onto a regular $T_1$-space having a $λ$-base. Both of these theorems may be proved from a unified point of view which encompasses certain non first countable situations. This is carried out in [14]. Here it seems preferable to give a direct proof with appropriate references to [13] rather than use the general mapping lemma of [14].

References


Models of second order arithmetic with definable Skolem functions

by

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Let $A_2$ be the axiomatic system of second order arithmetic as described in [2].

In the study of the problem whether the standard part of a model of $A_2$ is itself a model of $A_2$, we introduce the following model theoretic concept: Let $A$ be a structure of type $σ$ and $P$ a singular predicate of $σ$. Let $B$ be another structure of type $σ'$ such that $A$ is a reduct of $B$. We say that $B$ is an $S$-structure for $A$ and $P$ if

1° all the Skolem functions of $B$ are definable in $B$;

2° each subset of $P^A$ (the interpretation of $P$ in $A$) which is parametrically definable in $B$ is also definable in $A$. (See [3].)

Using Lévy's model for $A_2$ (see e.g. [4], pp. 241–247) we can easily exhibit an $ω$-model $A$ in which all the axioms of $A_2$, with the exception of the axiom of choice, are valid such that no $S$-structure exists for $A$ and the predicate $N(·)$. For $ω$-models of the full system $A_2$, the situation is different: we shall prove the following

THEOREM. If $M$ is a denumerable $ω$-model for $A_2$, then there exists an $S$-structure for $M$ and the predicate $N(·)$.

Proof of this theorem will occupy the rest of this paper. We shall use a very primitive form of the forcing argument. Our proof was influenced mainly by the result of Felgner [1].

LEMMA 1. The following scheme is provable in $A_2$ (cf. (iii) below for the meaning of $δ^0$):

$\begin{align*}
S(\omega, ϵ, n, m, g, ϵ) & \rightarrow (\sigma)_{σ(0)} \rightarrow (\sigma)_{σ(1)} \rightarrow (\sigma)_{σ(2)} \rightarrow (\sigma)_{σ(3)} \rightarrow \cdots \\
& \rightarrow (\sigma)_{σ(n)} \rightarrow (\sigma)_{σ(n+1)} \rightarrow (\sigma)_{σ(n+2)}.
\end{align*}$

Read $\sigma$ as a vertex of $G(σ)$ and $σ$ as an $n$th extension of $σ$ for $B(σ, ϵ) \& D(σ, ϵ)$.

The scheme can then be expressed as follows. If for every integer $n$, every vertex has an $n$th extension which is also a vertex then for every
vertex \( v \) there is a \( z \) such that \( z^{(0)} = w, z^{(1)}, \ldots, z^{(n)} \) is an infinite sequence of vertices and \( z^{(n+1)} \) is an \( n \)th extension of \( z^{(n)} \). In this formulation the scheme becomes obvious and one sees immediately how to prove it with the help of the axiom of choice. Scrutinizing the proof we convince ourselves that it can be repeated in the formal system \( A_4 \).

Let \( L \) be the language of \( A_4 \) as described in [2]. We assume that the logical constants of \( L \) are \( \& \), \( \lor \), \( 
abla \), \( \forall \), \( \Rightarrow \), \( \models \) and the existential and general quantifiers. In the formulae of \( L \) we shall admit only limited quantifiers \((E_n)_{\forall} \) and \((E_n)_{\exists} \); “there is an integer” and “there is a set” and limited general quantifiers: \((a)_{\forall} \), \((a)_{\exists} \). If the formula \((E_n)_x A(x, y, \ldots, t)\) is provable in \( A_2 \), then we shall allow a new term \( a(y, \ldots, t) \) and an axiom \( A(x, a(y, \ldots, t), y, \ldots, t) \). The system \( A_4 \) enriched by the new terms and axioms is an inessential extension of \( A_2 \). We shall treat formulae containing the defined terms as abbreviations of formulae of the language \( L \).

An \( \omega \)-model \( M \) of \( A_4 \) will always be identified with the family of its sets. If \( \Phi \) is a formula with, say, \( 3 \) free variables, then we denote by \( \Phi^\omega \) a ternary relation which holds between 3 elements of the model (integers or sets) if they satisfy \( \Phi \) in \( M \). An \( \omega \) varying relation \( R \) is parametrically definable in \( M \) if there is a formula \( \Phi \) with \( k \geq n \) free variables and elements \( a_0, \ldots, a_n \) of the model such that for arbitrary \( a_0, \ldots, a_n \) the relation \( R \) holds between \( a_0, \ldots, a_n \) if and only if they belong to the model and \( M \models \Phi(a_0, \ldots, a_n, a_{n+1}, \ldots, a_k) \). No other kind of definability will be involved in our discussion, we shall often omit the word “parametrically”.

We now introduce a series of abbreviations:

(i) **Pairs of integers.** A pair of integers \( m, n \) is defined as \( J(m, n) = \frac{1}{2}(m + n)(m + n + 1) + m \).

(ii) **Pairs of sets** (cf. [5]). A pair of sets \( x, y \subseteq N \) is defined as the set \( x \cup y = (2a + 1) \cup (2b : a \in x \cap y \cup (2b : b \in y) \cup x \cap y \cup y \cup b \)

(iii) **Coding of infinite sequences of sets.** For any integer \( n \) and set \( x \) we put \( x^{(n)} = \bigcup_{j \geq 1} J(n, m) \in x \). Instead of \( \exists y \models (x)_{\exists} \) we write \( y \in x \).

(iv) **Relations, domains and ranges.** We write \( \models J(m, n) \in x \); this formula is read: \( m \) holds the relation \( n \) to \( x \). Thus every set of integers can be conceived as a binary relation. We define \( \text{Dom}(x) \) (domain of \( x \)) and \( \text{Rng}(x) \) (range of \( x \)) in the usual way.

(v) **One-one mappings; isomorphism.** We shall abbreviate as \( \text{Fn}(f) \) a formula which says that \( f \) maps a set of integers onto a set of integers and is one-to-one and as \( x \equiv \gamma \) the formula \( \text{Fn}(f) \land \gamma \land (\forall u)(u \in f \gamma (u) \in x \land (u, f(u))) \); this formula says that \( f \) establishes an isomorphism between \( x \) restricted to the domain of \( f \) with \( x \) restricted to the range of \( f \).

(vi) **Well ordering.** Formulae \( \text{B} \) and \( \text{B}^\omega \). We abbreviate the formula “\( x \) is an irreflexive well-ordering of \( N \)” by \( \text{B}(x) \). Moreover we define \( \text{B}^\omega(x, y, n) : \text{Fn}(f) \land (\forall u)(\text{Fn}(f) \land (\forall v)(v \in \text{Dom}(f) \land (u, v) \in f)) \).

These formulae say that the range (or domain) of \( f \) coincides with the set of \( x \)-predecessors of \( n \).

**Lemma 2.** The following formula is provable in \( A_4 \):

\[ \text{Bord}(x) \land \text{Bord}(y) \rightarrow \text{B}(f) \models ((x \rightarrow y) \land (A \lor B \lor C)) \]

where \( A, B, C \) are formulae: \( \text{Bg}(f) = \text{Dom}(f) = N, \{\text{Dom}(f) = N\} \), \( \text{Bg}(f) = \{\text{Dom}(f) = N\} \land \text{Bg}(f) = N, \text{Bg}(f) = N \land \text{Bg}(f) = N \).

The formula says of course that if \( x, y \) are two well-orderings of \( N \), then they are similar, or one is isomorphic to an initial segment of the other. In all cases the isomorphism is determined uniquely. A formal proof of the formula is essentially the same as the one given in elementary set theory.

(vii) **Partial ordering of conditions.** Let \( x \prec y \) be the formula:

\[ \text{Cond}(x) \land \text{Cond}(y) \land (\forall f)(\varphi_{<}(f)(x) \rightarrow ((x < f) \lor (x = f))) \quad (\forall x) \]

Intuitively speaking a **condition** is a sequence \( (x)_{\forall}, (x)_{\exists}, (x)_{\exists} \), well ordered by the relation: \( (x)_{\exists} \) precedes \( (x)_{\forall} \) if \( x \notin f \in x \).

This means intuitively that the well-ordering \( (x)_{\forall} \) is similar to an initial segment of \( (x)_{\forall} \), and that \( (x)_{\forall} \) is an isomorphism of \( N \) into \( N \) which establishes the isomorphism of \( x \) and a segment of \( y \).

**Lemma 3.** The following formulae are provable in \( A_4 \):

(a) \( x < y \) \( x \prec y \);

(b) \( \text{Cond}(x) \rightarrow \neg (x < x) \);

(c) \( \text{Cond}(x) \land \text{S}(x) \land (\forall x)(\text{S}(y) \rightarrow (x < y) \land (x = y)) \)

(d) \( \text{S}(x) \land (\forall x)(x < x) \rightarrow (\forall f)(\varphi_{<}(f)(x < y) \land (x = y)) \).

(a) and (b) are obvious; (c) is proved by taking \( y \) so that the order type of \( y \) be \( a + 1 \) where \( a \) is the order type of \( x \) and that under the
ordering \((y)\), the last term of the sequence \((y^n)\), \(n = 0, 1, \ldots\) be equal to \(x\) whereas the previous terms be equal to consecutive terms of \((x)\). Finally (d) is proved as follows. From \(a^{0} < a^{d+1}\) we infer that there is a similarity mapping \(\epsilon_n\) of \((N, (a^{0}))(x)\) onto a segment \(O_{\epsilon_n}\) of \((N, (a^{0}))(x)\) such that \(a^{d+2(n+1)}(x) = a^{d+2(n+1)}(x)\) for each \(n\). Let \(N/N_0 \cup N_0 \cup \ldots\) be a decomposition of \(N\) into sets equipotent with \(N_0\) and let \(f_n\) be a one-to-one mapping of \(N_0\) onto \(N_0\). We take for \(y\) a condition \(\delta\) such that \((m, s) \in w\) \(\iff (i, j) \in f_n\) and \(f_n(n)\) and \(s^{(n)}\) \(= (a^{d+2(n+1)}(x))\) where \(m \in N_0\) and \(n \in N_0\).

(ix) Forcing. We construct a new language \(L'\) which differs from \(L\) only by containing a new binary predicate symbol \(\exists R\). For each formula \(\Phi\) of \(L\) we construct a new formula \(F_\Phi\) which has one more free variable ("the new variable") than \(\Phi\). The definition of \(F_\Phi\) is by induction.

If \(\Phi\) is an atomic formula of \(L\) then \(F_\Phi\) is the formula \(\text{Cond}(x) \& \Phi\) where \(x\) is the first variable not in \(\Phi\).

If \(\Phi\) is the formula \(u \not\in v\), then \(F_\Phi\) is the formula

\[
\text{Cond}(x) \& \text{S}(u) \& \text{S}(v) \& \text{E}((\exists y)(F(y))) \& (u = (x,y)) \& (v = (x,y)')
\]

where \(x\) is the first variable different from the variables \(u, v\).

If \(\Phi\) is \(\neg \Psi\), then \(F_\Phi\) is the formula \((x')_F(\neg u < x') \rightarrow \neg F((x')_F)\) where \(x'\) is the first variable which does not occur in \(F_\Psi\) and is different from \(x\) and \(F(x))_F\), \(F_\Psi\) result from \(F_\Phi\) by substituting \(x'\) for its new variable.

If \(\Phi\) is \(\Psi \lor \Theta\) then \(F_\Phi\) is the formula \((x')_F(\Psi(x') \lor \Theta((x')_F)\) where \(x'\) is the first variable which occurs neither in \(F_\Psi\) nor in \(F_\Phi\) and \(F_\Theta\) respectively, \(F_\Psi\) arise from \(F_\Psi\) by substituting \(x\) for their new variables.

If \(\Phi\) is \(\Psi \rightarrow \Theta\) then \(F_\Phi\) is the formula \((x')_F(\Psi(x') \rightarrow \Theta((x')_F)\) where \(x'\) are different variables which occur neither in \(F_\Psi\) nor in \(F_\Phi\) and \(F_\Theta\). \(F_\Psi\) arise from \(F_\Psi\), \(F_\Theta\) by substitution of \(x\) for their new variables of these formulas.

If \(\Phi\) is \(\Psi \lor \Theta\), then \(F_\Phi\) is \(F_\Psi\) \& \(F_\Theta\) \& \((x')_F(\Psi(x') \lor \Theta((x')_F)\) etc. are defined similarly as above.

If \(\Phi\) is \(\text{E}(\Psi)\), then \(F_\Phi\) is \(\text{E}(\Psi)\) \& \(F_\Psi\).

If \(\Phi\) is \((u)_F\) or \((u)'_F\) then \(F_\Phi\) is \((u)\) \& \(F_\Psi\) or \((u)'_F\) \& \(F_\Psi\).

(x) Decidability. The formula \(F_\Psi \lor \neg F_\Psi\) will be written as \(F_\Psi\).

The use of symbols \(F_\Psi\), \(F_\Theta\) is often cumbersome and we shall in most places write them as \(x \not\in \Phi\) or \(x = \Phi\). Instead of \(M \models F_\Psi(e_1, e_2, \ldots, e_n)\) or \(M \models \text{Dec}(e_1, e_2, \ldots, e_n)\) we shall then write \(\text{E}(\Phi)\) \& \(F_\Psi\) \& \(\text{Dec}(e_1, e_2, \ldots, e_n)\).

**Lemma 4.** If \(\Phi\) is a formula of \(L'\) then the formulae

(a) \((x < y) \& (x \not\in \Phi) \rightarrow (y \not\in \Phi)\),

(b) \(\text{Cond}(x) \rightarrow (x \in \Phi)\),

are provable in \(A_2\). Moreover if \(\Phi\) is a formula of \(L\), then the formula

(c) \(\text{Cond}(x) \rightarrow (x \not\in \Phi) = \Phi\)

is provable in \(A_2\).

Proof of this lemma is immediate.

(xi) Generic sets. Let \(M\) be an \(\omega\)-model of \(A_2\). A set \(\Gamma \subseteq \text{Cond}_M\) is dense in \(\text{Cond}_M\) if for every \(x\) in \(\text{Cond}_M\) there is an \(x'\) in \(\Gamma\) such that \(x < x'\). A set \(\Gamma \subseteq \text{Cond}_M\) is generic for \(M\) if it satisfies the usual three conditions: (1) for every \(x, y, z\) in \(M\) there is a \(z\) in \(M\) such that \(x > x'\) and \(y > y'\); (2) if \(x < z\) and \(y < z\) then \(y < x\); (3) if \(\Delta\) is a subset of \(\text{Cond}_M\) which is dense and definable in \(M\) then \(Q\) and \(G\) intersect with each other.

**Lemma 5.** For every denumerable \(\omega\)-model \(M\) there exist generic sets.

Proof: obvious.

We fix a denumerable \(\omega\)-model \(M\) and a set \(\Delta\) generic for \(M\) and define a relation \(\sim\) as follows: Let \(\Phi\) be the formula \(v \in \text{C}_2\) and let \(v\) be the new variable of \(F_\Phi\). We put for \(x, y, z\) in \(M\):

\[(x \sim y) \iff \text{there is a } z \text{ in } M \text{ such that } M \models F_\Phi[x, y, z].\]

Relation \(\sim\) together with \(M\) determines a model \((M, \sim)\) of the language \(L'\). We are going to prove that this model is the required \(\mathbf{S}\)-structure for \(M\). First we must establish the truth-lemma:

**Lemma 6.** Let \(\Phi\) be a formula of \(L'\) with \(k + 1\) free variables and let \(\bar{v}\) be a \(k\)-tuple of integers and \(\bar{v}\) an \(l\)-tuple of elements of \(M\). Then

\[(\text{E}[[\bar{v}]](M \models F_\Phi[\bar{x}, \bar{v}, \bar{y}]) = (M, \sim) \models \Phi[\bar{v}, \bar{v}]).\]

We omit the -time of this lemma.

**Lemma 7.** \(\sigma\) any of the following formulae

(i) \(S(v) \& \sigma\) \& \((v \not\in \text{C}_2) \rightarrow \neg (v = v)\),

(ii) \(S(v) \& \sigma\) \& \((v = v) \rightarrow \neg (v \not\in \text{C}_2)\),

(iii) \(S(v) \& \sigma\) \& \((v = v) \rightarrow \neg (v \not\in \text{C}_2)\),

\(f\) is a \(M\)-model of \(\Phi[\bar{x}, v, v, \bar{y}]\) for any \(x\) in \(\text{Cond}_M\).

Proof of (i). If \(x'\) is an extension of \(x\) which forces the antecedent then no extension of \(x'\) can force the formula \(u = v\) because otherwise we would infer \(u = v\) which is incompatible with \(x' \not\in u \not\in v\).

(ii) Follows from the definition of forcing.

(iii) Let \(\bar{x}'\) be an extension of \(x\) which forces the antecedent of (iii) and let \(x'\) be an extension of \(x\). Let \(x''\) be an extension of \(x'\) such that both \(u\) and \(v\) stand in relation \(\sigma\) to \(\sigma(x'')(x)\) (see Lemma 5(c)). It follows that
there are integers $m, n$ such that $u = (a''_n)^0$, $v = (a''_m)^0$ and $m \neq n$. Hence either $M \models \{[(u''_n)]^0, m\}$ or $M \models \{[(u''_m)]^0, m\}$ and hence either $a''_xM \models E u$ or $a''_x u M E R u$. Thus we have proved $a''_x M \models \{[(u''_x)]^0, E u R u \}$.

In the next lemma $\Phi$ is a formula of $L'$ with the free variables $u, v = (v_1, \ldots, v_k)$, $w = (w_1, \ldots, w_l)$, $s$ is a variable which does not occur in $\Phi$, $\Phi(u')$ is a result of substitution of $u'$ for $u$ in $\Phi$ and $\Psi$ is the formula $\Phi \rightarrow \neg \{[(E u)_{u'}^0](\Phi \land (w''_{x} u' \rightarrow \neg \Phi(u'))].$

**Lemma 8.** If $x \in Con_M$, $i = (i_1, \ldots, i_k)$ is a sequence of integers and $\bar{a} = (a_1, \ldots, a_l)$ is an element of $M$ and $m$ is an integer then $x \models M [\Psi]^m$.

**Proof.** Since $i, \bar{a}$ do not change throughout the proof, we shall not write them at all. Let $x'$ be an extension of $x$ which forces $\Phi[(a''_m)^0]$. We have to show that whenever $y > x'$, there is $a \in Con_M$ such that $c \succeq y$ and $c \models M [\Psi]^m$. Let $c \models M [\Psi]^m$. Let $\Phi[(a''_m)^0]$ and $\Phi[(a''_m)^0]$ be a function in $M$ (i.e., a set of ordered pairs) which enumerates (possibly with repetitions) the set consisting of $m$ and of all integers which precede $m$ in the ordering $(\bar{a})$. We shall first determine a condition $\Phi[(a''_m)^0]$ for each $c$. Put in Lemma 1 Compl for $G, u < u' \models (\Phi[(a''_m)^0])$ and $\Phi[(a''_m)^0]$ as $\Phi[(a''_m)^0]$ and $\Phi[(a''_m)^0]$ for each $n$. Using Lemma 3(d) we arrive at a condition $c$ which extends each $a''_m$. Hence $c$ is the required condition.

Let $p$ be the earliest integer in the ordering $(\bar{a})$, such that $c \models M [\Phi[(a''_m)^0]]$. Such a $p$ exists because $\bar{a}$ and hence $c$ forces $\Phi[(a''_m)^0]$ and $\Phi[(a''_m)^0]$ are represented as $\Phi[(a''_m)^0]$ for an integer $p$ because $c \succeq x$. Also notice that if $q$ precedes $p$ in the ordering $(\bar{a})$, then $\Phi[(a''_m)^0]$ has the form $\Phi[(a''_m)^0]$ whenever $p$ precedes $m$ in the ordering $(\bar{a})$. This is so because the ordering $(\bar{a})$ is similar to a segment of $(\bar{a})$, and $\Phi[(a''_m)^0]$ whenever $n$ and $q$ correspond to each other under the similarity mapping.

The lemma will be proved if we show that $c$ forces $\Phi[(a''_m)^0]$ and $\Phi[(a''_m)^0]$ is equivalent to $\Phi[(a''_m)^0]$. The former formula is evident because of the definition of $M$. Now select an arbitrary set in $M$ and assume that $i > x$ and $i$ forces $\Phi[(a''_m)^0]$. Since $i \succeq y > x$ there is in $M$ a map $f$ of integers into integers such that the ordering $(\bar{a})$ is mapped onto a segment of $(\bar{a})$ and $\Phi[(a''_m)^0] = \Phi[(a''_m)^0] = \Phi[(a''_m)^0]$. Since $\Phi[(a''_m)^0] = \Phi[(a''_m)^0]$ it follows $f(p)$ because no two sets of the form $\Phi[(a''_m)^0]$ equal to each other. Thus $j$ and therefore also $i$ belong to the segment onto which $f$ maps the integers and we can put $i = f(p)$ wherever $p$ precedes $p$ in the ordering $(\bar{a})$, and obviously $p \neq 0$. Since $p$ was the earliest integer in the ordering $(\bar{a})$, for which $c$ forces $\Phi[(a''_m)^0]$ it follows that $c$ non $\models M [\Phi[(a''_m)^0]]$. As we remarked above $(a''_m)^0$ has the form $[(a''_m)^0]$ where $n$ precedes $m$ in the ordering $(\bar{a})$. Hence $c$ decides $\Phi[(a''_m)^0]$ and therefore $c \models M [\Phi[(a''_m)^0]]$. Since $i > x$ and $(a''_m)^0 = (a''_m)^0$ we infer that $x \models M [\Phi[(a''_m)^0]]$ and $\Phi[(a''_m)^0]$. The lemma is thus proved.

**Corollary 8.** For each formula $\Phi$ in which the variable $u$ is free, the formula

\[ (\forall u)[\Phi \rightarrow (\exists u') (M \models \{[\Phi \land (w''_{x} u' \rightarrow \neg \Phi(u'))] \})] \]

is valid in $(M, <)$.

**Proof.** Let $\bar{a}$ be any valuation of the free variables of $\Phi$ which are different from $u$. It is sufficient to show that any condition $y$ forces $(\forall u)[\Phi \land (w''_{x} u' \rightarrow \neg \Phi(u'))]$, where $\Phi$ is defined as in Lemma 8.

Because of the double negation after the quantifier this assertion is equivalent to the following: for every set $x$ in $M$ and every $x' > x$ there is a condition $x \succeq M x'$ such that $x \models M \Psi(x', \bar{a}, \bar{a})$. If $x' = y$ and $x'$ are given, then by Lemma 3(c) we can find an $x > x'$ such that $x \models M \Psi(x, \bar{a}, \bar{a})$. This is precisely the formula proved in Lemma 8.

**Lemma 10.** All Skolem functions for $(M, <)$ are definable in $(M, <)$.

**Proof.** Let $\Phi$ be a formula of $L'$ and let $u$ be a free variable of $\Phi$ whereas $w$ is a variable which does not occur in $\Phi$. To say that the Skolem function for $\Phi$ is definable amounts to the following: there exists a formula $\Psi'$ with the same free variables as $\Phi$ such that the formulae:

(i) $\Psi' \models \Phi$;
(ii) $\Psi'(u) \land \Psi'(w) \rightarrow (u = w)$;
(iii) $\Phi \rightarrow (E u) \Psi'$

are valid in $(M, <)$.

Let us denote by $<_a$ the usual arithmetical inequality. We define $\Psi'$ as $\Psi' \land \Psi''$, where

\[ \Psi'_{1} = [(E u), \Phi] \land [(u = w) \land \Phi] \land [(w'_{x} u' \rightarrow \neg \Phi(u'))], \]

\[ \Psi'_{2} = [(\neg (E u), \Phi) \land \Psi(u') \land \Phi] \land [(w'_{x} u' \rightarrow \neg \Phi(u'))]. \]

The validity of (i) is obvious. To prove (ii) we argue as follows: Let $\bar{a}$ be a valuation of the free variables of $\Phi$. Either there exists an integer $n$ such that $(M, <) \models \Phi(n, \bar{a}, \bar{a})$ or not. In the former case there is a smallest such integer $n$, and therefore $(M, <) \models \Phi(n, \bar{a}, \bar{a})$; in the latter case either no set $x$ in $M$ satisfies $\Phi(x, \bar{a}, \bar{a})$ in $(M, <)$ and (iii) is valid or there is such a set $x$. If $(M, <) \models \Phi(n, \bar{a}, \bar{a})$ then by Corollary 9 there is a set $\bar{a}$ which satisfies $(M, <) \models \Phi(n, \bar{a}, \bar{a})$. Finally (ii) is proved as follows. If $(M, <) \models \Phi(n, \bar{a}, \bar{a})$ then $\bar{a}$ is the least integer with respect to $\Phi$ and finally this is the unique element such
that \((M, \prec) \models \Psi[n, i, \xi]\) and (ii) is obvious. If no integer satisfies the previous condition and \((M, \prec) \models \Phi[x, z, i, \xi] \& \Psi'[x, z, i, \xi]\) then the formula 
\[\forall \eta^*[\eta \in \mathcal{E} \land \eta \models \Phi]\] 
and \(\Phi\) is satisfied in \((M, \prec)\) by the valuations \((x, i, \xi)\) and also by \((x', i, \xi)\) which shows that neither \(s \prec s'\) nor \(s' \prec s\). Thus it follows that \(s = s'\).

**Lemma 11.** If \(X \subseteq N\) and \(X\) is parmetrical definable in \((M, \prec)\) then \(X\) belongs to \(M\) and hence is parmetrical definable in \(M\).

**Proof.** Let \(\Phi\) be a formula and \(\bar{\eta}\) a sequence of parameters such that \(n \in X = (M, \prec) \models \Phi[n, \bar{\eta}]\). Put

\[Q = \{ a \in \text{Cond}_M : (n, a) \models \Phi[n, \bar{\eta}] \} \]

This set is obviously definable (parametrically) in \(M\). We show first that \(Q\) is dense. Thus let \(c\) be an arbitrary element of \(\text{Cond}_M\). By Lemma 3(d) the formula

\[(n, a) \models \exists x (a > c) \rightarrow [(y > a) \& (y \in \Phi[n, \bar{\eta}])]
\]

is true in \(M\) for the values \(c, \bar{\eta}\) of the free variables. Using Lemmas 1 and 3(d) we infer that there is a condition \(x \in \text{Cond}_M\) such that \(x > a\) and \(\forall b \in \Phi[n, \bar{\eta}]\) for each integer \(n\). This establishes the density of \(Q\).

Now select \(a \in Q \cap N\). If \(n\) is in \(X\), then there is a \(a_n\) in \(Q\) such that \(a_n \models \Phi[n, \bar{\eta}]\). Since \(a_n\) and \(a_n\) have a common extension in \(G\) and \(G\) decides \(\Phi[n, \bar{\eta}]\) we infer \(\Phi[n, \bar{\eta}]\). Similarly \(\Phi[n, \bar{\eta}]\) if \(n \notin X\). Thus \(n \in X = g \models \Phi[n, \bar{\eta}]\) for each \(n\) which proves that \(X\) is parmetrical definable in \(M\).

The theorem formulated at the beginning of the paper follows directly from Lemmata 10 and 11.

**Appendix**

In order to show an application of the theorem proved above we state a result which generalizes a theorem proved in [3]. We omit most proofs because they are the same as in [3].

Let \(\mathcal{M} = (U, x, N_P, E_N, E_\mathcal{M}, A_\mathcal{M}, P_\mathcal{M})\) be a model of \(A_\mathcal{M}\), the \(n\)th element of \(N_P\) is denoted by \(\nu_n(\mathcal{M})\); these elements are called standard integers of \(\mathcal{M}\).

In the case of first order arithmetic the standard integers of any model form themselves a model. Our aim is to show that this is not necessarily the case for \(A_\mathcal{M}\).

First we define the standard part \(\mathcal{M}^*\) of \(\mathcal{M}\). This is a structure \((U^*, N^*, E^*, A_*, P^*)\) where \(U^* = N^* \times S^*\), \(N^*\) is the set of integers, \(A_*\), \(P^*\) are the relations \(x + y = x, x \cdot y = y, e\) is the usual set-theoretic relation of belonging to a set and \(S^*\) is the family of sets \(\{ u \in \mathcal{N} : \nu_u(\mathcal{M}) \in E_\mathcal{M}\}\) where \(E_\mathcal{M}\) ranges over \(E_\mathcal{M}\).

In case when \(\mathcal{N} \subseteq \mathcal{N}\), \(E_\mathcal{M}\) is the \(e\)-relation and \(S^*\) consists of subsets of \(N_P\), we can describe \(\mathcal{M}^*\) as the structure obtained from \(\mathcal{M}\) by removing all unnatural numbers.

**Theorem.** For each \(\omega\)-model \(\mathcal{M}\) of \(A_\mathcal{M}\) there is an elementarily equivalent model whose standard part is not a model of \(A_\mathcal{M}\).

**Proof.** of this theorem will be divided into four parts. It will obviously be sufficient to deal with a denumerable model.

**I. Definitions.** We start with a given \(\omega\)-model \(\mathcal{M}\) and denote by \(\mathcal{M}_\omega = \langle \omega, 2, \omega, \omega \rightarrow \omega, \omega, x_1, x_2, \ldots \rangle\) its isomorphic image obtained by a one-to-one mapping of \(\mathcal{M}\) onto \(\omega\) and of \(S^*\) onto \(\omega \cdot 2\). We can arrange the mapping so that the image of \(\nu_n(\mathcal{M})\) is \(n\). Thus \(\nu_n(\mathcal{M}_\omega) = n\).

By the theorem proved in the main body of the paper there is a relational system \(\mathcal{L}_\omega = \langle \mathcal{L}_\omega, \leq_\omega \rangle\) which is an \(S\)-structure for \(\mathcal{M}_\omega\) and the predicate \(\forall X(\cdot)\). We shall use the letter \(\mathcal{L}\) with indices to denote relational structures of the same type as \(\mathcal{L}_\omega\) while the letter \(\mathcal{M}\) with the same indices will denote the reduct of \(\mathcal{L}\) to the language of \(A_\mathcal{M}\).

A set \(b \subseteq \omega\) will be called definable if there exists a formula \(\Phi\) (of the language appropriate to \(\mathcal{L}_\omega\)) in which at least one variable, e.g. \(x\), is free and a sequence of parameters \(\varphi\) : \(\mathcal{L}(\Phi) = x \rightarrow \omega \cdot 2\) such that the equivalence \(n \in b \iff \mathcal{L}_\omega \models \Phi[\langle \langle x, n \rangle, \varphi \rangle]\) holds for any integer \(n\). \(\mathcal{L}(\Phi)\) denotes here the set of free variables of \(\Phi\); we shall identify variables with their Gödel numbers.

The family of definable subsets of \(\omega\) will be denoted by \(\mathcal{F}\).

A sequence \(s : \omega \rightarrow \omega \rightarrow \omega\) will be called definable if there exists a formula \(\Omega\) with at least two free variables \(x, y\) and a sequence of parameters \(\varphi\) : \(\mathcal{L}(\Phi) = x \rightarrow \omega \cdot 2\) such that the equivalence \(\langle n, \varphi \rangle \models \mathcal{L}_\omega \models \langle \langle x, y \rangle, \varphi \rangle\) holds for any integers \(n\) and \(\varphi\). We define similarly the concept of a definable sequence of integers (i.e. of elements of \(\omega\).

1. If \(b \in \mathcal{F}\), then there is a \(\beta, \alpha \leq \beta < \omega \cdot 2\) such that \(n \in b = n_\beta \beta\) for each \(n\) in \(\omega\).

**Proof.** of this lemma results from the fact that \(\mathcal{L}_\omega\) is an \(S\)-structure for \(\mathcal{M}_\omega\) and the predicate \(\forall X(\cdot)\) and that the axiom scheme of comprehension is valid in \(\mathcal{M}\).

**II. Definable reduced powers of \(\mathcal{L}\).** Since \(B\) is a Boolean algebra we can consider its filters. Let \(\mathcal{F}\) be an ultrafilter of \(B\) and let \(\sim\) be the following relation between definable sequences: \(s' \sim s'' = (n : s'_n = s''_n) \in \mathcal{F}\).
For a definable set \( S \) we denote by \( \bar{S} \) the set of all definable sequences \( x' \) satisfying \( S \). Let \( \bar{U} \) be the family of all the sets \( \bar{S} \) with \( x \) ranging over definable sequences. The definable reduced power of \( \bar{U} \) will be the relational structure \( \bar{U}(\bar{F}) \) with the universe \( \bar{U} \) and with the interpretations of the predicates defined as in the ordinary reduced powers (see \( \bar{F} \)).

For example, \( \bar{S} \) is interpreted as the family of those \( \bar{v} \) in \( \bar{U} \) for which: \( \langle x, \bar{v} \rangle \in \bar{F} \) and \( \bar{F} \) as the relation \( \bar{e}_x \), the interpretation of \( \bar{e}_x \) in \( \bar{U} \times \bar{U} \).

L₁ and \( L_2 \) are elementarily equivalent because the lemma of Loś is valid for \( L_1 \); in the proof of this lemma we use the fact that all the Skolem functions of \( L_1 \) are definable in \( L_2 \).

**Definition.** Let \( M_1(F) \) be the standard part of \( M_1(F) \).

Let \( A \) be a set \( X \) belongs to \( M_1(F) \). Let \( X \) be a set of the form \( E \). By 1 we can replace \( E \) by \( E \). Notice now that if \( E \), then \( E \) is a definable sequence in \( X \) and therefore \( X \in E \). Note that \( \forall \phi \rightarrow (\exists \phi \in E) \) is provable in \( A \) in view of the axiom of extensionality and comprehension. It follows that for each \( \phi \) in \( \bar{U} \) there is exactly one \( \phi \) such that \( \phi \in \bar{U} \) and \( \phi \rightarrow (\exists \phi \in E) \). Hence \( \forall \phi \rightarrow (\exists \phi \in E) \) is provable in \( A \). This proves that \( (\exists \phi \in E) \) is in the standard part of \( M_1(F) \).

Let \( \bar{U} \) be a set which belongs to \( \bar{U}(\bar{F}) \). Hence \( \exists \phi \in E \) and \( \exists \phi \in E \). If \( \forall \phi \rightarrow (\exists \phi \in E) \) is provable in \( A \), then \( \forall \phi \rightarrow (\exists \phi \in E) \) is provable in \( A \). If \( \forall \phi \rightarrow (\exists \phi \in E) \) is provable in \( A \), then \( \forall \phi \rightarrow (\exists \phi \in E) \) is provable in \( A \).

Lemma 2 is thus proved.

In the sequel we denote the set \( \{ (x, 0) \in E \} \) by \( B(x) \).

**III. Codes of definable sets.** We put \( \bar{U} = 2x + 1 \) for \( x \in F \) and \( |x| = 2(x - 1) + 2 \) for \( x > 0 \). If \( \phi \) is a finite function \( A \rightarrow \omega \) with domain \( \omega \), then we call the product \( \prod_{x} \bar{F}(\bar{U}) \) the code of \( \phi \); \( \bar{F} \) is of course the \( i \)th prime.

Let \( B \) be the (primitive recursive) set of integers \( m = J(m_1, m_2) \) such that \( m_1 \) is the Gödel number of a formula \( \phi \) (to be denoted in the sequel by \( \bar{U} \)) of the language of \( L_1 \) with \( x \in F(\bar{U}) \) and \( m_2 \) is the code of a sequence \( \bar{U} = \{ x \} \rightarrow \omega \). Elements of \( C \) are called codes of definable sets. Which set is coded by \( m \) cannot be read of \( f \) from \( m \) alone. In order to define this set we use the set \( \bar{S} \).

\[
\bar{S} = \{ \bar{S}(n, m) \in \bar{U} \mid \exists \bar{U} \}.
\]

The set coded by \( m \) is thus \( \{ n \in \bar{U} \mid \bar{U}(\bar{U}) \} \). The least \( m \) which is a code of a set \( S \) we call a distinguished code of \( S \) and denote by \( C \) the set of distinguished codes. Each \( D \) in \( \bar{U} \) has exactly one distinguished code and \( C \) is arithmetical in \( \bar{S} \). Henceforth we abbreviate "arithmetical in \( \bar{S} \)" by \( A \).

We shall say that a filter \( \bar{U} \subseteq \bar{U} \) or a base \( \bar{U} \) of such a filter is an \( A \)-filter or an \( A \)-base if the distinguished codes of the elements of \( \bar{U} \) (or of \( \bar{U} \)) form an \( A \)-set.

Let \( \bar{U} \) be an \( A \)-base of a filter \( \bar{U} \subseteq \bar{U} \). The distinguished codes of the elements of \( \bar{U} \) can be enumerated by an \( A \)-function. \( B \) being denumerable we obviously can extend \( \bar{U} \) to an ultrafilter in an effective way and if we use in the proof an enumeration of \( \bar{U} \) according to the ordering of the distinguished codes of its elements we can convince ourselves that \( \bar{U} \) can be extended to an \( A \)-ultrafilter.

From the definition of \( \bar{U}(\bar{U}) \) we immediately see that if \( \bar{U} \) is an \( A \)-filter, then \( \bar{U}(\bar{U}) \) is an \( A \)-set. From these we infer.

**IV. Construction of \( \bar{U} \) and \( \bar{U} \).** Let \( \bar{U} \) be an \( A \)-base of a filter \( \bar{U} \subseteq \bar{U} \). Assume that there is an ultrafilter \( \bar{U} \subseteq \bar{U} \) such that all the sets \( \bar{U} \) of \( \bar{U} \), \( \bar{U} \in \bar{U} \), are arithmetical in \( \bar{U} \) and the structure \( M_1(F) \) is not a model of \( A \).

Proof. Take an \( A \)-ultrafilter \( \bar{U} \subseteq \bar{U} \). By assumption \( A \) is arithmetical in \( \bar{U} \). Take any \( \bar{U} \in \bar{U} \). Since no \( \omega \)-model of \( A \) has the property that all of its sets are arithmetical in one selected set of the model, it follows that \( M_1(F) \) is not a model of \( A \).

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It is easy to determine the distinguished codes of $D_a$ and to prove that they form an $A$-set. Hence $X_a$ is an $A$-base of $F_a$.

Let $b = \{ f(m, a); (m > 0) \}$ and $a = a'_0 \pmod{2^n}$. Since $b$ is primitive recursive, it is definable. It remains to show that whenever $F$ is an ultrafilter and $F \supseteq X_a$, then $\text{Staf}$ is arithmetical in $R_F(b)$. Thus assume that each $D_a$ belongs to $F$. Since

\[ q \in R_F(b) \iff (i > 0; i = a'_i \pmod{2^n}) \in F \]

we obtain taking $q = 2^n + a'_n$

\[ 2^n + a'_n \in R_F(b) = D_a \in F \]

whence $2^n + a'_n \in R_F(b)$. We now show that $2^n + a'_n$ is a unique element of $R_F(b)$ such that $2^n + a'_n < 2^{n+1}$. To see this we notice that if $m \in R_F(b)$ and $2^n < m < 2^{n+1}$, then $m = a'_n < 2^n$ and

\[ (i > 0; i = a'_m \pmod{2^n}) \in F \]

This set must intersect $D_a$ since they both belong to $F$. It follows that $a'_m = a'_m \pmod{2^n}$ and since both $a'_n, e'_n$ are $< 2^n$ we obtain $a'_m = e'_m$ and $m = 2^n + a'_n$.

The last term $e_{n-1}$ of $e_n$ where $n > 0$ can therefore be defined as the integral part of $2^{n-1}$ where $x$ is a unique integer $< 2^n$ such that $2^n + x \in R_F(b)$. Since $x \in \text{Staf} = e_n = 1$ it follows that $x$ is arithmetical in $R_F(b)$ and the proof is finished.

In [3] the theorem was proved only for models which are elementarily equivalent to the principal model. It would be interesting to verify whether it holds for $\alpha$-models of the system $\Delta_4$ resulting from $A_3$ by omitting the choice axiom.

References