onto a regular T_0-space having a λ-base. Both of these theorems may be proved from a unified point of view which encompasses certain non first countable situations. This is carried out in [14]. Here it seems preferable to give a direct proof with appropriate references to [13] rather than use the general mapping lemma of [14].

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Models of second order arithmetic with definable Skolem functions

by

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Let A_2 be the axiomatic system of second order arithmetic as described in [2].

In the study of the problem whether the standard part of a model of A_2 is itself a model of A_2 we introduced the following model theoretic concept: Let A be a structure of type σ and P a singular predicate of σ. Let B be another structure of type σ' such that A is a reduct of B. We say that B is an *S-structure* for A and P if

1° all the Skolem functions of B are definable in B;

2° each subset of P^A (the interpretation of P in A) which is parametrically definable in B is so definable in A. (See [3].)

Using Lévy's model for A_2 (see e.g. [4], pp. 241–247) we can easily exhibit an ω-model A in which all the axioms of A_2 with the exception of the axiom of choice are valid such that no S-structure exists for A and the predicate $N(\cdot)$. For ω-models of the full system A_2 the situation is different: we shall prove the following

THEOREM. *If M is a denumerable ω-model for A_2 then there exists an S-structure for M and the predicate $N(\cdot)$.*

Proof of this theorem will occupy the rest of this paper. We shall use a very primitive form of the forcing argument. Our proof was influenced mainly by the result of Felgner [1].

LEMMA 1. *The following scheme is provable in A_2* (*cf.* (iii) *below for the meaning of* $z^{(n)}$):

$$S(w)\ \&\ C(w)\ \&\ (n)_N(c)_S\{C(c)\to(\mathrm{E}\bar{c})_S[C(\bar{c})\ \&\ B(c,\bar{c})\ \&\ D(n,\bar{c})]\}$$
$$\to(\mathrm{E}z)_S\{(z^{(0)}=w)\ \&\ (n)_N[C(z^{(n)})\ \&\ B(z^{(n)},z^{(n+1)})\ \&\ D(n,z^{(n+1)})]\}.$$

Read "c is a vertex" for $C(c)$ and "$\bar{c}$ is an nth extension of c" for $B(c,\bar{c})\ \&\ D(n,\bar{c})$.

The scheme can then be expressed as follows. If for every integer n every vertex has an nth extension which is also a vertex then for every vertex w there is a z such that $z^{(0)} = w, z^{(1)}, \dots, z^{(n)}, \dots$ is an infinite sequence of vertices and $z^{(n+1)}$ is an nth extension of $z^{(n)}$. In this formulation the scheme becomes obvious and one sees immediately how to prove it with the help of the axiom of choice. Scrutinizing the proof we convince ourselves that it can be repeated in the formal system A_2.

Let L be the language of A_2 as described in [2]. We assume that the logical constants of L are &, $\vee$, $\to$, $\equiv$, $\neg$ and the existential and general quantifiers. In the formulae of L we shall admit only limited quantifiers $(\mathrm{E}x)_N$ and $(\mathrm{E}x)_S$: "there is an integer" and "there is a set" and limited general quantifiers: $(x)_N$, $(x)_S$. If the formula $(\mathrm{E}!\,x)\,A(x, y, \dots, t)$ is provable in A_2, then we shall allow a new term $a(y, \dots, t)$ and an axiom $A\big(a(y, \dots, t), y, \dots, t\big)$. The system A_2 enriched by the new terms and axioms is an inessential extension of A_2. We shall treat formulae containing the defined terms as abbreviations of formulae of the language L.

An ω-model M of A_2 will always be identified with the family of its sets. If Φ is a formula with, say, 3 free variables, then we denote by Φ_M a ternary relation which holds between 3 elements of the model (integers or sets) if they satisfy Φ in M. An $n$ary relation R is parametrically definable in M if there is a formula Φ with $k \geqslant n$ free variables and elements $p_{n+1}, \dots, p_k$ of the model such that for arbitrary $a_1, \dots, a_n$ the relation R holds between $a_1, \dots, a_n$ if and only if they belong to the model and $M \models \Phi[a_1, \dots, a_n, p_{n+1}, \dots, p_k]$. As no other kind of definability will be involved in our discussion, we shall often omit the word "parametrically".

We now introduce a series of abbreviations:

(i) *Pairs of integers.* A pair of integers m, n is defined as $J(m, n) = \frac{1}{2}(m+n)(m+n+1)+m$.

(ii) *Pairs of sets* (cf. [5]). A pair of sets $x, y \subseteq N$ is defined as the set $x; y = \{2a+1\colon a \in x\} \cup \{2b\colon b \in y\}$. Each set z can be uniquely represented as $x; y$ and we put $x = (z)_1$, $y = (z)_2$.

(iii) *Coding of infinite sequences of sets.* For any integer n and set x we put $x^{(n)} = \{m\colon J(n, m) \in x\}$. Instead of $(\mathrm{E}n)(y = x^{(n)})$ we write $y \varepsilon x$.

(iv) *Relations, domains and ranges.* We write mxn for $J(m, n) \in x$; this formula is read: m bears the relation x to n. Thus every set of integers can be conceived as a binary relation. We define $\mathrm{Dom}(x)$ (domain of x) and $\mathrm{Rg}(x)$ (range of x) in the usual way.

(v) *One-one mappings; isomorphism.* We shall abbreviate as $\mathrm{Fn}(f)$ a formula which says that f maps a set of integers onto a set of integers and is one-to-one and as $x_1 \underset{f}{\simeq} x_2$ the formula $\mathrm{Fn}(f)$ & $(u_1)_N(u_2)_N(v_1)_N(v_2)_N \{[(u_1 f u_2)$ & $(v_1 f v_2)] \to [(u_1 x_1 v_1) \equiv (u_2 x_2 v_2)]\}$; this formula says that f establishes an isomorphism between x_1 restricted to the domain of f with x_2 restricted to the range of f.

(vi) *Well ordering. Formulae $\vec{R}$ and $\overleftarrow{R}$.* We abbreviate the formula "x is an irreflexive well-ordering of N" by $\mathrm{Bord}(x)$. Moreover we define

$$\vec{R}(f, x, n)\colon\ \mathrm{Fn}(f)\ \&\ (j)_N\big[\big(j \in \mathrm{Rg}(f)\big) \equiv (jxn)\big],$$
$$\overleftarrow{R}(f, x, n)\colon\ \mathrm{Fn}(f)\ \&\ (i)_N\big[\big(i \in \mathrm{Dom}(f)\big) \equiv (ixn)\big].$$

These formulae say that the range (or domain) of f coincides with the set of x-predecessors of n.

LEMMA 2. *The following formula is provable in A_2:*

$$\mathrm{Bord}(x)\ \&\ \mathrm{Bord}(y) \to (\mathrm{E}!\,f)[(x \underset{f}{\simeq} y)\ \&\ (A \vee B \vee C)]$$

where A, B, C are formulae: $\mathrm{Rg}(f) = \mathrm{Dom}(f) = N$, $\big(\mathrm{Dom}(f) = N\big)$ & & $(\mathrm{E}n)_N \vec{R}(f, y, n)$, $\big(\mathrm{Rg}(f) = N\big)$ & $(\mathrm{E}n)_N \overleftarrow{R}(f, x, n)$.

The formula says of course that if x, y are two well-orderings of N, then they are either similar, or one is isomorphic to an initial segment of the other. In all cases the isomorphism is determined uniquely. A formal proof of the formula is essentially the same as the one given in elementary set theory.

(vii) Let Cond be the formula (with one free variable x)

$$\mathrm{Bord}\big((x)_1\big)\ \&\ (i)_N(j)_N\big[(i = j) \vee \big((x)_2^{(i)} \neq (x)_2^{(j)}\big)\big].$$

Intuitively speaking a *condition* is a sequence $(x)_2^{(0)}, (x)_2^{(1)}, (x)_2^{(2)}, \dots$ well ordered by the relation: $(x)_2^{(i)}$ precedes $(x)_2^{(j)}$ if $J(i, j) \in (x)_1$.

(viii) *Partial ordering of conditions.* Let $x < y$ be the formula:

$$\mathrm{Cond}(x)\ \&\ \mathrm{Cond}(y)\ \&\ (\mathrm{E}f)_S(\mathrm{E}n)_N\{[(x)_1 \underset{f}{\simeq} (y)_1]\ \&\ \big(\mathrm{Dom}(f) = N\big)\ \&$$
$$\&\ \vec{R}\big(f, (y)_1, n\big)\ \&\ (i)_N(j)_N\big[(ifj) \to \big((x)_2^{(i)} = (y)_2^{(j)}\big)\big]\}.$$

The intuitive meaning of this formula is that the well-ordering $(x)_1$ is similar to an initial segment of $(y)_1$ and that $(x)_2^{(i)} = (y)_2^{(f(i))}$ where f is an isomorphic mapping of N into N which establishes the isomorphism of $(x)_1$ and a segment of $(y)_1$.

LEMMA 3. *The following formulae are provable in A_2:*

(a) $(x < y)$ & $(y < z) \to (x < z)$;

(b) $\mathrm{Cond}(x) \to \neg(x < x)$;

(c) $\mathrm{Cond}(x)$ & $S(s)$ & $\neg\big(s \varepsilon (x)_2\big) \to (\mathrm{E}y)_S\big[(x < y)\ \&\ \big(s \varepsilon (y)_2\big)\big]$;

(d) $S(x)$ & $(n)_N(x^{(n)} < x^{(n+1)}) \to (\mathrm{E}y)_S(n)_N(x^{(n)} < y)$.

(a) and (b) are obvious; (c) is proved by taking y so that the order type of $(y)_1$ be $\alpha+1$ where α is the order type of $(x)_1$ and that under the ordering $(y)_1$ the last term of the sequence $(y)_2^{(n)}$, $n = 0, 1, \ldots$ be equal to s whereas the previous terms be equal to consecutive terms of $(x)_2$. Finally (d) is proved as follows. From $x^{(i)} < x^{(i+1)}$ we infer that there is a similarity mapping ε_i of $\langle N, (x^{(i)})_1\rangle$ onto a segment O_{i+1} of $\langle N, (x^{(i+1)})_1\rangle$ such that $(x^{(i)})_2^{(k)} = (x^{(i+1)})_2^{(\varepsilon_i(k))}$ for each k. Let $N = N_0 \cup N_1 \cup \ldots$ be a decomposition of N into sets equipotent with $N - O_i$ and let f_i be a one-one mapping of N_i onto $N - O_i$. We take for y a condition $u; v$ such that $(mun) \equiv \big[(i<j) \vee (i=j) \,\&\, f_i(m)(x^{(i)})_1 f_j(n)\big]$ and $v^{(n)} = (x^{(i)})_2^{(f_i(m))}$ where $m \in N_i$ and $n \in N_j$.

(ix) *Forcing*. We construct a new language L' which differs from L only by containing a new binary predicate symbol "R". For each formula Φ of L we construct a new formula F_Φ which has one more free variable ("the new variable") than Φ. The definition of F_Φ is by induction.

If Φ is an atomic formula of L then F_Φ is the formula $\mathrm{Cond}(x) \,\&\, \Phi$ where x is the first variable not in Φ.

If Φ is the formula uRv, then F_Φ is the formula

$$\mathrm{Cond}(x) \,\&\, S(u) \,\&\, S(v) \,\&\, (\mathrm{E}i)_N(\mathrm{E}j)_N\big(i(x)_1 j\big) \,\&\, \big(u = (x)_2^{(i)}\big) \,\&\, \big(v = (x)_2^{(j)}\big)$$

where x is the first variable different from the variables u, v.

If Φ is $\neg\Psi$, then F_Φ is the formula $(x')_S[(x < x') \to \neg F_\Psi(x')]$ where x' is the first variable which does not occur in F_Ψ and is different from x and $F_\Psi(x')$ result from F_Ψ by substituting x' for its new variable.

If Φ is $\Psi \,\&\, \Theta$ or $\Psi \vee \Theta$ then F_Φ is the formula $F_\Psi(x) \,\&\, F_\Theta(x)$ or $F_\Psi(x) \vee \vee F_\Theta(x)$ where x is the first variable which occurs neither in F_Ψ nor in F_Θ and $F_\Psi(x)$ resp. $F_\Theta(x)$ arise from F_Ψ, F_Θ by substituting x for their new variables.

If Φ is $\Psi \to \Theta$ then F_Φ is the formula $(x')_S[x < x' \,\&\, F_\Psi(x') \to F_\Theta(x')]$ where x, x' are different variables which occur neither in F_Ψ nor in F_Θ and $F_\Psi(x')$. $F_\Theta(x')$ arise from F_Ψ, F_Θ by substitution of x for the new variables of these formulae.

If Φ is $\Psi \equiv \Theta$, then F_Φ is $F_\Psi(x) \,\&\, F_\Theta(x) \vee (x')_S[(x < x') \to (\neg F_\Psi(x') \,\&\, \&\, \neg F_\Theta(x'))]$ where $x, x', F_\Psi(x)$ etc. are defined similarly as above.

If Φ is $(\mathrm{E}u)_N\Psi$ or $(\mathrm{E}u)_S\Psi$, then F_Φ is $(\mathrm{E}u)_N F_\Psi$ or $(\mathrm{E}u)_S F_\Psi$.

If Φ is $(u)_N\Psi$ or $(u)_S\Psi$ then F_Φ is $(u)_N F_\Psi$ or $(u)_S F_\Psi$.

(x) *Decidability*. The formula $F_\Phi \vee F_{\neg\Phi}$ will be written as Dec_Φ.

The use of symbols $F_\Phi, \mathrm{Dec}_\Phi$ is often cumbersome and we shall in most places write them as $x \Vdash \Phi$ or $x \| \Phi$. Istead of $M \models F_\Phi[p, a_1, \ldots, a_n]$ or $M \models \mathrm{Dec}_\Phi[p, a_1, \ldots, a_n]$ we shall then write $p \Vdash_M \Phi[a_1, \ldots, a_n]$ or $p \|_M \Phi[a_1, \ldots, a_n]$.

Lemma 4. *If Φ is a formula of L' then the formulae*

(a) $(x < y) \,\&\, (x \Vdash \Phi) \to (y \Vdash \Phi)$,

(b) $\mathrm{Cond}(x) \to (\mathrm{E}y)[(x < y) \,\&\, (y \| \Phi)]$,

are provable in A_2. Moreover if Φ is a formula of L, then the formula

(c) $\mathrm{Cond}(x) \to [(x \Vdash \Phi) \equiv \Phi]$

is provable in A_2.

Proof of this lemma is immediate.

(xi) *Generic sets*. Let M be an ω-model of A_2. A set $Q \subseteq \mathrm{Cond}_M$ is dense in Cond_M if for every x in Cond_M there is an x' in Q such that $x <_M x'$. A set $G \subset \mathrm{Cond}_M$ is generic for M if it satisfies the usual three conditions: (1) for every x, y in G there is a z in G such that $z >_M x$ and $z >_M y$; (2) if $x \in G$ and $y <_M x$ then $y \in G$; (3) if Q is a subset of Cond_M which is dense and definable in M then Q and G intersect with each other.

Leema 5. *For every denumerable ω-model M there exist generic sets.*

Proof: obvious.

We fix a denumerable ω-model M and a set G generic for M and define a relation $\lesssim$ as follows: Let Φ be the formula $v_1 R v_2$ and let v_0 be the new variable of F_Φ. We put for x, y in M:

$$(x \lesssim y) \equiv \text{there is a } g \text{ in } G \text{ such that } M \models F_\Phi[g, y, x].$$

Relation $\lesssim$ together with M determines a model $(M, \lesssim)$ of the language L'. We are going to prove that this model is the required S-structure for M. First we must establish the truth-lemma:

Lemma 6. *Let Φ be a formula of L' with $k + l$ free variables and let $\bar{n}$ be a k-tuple of integers and $\bar{s}$ an l-tuple of elements of M. Then*

$$(\mathrm{E}x)_G(M \models F_\Phi[x, \bar{n}, \bar{s}]) \equiv (M, \lesssim) \models \Phi[\bar{n}, \bar{s}].$$

We omit the [illegible]tine proof of this lemma.

Lemma 7. *I*[illegible]*s any of the following formulae*

(i) $S(v_0) \,\&\,$ [illegible] $\&\, (v_0 R v_1) \to \neg(v_0 = v_1)$,

(ii) $S(v_0) \,\&\,$ [illegible] $\&\, S(v_2) \,\&\, (v_0 R v_1) \,\&\, (v_1 R v_2) \to (v_0 R v_2)$,

(iii) $S(v_0) \,\&\, S(v_1) \,\&\, \neg(v_0 = v_1) \to \neg\neg[(v_0 R v_1) \vee (v_1 R v_0)]$

and if $u, v, w \in M$ then $x \Vdash_M \Phi[u, v, w]$ for any x in Cond_M.

Proof of (i). If x' is an extension of x which forces the antecedent then no extension of x' can force the formula $u = v$ because otherwise we would infer $u = v$ which is incompatible with $x' \Vdash_M uRv$.

(ii) Follows from the definition of forcing.

(iii) Let x' be an extension of x which forces the antecedent of (iii) and let x^* be an extension of x'. Let x'' be an extension of x^* such that both u and v stand in relation ε to $(x'')_2$ (see Lemma 3(c)). It follows that there are integers m, n such that $u=(x'')_2^{(m)}$, $v=(x'')_2^{(n)}$ and $m\neq n$. Hence either $M\models\big(n(x'')_1 m\big)$ or $M\models\big(n(x'')_1 m\big)$ and hence either $x''\models_M uRv$ or $x''\models_M vRu$. Thus we have proved $x'\Vdash_M \neg\neg[(uRv)\vee(vRu)]$.

In the next lemma Φ is a formula of L' with the free variables $u,\bar v=(v_1,\ldots,v_k)$, $\bar w=(w_1,\ldots,w_l)$, u' is a variable which does not occur in Φ, $\Phi(u')$ is a result of substitution of u' for u in Φ and Ψ is the formula

$$\Phi\to\neg\neg(\mathrm{E}u)_S\{\Phi\ \&\ (u')_S[u'Ru\to\neg\Phi(u')]\}.$$

LEMMA 8. *If $x\in\mathrm{Cond}_M$, $\bar i=(i_1,\ldots,i_k)$ is a sequence of integers and $\bar s=(s_1,\ldots,s_l)$ is a sequence of elements of M and m is an integer then $x\Vdash_M\Psi[(x)_2^{(m)},\bar i,\bar s]$.*

Proof. Since $\bar i,\bar s$ do not change throughout the proof, we shall not write them at all. Let x' be an extension of x which forces $\Phi[(x)_2^{(m)}]$. We have to show that whenever $y>_M x'$, there is a c in Cond_M such that $c>_M y$ and $c\Vdash_M(\mathrm{E}u)_S\{\Phi\ \&\ (u')_S[u'Ru\to\neg\Phi(u')]\}$. Thus let $y>_M x'$ and let e be a function in M (i.e. a set of ordered pairs) which enumerates (possibly with repetitions) the set consisting of m and of all integers which precede m in the ordering $(x)_1$. We shall first determine a condition $c>_M y$ such that $c\|_M\Phi[(x)_2^{(e(n))}]$ for each n. Put in Lemma 1 Cond_M for C, $u<\bar u$ for $B(u,\bar u)$ and $\bar u\|\Phi\big((x)_2^{(e(n))}\big)$ for $D(n,\bar u)$. The antecedent of the formula in Lemma 1 is satisfied in M if we interpret w as y (cf. Lemma 4(b)). It follows that there exists a z in S_M such that $z^{(0)}=y$, $z^{(n)}\|_M\Phi[(x)_2^{(e(n))}]$ and $z^{(n)}<_M z^{(n+1)}$ for each n. Using Lemma 3(d) we arrive at a condition c which extends each $z^{(n)}$. Hence c is the required condition. Let p be the earliest integer in the ordering $(c)_1$ such that $c\Vdash_M\Phi[(c)_2^{(p)}]$. Such a p exists because x' and hence c forces $\Phi[(x)_2^{(m)}]$ and $(x)_2^{(m)}$ can be represented as $(c)_2^{(p)}$ for an integer p because $c>_M x$. Also notice that if q precedes p in the ordering $(c)_1$, then $(c)_2^{(q)}$ has the form $(x)_2^{(n)}$ where n precedes m in the ordering $(x)_1$. This is so because the ordering $(x)_1$ is similar to a segment of $(c)_1$ and $(x)_2^{(n)}=(c)_2^{(q)}$ whenever n and q correspond to each other under the similarity mapping.

The lemma will be proved if we show that c forces $\Phi[(c)_2^{(p)}]$ and c forces $(u')_S[u'R(c)_2^{(p)}\to\neg\Phi(u')]$. The first formula is evident because of the definition of c. Now select an arbitrary set s in M and assume that $t>_M c$ and t forces $sR(c)_2^{(p)}$. Since $t>_M c$ there is in M a map f of integers into integers such that the ordering $(c)_1$ is mapped onto a segment of $(t)_1$ and $(c)_2^{(k)}=(t)_2^{(f(k))}$ for each k. Since $t\Vdash_M sR(c)_2^{(p)}$ there are integers i,j such that i precedes j in the ordering $(t)_1$ and $(t)_2^{(i)}=s$, $(t)_2^{(j)}=(c)_2^{(p)}$. Since $(c)_2^{(p)}=(t)_2^{(f(p))}$ it follows $j=f(p)$ because no two sets of the form $(t)_2^{(k)}$ are equal to each other. Thus j and therefore also i belong to the segment onto which f maps the integers and we can put $i=f(q)$ where q precedes p in the ordering $(c)_1$ and obviously $p\neq q$. Since p was the earliest integer (in the ordering $(c)_1$) for which c forces $\Phi[(c)_2^{(p)}]$ it follows

that c non $\Vdash_M\Phi[(c)_2^{(q)}]$. As we remarked above $(c)_2^{(q)}$ has the form $(x)_2^{(n)}$ where n precedes m in the ordering $(x)_1$. Hence c decides $\Phi[(c)_2^{(q)}]$ and therefore $c\Vdash_M\neg\Phi[(c)_2^{(q)}]$. Since $s=(t)_2^{(f(q))}=(c)_2^{(q)}$ we infer that $c\Vdash_M\neg\Phi[s]$ and $t\Vdash_M\neg\Phi[s]$. The lemma is thus proved.

COROLLARY 9. *For each formula Φ in which the variable u is free, the formula*

$$(u)_S\big(\Phi\to(\mathrm{E}u)_S\{\Phi\ \&\ (u')_S[u'Ru\to\neg\Phi(u')]\}\big)$$

is valid in $(M,\precsim)$.

Proof. Let $\bar i,\bar s$ be any valuation of the free variables of Φ which are different from u. It is sufficient to show that any condition y forces $(u)_S\neg\neg\Psi[\bar i,\bar s]$ where Ψ is defined as in Lemma 8.

Because of the double negation after the quantifier this assertion is equivalent to the following: for every set t in M and every $x'>_M y$ there is a condition $x>_M x'$ such that $x\Vdash_M\Psi[t,\bar i,\bar s]$. If t and x' are given, then by Lemma 3(c) we can find an $x>_M x'$ and an integer m such that $t=(x)_2^{(m)}$ and the formula to be proved becomes $x\Vdash_M\Psi[(x)_2^{(m)},\bar i,\bar s]$. This is precisely the formula proved in Lemma 8.

LEMMA 10. *All Skolem functions for $(M,\precsim)$ are definable in $(M,\precsim)$.*

Proof. Let Φ be a formula of L' and let u be a free variable of Φ whereas u' is a variable which does not occur in Φ. To say that the Skolem function for Φ is definable amounts to the following: there exists a formula Ψ with the same free variables as Φ such that the formulae:

(i) $\Psi\to\Phi$;

(ii) $\Psi(u)\ \&\ \Psi(u')\to(u=u')$;

(iii) $\Phi\to(\mathrm{E}u)\Psi$

are valid in $(M,\precsim)$.

Let us denote by $<_a$ the usual arithmetical inequality. We define Ψ as $\Psi_1\vee\Psi_2$ where

$$\Psi_1\colon\ [(\mathrm{E}u)_N\Phi]\ \&\ N(u)\ \&\ \Phi\ \&\ (u')_N[u'<_a u\to\neg\Phi(u')],$$

$$\Psi_2\colon\ [\neg(\mathrm{E}u)_N\Phi]\ \&\ S(u)\ \&\ \Phi\ \&\ (u')_S[u'Ru\to\neg\Phi(u')].$$

The validity of (i) is obvious. To prove (iii) we argue as follows: Let $\bar i,\bar s$ be a valuation of the free variables of Φ. Either there exists an integer n such that $(M,\precsim)\models\Phi[n,\bar i,\bar s]$ or not. In the former case there is a smallest such integer n_0 and therefore $(M,\precsim)\models\Psi_1[n_0,\bar i,\bar s]$; in the latter case either no set s in M satisfies $\Phi[s,\bar i,\bar s]$ in $(M,\precsim)$ and (iii) is valid or there is such a set s. If $(M,\precsim)\models\Phi[s,\bar i,\bar s]$ then by Corollary 9 there is a set s_0 which satisfies $(M,\precsim)\models\Psi_2[s_0,\bar i,\bar s]$.

Finally (ii) is proved as follows. If $(M,\precsim)\models\Phi[n,\bar i,\bar s]$ where n is the least integer with this property then n is the unique element such that $(M, \precsim) \models \Psi[n, \bar{i}, \bar{s}]$ and (ii) is obvious. If no integer satisfies the previous condition and $(M, \precsim) \models \Psi[s, \bar{i}, \bar{s}] \,\&\, \Psi[s', \bar{i}, \bar{s}]$ then the formula $(u')[u'Ru \to \neg\Phi(u')] \,\&\, \Phi$ is satisfied in $(M, \precsim)$ by the valuations $(s, \bar{i}, \bar{s})$ and also by $(s', \bar{i}, \bar{s})$ which shows that neither $s \precsim s'$ nor $s' \precsim s$. Thus it follows that $s = s'$.

LEMMA 11. *If $X \subseteq N$ and X is parametrically definable in $(M, \precsim)$ then X belongs to M and hence is parametrically definable in M.*

Proof. Let Φ be a formula and $\bar{p}$ a sequence of parameters such that $n \in X \equiv (M, \precsim) \models \Phi[n, \bar{p}]$. Put

$$Q = \{c \in \mathrm{Cond}_M\colon\ (n)_N c \|_M \Phi[n, \bar{p}]\}.$$

This set is obviously definable (parametrically) in M. We show first that Q is dense. Thus let c be an arbitrary element of Cond_M. By Lemma 3(d) the formula

$$(n)_N(x)_S\{(x > c) \to (\mathrm{E}y)[(y > x) \,\&\, (y\|\Phi(n, \bar{p}))]\}$$

is true in M for the values $c, \bar{p}$ of the free variables. Using Lemmas 1 and 3(d) we infer that there is a condition x in Cond_M such that $x >_M c$ and $x\|_M \Phi[n, \bar{p}]$ for each integer n. This establishes the density of Q.

Now select a g in $Q \cap G$. If n is in X, then there is a g_n in G such that $g_n \Vdash_M \Phi[n, \bar{p}]$. Since g and g_n have a common extension in G and g decides $\Phi[n, \bar{p}]$ we infer $g \Vdash_M \Phi[n, \bar{p}]$. Similarly $g \Vdash_M \neg\Phi[n, \bar{p}]$ if $n \notin X$. Thus $n \in X \equiv g \Vdash_M \Phi[n, \bar{p}]$ i.e. $n \in X \equiv M \models F_\Phi[g, n, \bar{p}]$ for each n which proves that X is parametrically definable in M.

The theorem formulated at the beginning of the paper follows directly from Lemmata 10 and 11.

Appendix

In order to show an application of the theorem proved above we state a result which generalizes a theorem proved in [3]. We omit most proofs because they are the same as in [3].

Let $\mathcal{M} = \langle U_{\mathcal{M}}, N_{\mathcal{M}}, S_{\mathcal{M}}, E_{\mathcal{M}}, A_{\mathcal{M}}, P_{\mathcal{M}}\rangle$ be a model of A_2. The nth element of $N_{\mathcal{M}}$ is denoted by $\nu_n(\mathcal{M})$; these elements are called standard integers of $\mathcal{M}$.

In the case of first order arithmetic the standard integers of any model form themselves a model. Our aim is to show that this is not necessarily the case for A_2.

First we define the standard part $\mathcal{M}^*$ of $\mathcal{M}$. This is a structure $\langle U^*, N^*, S^*, \epsilon, A^*, P^*\rangle$ where $U^* = N^* \cup S^*$, N^* is the set of integers, A^*, P^* are the relations $x+y = z$, $x \cdot y = z$, ϵ is the usual set-theoretic

relation of belonging to a set and S^* is the family of sets $\{n \in N^*\colon \nu_n(\mathcal{M})\, E_{\mathcal{M}} X\}$ where X ranges over $S_{\mathcal{M}}$.

In case when $N^* \subset N_{\mathcal{M}}$, $E_{\mathcal{M}}$ is the ϵ-relation and $S_{\mathcal{M}}$ consists of subsets of $N_{\mathcal{M}}$ we can describe $\mathcal{M}^*$ as the structure obtained from $\mathcal{M}$ by removing all unnatural numbers.

THEOREM. *For each ω-model $\mathcal{M}$ of A_2 there is an elementarily equivalent model whose standard part is not a model of A_2.*

Proof of this theorem will be divided into 4 parts. It will obviously be sufficient to deal with a denumerable model.

I. Definitions. We start with a given ω-model $\mathcal{M}$ and denote by

$$\mathcal{M}_1 = \langle \omega\cdot 2, \omega, \omega\cdot 2 - \omega, \varepsilon_1, A_1, P_1\rangle$$

its isomorphic image obtained by a one-to-one mapping of $N_{\mathcal{M}}$ onto ω and of $S_{\mathcal{M}}$ onto $\omega\cdot 2 - \omega$. We can arrange the mapping so that the image of $\nu_n(\mathcal{M})$ be n. Thus $\nu_n(\mathcal{M}_1) = n$.

By the theorem proved in the main body of the paper there is a relational system $\mathfrak{L}_1 = (\mathcal{M}_1, \leqslant_1)$ which is an S-structure for $\mathcal{M}_1$ and the predicate $N(\cdot)$. We shall use the letter $\mathfrak{L}$ with indices to denote relational structures of the same type as $\mathfrak{L}_1$ while the letter $\mathcal{M}$ with the same indices will denote the reduct of $\mathfrak{L}$ to the language of A_2.

A set $b \subset \omega$ will be called *definable* if there exists a formula Φ (of the language appropriate to $\mathfrak{L}_1$) in which at least one variable, e.g. x, is free and a sequence of parameters $\varphi\colon \mathrm{Fr}(\Phi) - \{x\} \to \omega\cdot 2$ such that the equivalence $n \in b \equiv \mathfrak{L}_1 \models \Phi[\{\langle x, n\rangle\} \cup \varphi]$ holds for any integer n. $\mathrm{Fr}(\Phi)$ denotes here the set of free variables of Φ; we shall identify variables with their Gödel numbers.

The family of definable subsets of ω will be denoted by B.

A sequence $s\colon \omega \to \omega\cdot 2 - \omega$ will be called *definable* if there exists a formula Ω with at least two free variables x, y and a sequence of parameters $\varphi\colon \mathrm{Fr}(\Phi) - \{x, y\} \to \omega\cdot 2$ such that the equivalence $(q \varepsilon_1 s_n) \equiv \mathfrak{L}_1 \models \Omega[\{\langle x, n\rangle, \langle y, q\rangle\} \cup \varphi]$ holds for any integers n and q. We define similarly the concept of a definable sequence of integers (i.e. of elements of ω).

1. *If $b \in B$, then there is a β, $\omega \leqslant \beta < \omega\cdot 2$, such that $n \in b \equiv n \varepsilon_1 \beta$ for each n in ω.*

Proof of this lemma results from the fact that $\mathfrak{L}_1$ is an S-structure for $\mathcal{M}_1$ and the predicate $N(\cdot)$ and that the axiom scheme of comprehension is valid in $\mathcal{M}$.

II. Definable reduced powers of $\mathfrak{L}_1$. Since B is a Boolean algebra we can consider its filters. Let F be an ultrafilter of B and let $\sim_F$ be the following relation between definable sequences: $s' \sim_F s'' \equiv \{n\colon s'_n = s''_n\} \in F$.

For a definable s we denote by $\widetilde{s}_F$ the set of all definable sequences s' equivalent to s. Let $\widetilde{U}_F$ be the family of all the sets $\widetilde{s}_F$ with s ranging over definable sequences. The definable reduced power of $\mathfrak{L}_1$ will be the relational structure $\mathfrak{L}_2(F)$ with the universe $\widetilde{U}_F$ and with the interpretations of the predicates defined as in the ordinary reduced powers (see [6]). E.g. S is interpreted as the family of those $\widetilde{s}_F$ in $\widetilde{U}_F$ for which $\{n: s_n \in \omega\cdot 2-\omega\} \in F$ and E as the relation $\varepsilon_2 = \{\langle \widetilde{s}_F, \widetilde{u}_F\rangle \in \widetilde{U}_F \times \widetilde{U}_F: \{n: s_n \varepsilon_1 u_n\} \in F\}$.

$\mathfrak{L}_1$ and $\mathfrak{L}_2(F)$ are elementarily equivalent because the lemma of Łoś is valid for $\mathfrak{L}_2(F)$; in the proof of this lemma we use the fact that all the Skolem functions of $\mathfrak{L}_1$ are definable in $\mathfrak{L}_1$.

Definition. Let $\mathcal{M}_3(F)$ be the standard part of $\mathcal{M}_2(F)$.

2. *A set X belongs to $S_{\mathcal{M}_3(F)}$ if and only if there is a b in B such that* $X = \{j \in \omega: \{i \in \omega: J(i,j) \in b\} \in F\}$.

Proof. Let X be the set of the form indicated. By 1 we can replace $J(i,j) \in b$ by $J(i,j)\varepsilon_1\beta$ where $\omega \leqslant \beta < \omega\cdot 2$. Notice now that if Θ is the formula $(v)_N[E(v,y) \equiv E(J(x,v),w)]$ then the formula $(w)_S(x)_N(\mathrm{E}!\,y)_S\,\Theta(x,y,w)$ is provable in A_2 in view of the axioms of extensionality and of comprehension. It follows that for each n in ω there is exactly one s_n such that $\omega \leqslant s_n < \omega\cdot 2$ and $\mathfrak{L}_1 \models \Theta[n, s_n, \beta]$. Hence $q\varepsilon_1 s_n \equiv J(n,q)\varepsilon_1\beta$ which proves that s is a definable sequence and hence $\widetilde{s}_F \in \widetilde{U}_F$. Let $\sigma^{(q)}$ be a sequence all of whose terms are equal to q. We easily verify that $(\widetilde{\sigma}^{(q)})_F\varepsilon_2\widetilde{s}_F \equiv q \in X$. Since $(\widetilde{\sigma}^{(q)})_F = \nu_q(\mathcal{M}_2(F))$, this proves that X is in the standard part of $\mathcal{M}_2(F)$.

Let now X be a set which belongs to (the universe of) the standard part of $\mathcal{M}_2(F)$. Hence $q \in X \equiv \nu_q(\mathcal{M}_2(F))\varepsilon_2\widetilde{s}_F \equiv (\widetilde{\sigma}^{(q)})_F\varepsilon_2\widetilde{s}_F \equiv \{n: q\varepsilon_1 s_n\} \in F$ where s is a definable sequence. Let Ω and a sequence φ of parameters define s and let Φ be the formula $\Omega(Kz, Lz, \ldots)$ where K, L are functions inverse to the pairing function J. The formula Φ and the sequence of parameters φ define a set b in B and $J(n,q) \in b \equiv q\varepsilon_1 s_n$. It follows $q \in X \equiv \{n: J(n,q) \in b\} \in F$.

Lemma 2 is thus proved.

In the sequel we denote the set $\{j \in \omega: \{i \in \omega: J(i,j) \in b\} \in F\}$ by $R_F(b)$.

III. Codes of definable sets. We put $|x| = 2x+1$ for x in ω and $|x| = 2(x-\omega)+2$ for $\omega \leqslant x < \omega\cdot 2$. If φ is a finite function $A \to \omega\cdot 2$ with domain $A \subseteq \omega$, then we call the product $\prod_{i\in A} p_i^{|\varphi(i)|}$ the code of φ; p_i is of course the ith prime.

Let C be the (primitive recursive) set of integers $m = J(m_1, m_2)$ such that m_1 is the Gödel number of a formula Φ (to be denoted in the sequel by Φ_m) of the language of $\mathfrak{L}_1$ with $x \in \mathrm{Fr}(\Phi_m)$ and m_2 is the code of a sequence φ_m: $\mathrm{Fr}(\Phi_m) - \{x\} \to \omega\cdot 2$. Elements of C are called *codes of definable sets*. Which set is coded by m cannot be read of f from m alone. In order to define this set we use the set

$$\mathrm{Stsf} = \{J(n,m): m \in C \,\&\, \mathfrak{L}_1 \models \Phi_m[\{\langle x, n\rangle\} \cup \varphi_m]\}.$$

The set coded by m is thus $\{n: J(n,m) \in \mathrm{Stsf}\}$. The least m which is a code of a set b we call a distinguished code of b and denote by C^* the set of distinguished codes. Each b in B has exactly one distinguished code and C^* is arithmetical in Stsf. Henceforth we abbreviate "arithmetical in Stsf" by $\mathcal{A}$.

We shall say that a filter $F_0 \subseteq B$ or a base X_0 of such a filter is an $\mathcal{A}$-filter or an $\mathcal{A}$-base if the distinguished codes of the elements of F_0 (or of X_0) form an $\mathcal{A}$-set.

Let X_0 be an $\mathcal{A}$-base of a filter $F_0 \subseteq B$. The distinguished codes of the elements of F_0 can be enumerated by an $\mathcal{A}$-function. B being denumerable we obviously can extend F_0 to an ultrafilter in an effective way and if we use in the proof an enumeration of B according to the ordering of the distinguished codes of its elements we can convince ourselves that F_0 can be extended to an $\mathcal{A}$-ultrafilter F.

From the definition of $R_F(b)$ we immediately see that if F is an $\mathcal{A}$-filter, then $R_F(b)$ is an $\mathcal{A}$-set. From these remarks we infer.

3. *Let $X_0 \subseteq B$ be an $\mathcal{A}$-base of a filter $F_0 \subseteq B$ and let $b_0 \in B$ be such that for every ultrafilter $F \supseteq F_0$ the set* Stsf *is arithmetical in $R_F(b_0)$. Under these assumptions there is an ultrafilter $F \supseteq F_0$ such that all the sets $R_F(b)$, b in B, are arithmetical in $R_F(b_0)$ and the structure $\mathcal{M}_3(F)$ is not a model of A_2.*

Proof. Take an $\mathcal{A}$-ultrafilter $F \supseteq F_0$. By assumption Stsf is arithmetical in $R_F(b_0)$ and hence so are all the sets $R_F(b)$ for b in B. Since no ω-model of A_2 has the property that all of its sets are arithmetical in one selected set of the model, it follows that $\mathcal{M}_3(F)$ is not a model of A_2.

IV. Construction of F_0 and b_0. In the final part of the proof we construct F_0 and b_0 satisfying the assumptions of 3. We consider a full binary tree consisting of finite (possibly void) sequence $e = \langle e(0), \ldots, e(n-1)\rangle$ of zeros and ones. For each e of the length n we put $e' = \sum_{j<n} 2^j e(j)$; this is an integer and $0 \leqslant e' < 2^n$. The immediate successors of e are sequences $e * \langle i\rangle$ where $i = 0, 1$ and $*$ denotes concatenation. Let $\{e_n\}_{n\in\omega}$ be the following branch in the tree: $e_0 = 0$ (the void sequence), $e_{n+1} = e_n * \langle \varepsilon_n\rangle$ where $\varepsilon_n = 0$ if $n \notin \mathrm{Stsf}$, $\varepsilon_n = 1$ if $n \in \mathrm{Stsf}$.

We represent each integer $m > 0$ in the form $2^{\overline{m}} + a'_m$ where a_m is a zero-one sequence of length $\overline{m}$.

For $n > 0$ let $D_n = \{x \in \omega: x \equiv e'_n \pmod{2^n}\}$. Each D_n is obviously non-void, definable and, since $e'_{n-1} \equiv e'_n \pmod{2^{n-1}}$ for $n > 0$, it is contained in D_{n-1}. Hence the family $X_0 = \{D_1, D_2, \ldots\}$ is a basis of a filter F_0.

It is easy to determine the distinguished codes of D_n and to prove that they form an $\mathcal{A}$-set. Hence X_0 is an $\mathcal{A}$-base of F_0.

Let $b_0 = \{J(m, n): (m > 0) \,\&\, m \equiv a'_n \ (\mathrm{mod}\ 2^{\overline{n}})\}$. Since b_0 is primitive recursive, it is definable. It remains to show that whenever F is an ultrafilter and $F \supseteq X_0$, then Stsf is arithmetical in $R_F(b_0)$.

Thus assume that each D_n belongs to F. Since

$$q \in R_F(b_0) \equiv \{i > 0 : i \equiv a'_q \ (\mathrm{mod}\ 2^{\bar{q}})\} \in F$$

we obtain taking $q = 2^n + e'_n$

$$2^n + e'_n \in R_F(b_0) \equiv D_n \in F$$

whence $2^n + e'_n \in R_F(b_0)$. We now show that $2^n + e'_n$ is a unique element n of $R_F(b_0)$ such that $2^n \leqslant m < 2^{n+1}$. To see this we notice that if $m \in R_F(b_0)$ and $2^n \leqslant m < 2^{n+1}$, then $\overline{m} = n$, $a'_m < 2^n$ and

$$\{i > 0 : \ i \equiv a'_m \ (\mathrm{mod}\ 2^n)\} \in F.$$

This set must intersect with D_n since they both belong to F. It follows that $a'_m \equiv e'_n \ (\mathrm{mod}\ 2^n)$ and since both a'_m, e'_n are $< 2^n$ we obtain $a'_m = e'_n$ and $m = 2^n + e'_n$.

The last term ε_{n-1} of e_n where $n > 0$ can therefore be defined as the integral part of $x/2^{n-1}$ where x is a unique integer $< 2^n$ such that $2^n + x \in R_F(b_0)$. Since $n \in \mathrm{Stsf} \equiv \varepsilon_n = 1$ it follows that Stsf is arithmetical in $R_F(b_0)$ and the proof is finished.

In [3] the theorem was proved only for models which are elementarily equivalent to the principal model. It would be interesting to verify whether it holds for ω-models of the system $\overline{A}_2$ resulting from A_2 by omitting the choice axiom.

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The restricted cancellation law in a Noether lattice

by

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In [2], R. P. Dilworth defined the concept of a Noether lattice. The definition is based on the idea of a principal element. $A \in L$ is a *principal element* if for all $B, C \in L$, $\big(B \wedge (C: A)\big)A = BA \wedge C$ and $(B \vee CA): A = B: A \vee C$, thus a principal element is a generalization of the idea of a principal ideal in a Noetherian ring. The ramifications of this concept have been investigated in [2], [5], [6], and [7].

In [3], R. Gilmer considered the restricted cancellation law (RCL) in commutative rings. An element A of a Noether lattice satisfies RCL if for any $B, C \in L$, $AB = AC \neq 0$ implies $B = C$. We show this condition is closely related to the idea of a weak join principal element. $A \in L$ is weak join principal if $BA: A = B \vee 0: A$.

In section 1, we consider a theorem of Gilmer [3] in which he characterizes a commutative ring in which every ideal satisfies RCL. In a Noether lattice L, we show a similar result holds when RCL is assumed on the prime elements of L. Such lattices are characterized as Dedekind or local with maximal M in which either $M^2 = 0$ or M is principal with $M^k = 0$ for some k.

In section 2, the situation in which (L, M) is a local Noether lattice with maximal M such that M satisfies RCL is investigated. With the aid of the lattice RL_n introduced by Bogart [1], these lattices are characterized. In addition, we show the maximal element M in (L, M) is join principal.

Finally, we consider a local Noether lattice in which the maximal is weak join principal. We investigate the distributive case in which the maximal has a minimal representation as the join of two principals.

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Section 1. In this section we will characterize Noether lattices in which every prime element satisfies the restricted cancellation law.

LEMMA 1.11. *If A satisfies RCL and $AB \leqslant AC \neq 0$ for some $B, C \in L$, then $B \leqslant C$.*