these are the only principals of L/E_i. If $j(i) \neq 1$, then $F_1 \leqslant P_1$ which is a contradiction. Hence, $j(i) = 1$ for $i = 2, \ldots, k+1$ and $F_1 \leqslant E_1 \vee E_i$, for $i = 2, \ldots, k+1$. Since $k \geqslant 2$, $F_1 \leqslant E_1 \vee E_2 \leqslant P_{k+1}$, and $F_1 \leqslant E_1 \vee E_3 \leqslant P_2, \ldots, F_1 \leqslant E_1 \vee E_i \leqslant P_{i-1}, \ldots, F_1 \leqslant E_1 \vee E_{k+1} \leqslant P_k$, so $F_1 \leqslant P_2 \wedge \ldots \wedge P_{k+1} = E_1$. So $F_1 = F_1 \wedge E_1 = (F_1 : E_1) E_1 = (T_1 \vee \ldots \vee T_j) E_1$, where T_i is principal for each i. By join irreducibility of principals in a local lattice, $F_1 = T_q E_1$. $F_1 = T_q E_1 \leqslant P^*$, $E_1 \nleqslant P^*$, so since P^* is prime, $T_q \leqslant P^*$ and $F_1 \leqslant MP^*$. Arrange $F_1, \ldots, F_m$, so that $F_1, \ldots, F_p$ have the property, $F_i \nleqslant P_j$ for some j and $F_{p+1}, \ldots, F_m \leqslant \bigwedge_{i=1}^{k+1} P_i$. By the above argument, if $j \leqslant p$, then $F_j \leqslant MP^*$. Hence, $P^* \leqslant MP^* \vee F_{p+1} \vee \ldots \vee F_m$ and $P^* = F_{p+1} \vee \ldots \vee F_m \leqslant \bigwedge_{i=1}^{k+1} P_i \leqslant P_i$, for each i. This is a contradiction. Therefore, $\{M, P_1, \ldots, P_{k+1}\}$ is the set of primes of L and $\bigwedge_{i=1}^{k+1} P_i = P_j E_j = 0$, for each j.

Furthermore, if C is principal in L, $C \neq I$, then $C \neq 0$ implies $C \nleqslant P_j$, for some j. Suppose $C \nleqslant P_1$, then as in the preceding argument, $C \leqslant P_j$, for $j = 2, \ldots, k+1$, and $C \leqslant E_1$. Choose n, such that $C \leqslant E_1^n$ and $C \nleqslant E_1^{n+1}$, then $C = C' E_1^n$, where C' is principal. If $C' \neq I$, then since $C' \neq 0$, $C' \nleqslant P_j$, for some j and $C' \leqslant E_j$. $j \neq 1$, since $C \nleqslant E_1^{n+1}$. If $j > 1$, then $C = C' E_1^n \leqslant E_j E_1 = 0$, contradicting $C \neq 0$. Hence $C' = I$ and $C = E_1^n$. Therefore, L is distributive. If $E_i^n = E_i^{n+j}$, $j > 0$, then $E_i^n = 0$ and $M^n = E_1^n \vee \ldots \vee \hat{E}_i^n \vee \ldots \vee E_{k+1}^n = P_i^{n-1} M \neq 0$. By RCL, $M^{n-1} = P_i^{n-1}$ which is a contradiction. Thus, $\{0, I, E_1, \ldots, E_{k+1}, E_1^i, \ldots, E_{k+1}^i, i \text{ an integer}\}$ is the complete set of principals of L and no two of these principals are equal. If $E_1^{n(1)} \vee \ldots \vee E_{k+1}^{n(k+1)} = E_1^{m(1)} \vee \ldots \vee E_{k+1}^{m(k+1)}$, then in L/E_1, $E_1 \vee E_2^{n(2)} \vee \ldots \vee E_{k+1}^{n(k+1)} = E_1 \vee E_2^{m(2)} \vee \ldots \vee E_{k+1}^{m(k+1)}$. By induction, $n(i) = m(i)$, for $i = 2, \ldots, k+1$. By considering L/E_2, $n(1) = m(1)$. Hence, every element

of L has a unique representation as the join of principals and clearly L is isomorphic to $RL_{k+1}/\bigvee_{i\neq j} X_i X_j$. q.e.d.

THEOREM 2.17. *If (L, M) is a local Noether lattice in which M satisfies RCL, then L is one of the following*

(i) $M^2 = 0$,

(ii) *special primary*,

(iii) *regular local altitude one*,

(iv) $RL_n/\bigvee_{i\neq j} X_i X_j$, *for some integer n*.

Proof. By Lemma 1.16, rank $M \leqslant 1$. If rank $M = 0$, then the result follows by Theorem 1.18. If rank $M = 1$ and 0 is not prime, then L is $RL_n/\bigvee_{i\neq j} X_i X_j$ by Theorem 2.16. q.e.d.

THEOREM 2.18. *If (L, M) is a local Noether lattice in which M satisfies RCL, then M is join principal.*

Proof. If $M^2 = 0$, then $(A \vee BM) : M = M = A : M \vee B$, if $A \neq M$. If $A = M$, then $(A \vee BM) : M = I = A : M \vee B$, and M is join principal. If L is special primary or regular local altitude one, then M is principal and hence, join principal. By Theorem 2.17, the only remaining possibility is that L be isomorphic to $RL_n/\bigvee_{i\neq j} X_i X_j$. We shall show that the maximal in this lattice is join principal.

First, we show $X \vee Y$ is join principal in RL_2/XY. The elements of RL_2/XY are of one of the following forms, $X^k \vee XY$, where $k \geqslant 1$, $Y^k \vee XY$, where $k \geqslant 1$, or $X^k \vee Y^n \vee XY$, where $k, n \geqslant 1$.

We omit this part of the proof since it is almost identical to the induction step proof. So, assume that the maximal in $RL_{n-1}/\bigvee_{i\neq j} X_i X_j$ is join principal and consider $RL_n/\bigvee_{i\neq j} X_i X_j$. For simplicity of notation, we will denote $\bigvee_{i\neq j} X_i X_j$ by 0^* and $RL_n/0^*$ by L^*. Also we note that $0^* : X_i = P_i$, for $i = 1, \ldots, n$. Furthermore in L^*, the multiplication is defined by $A \circ B = AB \vee 0^*$.

If $X_1^{k(1)} \vee \ldots \vee X_m^{k(m)} \vee 0^* \in L^*$ and $k(i) \geqslant 2$ for $i = 1, \ldots, m$, then $(X_1^{k(1)} \vee \ldots \vee X_m^{k(m)} \vee 0^*) : M = (X_1^{k(1)-1} \vee P_1) \wedge \ldots \wedge (X_m^{k(m)-1} \vee P_m) \wedge P_{m+1} \wedge \ldots \wedge P_n = X_1^{k(1)-1} \vee \ldots \vee X_m^{k(m)-1} \vee 0^*$.

Let $A, B \in L^*$. If $X_i \leqslant A \wedge B$, for any i, then since the maximal of L^*/X_i is join principal, $(A \circ M \vee B) : M = A \vee B : M$ in L^*/X_i, and hence, in L^*. Rearange the X_i, so that $A = X_1^{n(1)} \vee \ldots \vee X_m^{n(m)} \vee 0^*$ and $B = X_1^{k(1)} \vee \ldots \vee X_j^{k(j)} \vee 0^*$.

First, assume $n(i), k(i) \geqslant 2$. We have two cases, $m \leqslant j$ and $j \leqslant m$. Assume $m \leqslant j$.

$$(AM\vee B):\ M = (X_1^{n(1)+1}\vee\ldots\vee X_m^{n(m)+1}\vee 0^*\vee X_1^{k(1)}\vee\ldots\vee X_j^{k(j)}):\ M$$
$$= (X_1^{\min\{n(1)+1,k(1)\}}\vee\ldots\vee X_m^{\min\{n(m)+1,k(m)\}}\vee X_{m+1}^{k(m+1)}\vee\ldots\vee X_j^{k(j)}\vee 0^*):\ M$$
$$= X_1^{\min\{n(1)+1,k(1)\}-1}\vee\ldots\vee X_m^{\min\{n(m)+1,k(m)\}-1}\vee X_{m+1}^{k(m+1)-1}\vee\ldots\vee X_j^{k(j)-1}\vee 0^*.$$
$$A\vee B:\ M = X_1^{n(1)}\vee\ldots\vee X_m^{n(m)}\vee 0^*\vee X_1^{k(1)-1}\vee\ldots\vee X_j^{k(i)-1}$$
$$= X_1^{\min\{n(1),k(1)-1\}}\vee\ldots\vee X_m^{\min\{n(m),k(m)-1\}}\vee X_{m+1}^{k(m+1)-1}\vee\ldots\vee X_j^{k(j)-1}\vee 0^*.$$

Assume $j\leqslant m$.

$$(AM\vee B):\ M$$
$$= X_1^{\min\{n(1)+1,k(1)\}}\vee\ldots\vee X_j^{\min\{n(j)+1,k(j)\}}\vee X_{j+1}^{n(j+1)+1}\vee\ldots\vee X_m^{n(m)+1}\vee 0^*:\ M$$
$$= X_1^{\min\{n(1)+1,k(1)\}-1}\vee\ldots\vee X_j^{\min\{n(j)+1,k(1)\}-1}\vee X_{j+1}^{n(j+1)}\vee\ldots\vee X_m^{n(m)}\vee 0^*.$$
$$A\vee B:\ M = X_1^{\min\{n(1),k(1)-1\}}\vee\ldots\vee X_j^{\min\{n(j),k(j)-1\}}\vee X_{j+1}^{n(j+1)}\vee\ldots\vee X_m^{n(m)}\vee 0^*.$$

Hence, in either case, $(AM\vee B):\ M = A\vee B:\ M$.

We will outline the cases which remain. However, we omit the proofs since they are lengthy, but straight-forward, computations.

(a) $k(i)\geqslant 2$, for $i = 1, \ldots, j$, and $n(i) = 1$ for at least one i, $1\leqslant i\leqslant m$.

(b) $n(i)\geqslant 1$, for $i = 1, \ldots, m$, and $k(i) = 1$, for at least one i, $1\leqslant i\leqslant j$. Say $A = X_1^{n(1)}\vee\ldots\vee X_m^{n(m)}\vee 0^*$ and $B = X_1\vee\ldots\vee X_q\vee X_{q+1}^{k(q+1)}\vee\ldots\vee X_j^{k(j)}\vee 0^*$, by rearranging the X_i if necessary. In this case, we consider three possibilities, $m\leqslant q\leqslant j$, $q\leqslant m\leqslant j$, and $q\leqslant j\leqslant m$.

(c) Assume $A = X_1^{n(1)}\vee\ldots\vee X_m^{n(m)}\vee 0^*$, where $n(i)\geqslant 1$, and $B = X_{m+1}^{k(m+1)}\vee\ldots\vee X_q^{k(q)}\vee 0^*$. We consider two alternatives for B, either $k(j)\geqslant 2$, for $j = m+1, \ldots, q$, or $k(j) = 1$, for some j.

These are all the posibilities for A and B in L^*. Therefore $X_1\vee\ldots\vee X_n$ satisfies $(AM\vee B):\ M = A\vee B:\ M$, for any $A, B\in L^*$ and hence, is join principal. q.e.d.

Section 3. An element M is *weak join principal* (WJP) if $AM:\ M = A\vee 0:\ M$ for every $A\in L$. An element M, which satisfies RCL, in addition to being weak join principal, has the property that $AM\neq 0$ implies $0:\ M\leqslant A$. Theorem 2.18 shows that this additional condition on a weak join principal maximal element of a local Noether lattice insures that the maximal is, in fact, join principal. However a weak join principal maximal need not satisfy RCL and it also need not be join principal. A lattice which illustrates this is $RL_2/XY^2\vee X^2Y$. The maximal of this lattice $X\vee Y$ fails to satisfy $XY^2\vee X^2Y:\ X\vee Y = XY^2\vee X^2Y$ so by Lemma 2.11, $X\vee Y$ does not satisfy RCL in $RL_2/XY^2\vee X^2Y$. Furthermore $X\vee Y$ is not join principal since $\big(X\circ(X\vee Y)\big)\vee Y^2:\ (X\vee Y) = (X^2\vee XY\vee Y^2):\ (X\vee Y) = X\vee Y$ but $XY\big(Y^2:\ (X\vee Y)\big) = X\vee Y^2$. It is a straight forward

computation to verify that $X\vee Y$ is weak join principal in $RL_2/X^2Y\vee XY^2$. Furthermore the maximal in RL_2/XY, $RL_2/XY\vee X^j$, $RL_2/XY\vee X^j\vee Y^n$, $RL_2/X^2Y\vee XY^2\vee X^j$ and $RL_2/X^2Y\vee XY^2\vee X^j\vee Y^n$ where $j, n\geqslant 2$, can be shown to be weak join principal. In the following theorem we show that these are all the distributive local lattices with a maximal which is the join of two principals in which the maximal is weak join principal.

THEOREM 3.11. *If (L, M) is a distributive local Noether lattice in which $M = X\vee Y$ is a minimal representation of M as the join of principals and M is weak join principal, then L is one of the following RL_2/XY, $RL_2/XY\vee X^j$, $RL_2/XY\vee X^j\vee Y^n$, $RL_2/XY^2\vee X^2Y$, $RL_2/XY^2\vee X^2Y\vee X^j$, $RL_2/XY^2\vee X^2Y\vee \vee X^j\vee Y^n$, where $j, n\geqslant 2$.*

Proof. $M^3 = (X^2\vee Y^2)M$, so $M^2\leqslant X^2\vee Y^2\vee 0:\ M$ and $XY\leqslant X^2\vee \vee Y^2\vee 0:\ M$. By distributivity, $XY\leqslant X^2$, $XY\leqslant Y^2$, or $XY\leqslant 0:\ M$. If $XY\leqslant X^2$, then $Y\leqslant X\vee 0:\ X$ and $Y\nleqslant X$ implies $Y\leqslant 0: X$ and $XY = 0$. Similarly, if $XY\leqslant Y^2$, then $XY = 0$. If $XY\leqslant 0:\ M$, then $X^2Y = XY^2 = 0$. Furthermore, $M^3 = X^3\vee Y^3$ and $M^k = X^k\vee Y^k$, for $k\geqslant 3$, so as in Lemma 1.17, rank $M\leqslant 1$. The proof is divided into two main cases, rank $M = 0$ and rank $M = 1$.

Case 1. Since rank $M = 0$, M is the only prime of L. First, assume $XY = 0$. $X\nleqslant Y$, but $M^n = 0$, for some integer n, so choose j, such that, $X^j\leqslant Y$ and $X^{j-1}\nleqslant Y$, then $j\geqslant 2$. Since $M^j\leqslant Y$, $M^j = M^j\wedge Y = (M^j: Y)Y$. Now $Y\nleqslant M^2$, for then $M\leqslant X\vee M^2$ and $M\leqslant X$, contradicting $X\vee Y$ is a minimal base for M. Therefore, $Y\nleqslant M^j$, since $Y\nleqslant M^2$, and $M^j\leqslant MY$. M is weak join principal, so $M^{j-1}\leqslant Y\vee 0:\ M$. $X^{j-1}\leqslant 0:\ M$, since $X^{j-1}\nleqslant Y$, so $X^j = 0$. In the same way, choose n, such that $Y^n\leqslant X$, $Y^{n-1}\nleqslant X$, then $Y^n = 0$ and $n\geqslant 2$. Define $C^* = \{I, 0, X, X^2, \ldots, X^{j-1}, Y, Y^2, \ldots, Y^{n-1}\}$. Clearly, any principal in L is an element of C^*, since L is distributive and $XY = 0$. If $X^k, Y^p\in C^*$, then $X^k = Y^p\leqslant Y$ implies $k\geqslant j$, contradicting $X^k\in C^*$. Hence, the elements of C^* are distinct. Since these elements are distinct and L is distributive, joins of these elements are unique. For, if $X^h\vee Y^k = X^p\vee Y^q$, $1\leqslant h$, $p\leqslant j-1$, $1\leqslant k$, $q\leqslant n$, then $X^h\nleqslant Y^q$ implies $X^h\leqslant X^p$ and similarly, $X^p\leqslant X^h$, and $p = h$. Similarly, $q = k$. Hence, $j, n\geqslant 2$ and L is isomorphic to $RL_2/X^j\vee Y^n\vee XY$.

Now assume that $XY\neq 0$, then $X^2Y = XY^2 = 0$. There exist n and j both greater than one, so that $X^j = 0$, $X^{j-1}\nleqslant Y$, $Y^n = 0$, $Y^{n-1}\nleqslant X$. $C^* = \{0, I, X, X^2, \ldots, X^{j-1}, Y, Y^2, \ldots, Y^{n-1}, XY\}$ is the set of all principal elements of L. As in the preceding argument, these principals are distinct and their joins are unique. Therefore, L is isomorphic to $RL_2/X^2Y\vee \vee XY^2\vee X^j\vee Y^n$.

Case 2. Assume rank $M = 1$. Since L is distributive the only possible primes of L are 0, X, Y and $X\vee Y$. 0 is not prime, for then M satisfies RCL and is principal by Lemma 2.13, contradicting $M = X\vee Y$ is a minimal representation of M. Case 2 is divided into three subcases, (a) X and Y are prime, (b) X is prime and Y is not prime, (c) Y is prime and X is not prime.

(a) Assume X and Y are prime. If $XY = 0$, then $X \wedge Y = (X: Y)Y = XY = 0$ and M is not a prime of 0. Therefore, M satisfies RCL and hence L is isomorphic to RL_2/XY.

If $XY \neq 0$, then $X^2Y = XY^2 = 0$. If $k \geqslant 2$, then $XY^2 = 0 \leqslant X^k \leqslant X^2$ and $X \nleqslant X^2$, $Y^2 \nleqslant X$, so the only X-primary element of L is X. Similarly, Y is the only Y-primary element of L. Hence, $0 = X \wedge Y \wedge Q_M$, where Q_m is M-primary and is necessary to the decomposition since $XY \neq 0$. Let $Q_M = M^3$, $X \wedge Y \wedge M^3 = X^2Y \vee XY^2 = 0$ and since M is maximal, M^3 is primary for M. If $C^* = \{0, I, XY, X^i, Y^j\}$ where i and j are positive integers, then every principal of L is in C^* and the elements of C^* are distinct. By distributivity, joins of these elements are unique and L is isomorphic to $RL_2/X^2Y \vee XY^2$.

(b) Assume X is prime and Y is not prime. Every principal of L is a power of X or Y or equal to XY. Since Y is not prime, $X^k \leqslant Y$ for some k. Choose k, such that, $X^k \leqslant Y$ and $X^{k-1} \nleqslant Y$, then $k \geqslant 2$, since $X \nleqslant Y$. $M^k \leqslant Y$, so $M^k = (M^k: Y)Y$. $Y \nleqslant M^2$ implies $Y \nleqslant M^k$ and since M is weak join principal, $M^{k-1} \leqslant Y \vee 0: M$, since $M^k \leqslant MY$. $X^{k-1} \nleqslant Y$, so $X^{k-1} \leqslant 0: M$ and $X^k = 0$.

First, suppose that $XY = 0$. $C^* = \{0, I, X, X^2, \ldots, X^{k-1}, Y, Y^2, \ldots\}$ is the set of all principals of L and elements of C^* are all distinct. Furthermore, the joins of these principals are unique by distributivity. Hence, L is isomorphic to $RL_2/X^k \vee XY$.

Now, assume $XY \neq 0$, then $XY^2 = X^2Y = 0$. $C^* = \{0, I, X, X^2, \ldots \ldots, X^{k-1}, XY, Y, Y^2, \ldots\}$ is the complete set of principals of L. $Y^j \neq X^n$, since $Y \nleqslant X$, a prime. $Y^j \neq Y^n$ for $n \neq j$, since Y is not nilpotent. $XY \neq Y^n$, since $Y \nleqslant X$. $XY \neq X^j$, because if $XY = X$ then $X = 0$ and if $XY = X^j$, for $j > 1$, then $Y \vee 0: X = X^{j-1} \vee 0: X$ and $XY \neq 0$ implies $Y \leqslant X^{j-1} \leqslant X$, which is a contradiction. Hence the elements of C^* are distinct. As before, the joins of elements of C^* are unique and hence, L is isomorphic to $RL_2/X^2Y \vee XY^2 \vee X^k$ and $k \geqslant 2$.

(c) If Y is prime and X is not prime, then as in (b), L is isomorphic to $RL_2/XY \vee Y^n$ or $RL_2/X^2Y \vee XY^2 \vee Y^n$, where $n \geqslant 2$. q.e.d.

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Reçu par la Rédaction le 10. 4. 1971