Then h_0 is the inclusion of $V(W_1, \psi)$ into V_2, and h_1 is the restriction of fg to $V(W_1, \psi)$. This proves that $fg \simeq v$.

4. Proof of Theorem 1. For $i = 1, 2$, let $Z_i = Z_i(X, Y_i)$ be the set $Y_i \times [0, 1] - ((Y_i - X) \times \{0\})$ of § 3. Because similarity is transitive, the theorem will be proved if we can establish the following sequence of similarities:

$$U(X, Y_1) \simeq V(X, Z_1) \simeq U(X, Z_1) \simeq U(X, Z_2) \simeq V(X, Z_2) \simeq U(X, Y_2).$$

The first and last similarities follow from Lemma 3; the second and fourth follow from Lemma 2. The middle one is a consequence of Lemma 1 and the following result, which is based on a method of Fox [1].

LEMMA 4. *If Y is an ANR and $X \subset Y$, then $Z(X, Y)$ is an ANR.*

Proof. Write $Z = Z(X, Y)$. Let A be a closed subset of a metric space B, and suppose $f: A \to Z$ is a map. Because $Y \times [0, 1]$ is an ANR, there exists a neighborhood W of A in B and an extension $g: W \to Y \times \times [0, 1]$ of f. Let $\lambda: W \to [0, 1]$ be a map such that $\lambda^{-1}(0) = A$. Letting $\pi_1: Y \times [0, 1] \to Y$ and $\pi_2: Y \times [0, 1] \to [0, 1]$ denote the coordinate projections, define a map $F: W \to Z$ by

$$F(w) = (\pi_1 g(w), \min\{1, \pi_2 g(w) + \lambda(w)\}) \quad \text{for all } w \in W.$$

Then $F(W)$ is indeed contained in Z, and clearly F extends f; hence Z is an ANR.

We close by reformulating the results of the last two sections in the following manner, which might be of independent interest.

THEOREM 2. *If X is a subset of a metric space Y, then there exists a metric space Z containing X as a closed subset such that the systems $U(X, Y)$ and $U(X, Z)$ are of the same similarity type; if Y is an ANR, we may choose Z to be an ANR.*

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RUTGERS UNIVERSITY, DOUGLASS COLLEGE
New Brunswick, New Jersey

Reçu par la Rédaction le 31. 8. 1970

Topological completeness of first countable Hausdorff spaces I*

by

H. H. Wicke (Athens, Ohio) and **J. M. Worrell, Jr.** (Albuquerque, N. Mex.)

1. Introduction. This paper is the first of several which present a theory of topological completeness for first countable Hausdorff spaces. The completeness concept introduced here, called *basic completeness*, permits the development of a theory analogous in many respect to that elaborated classically for complete metric spaces. In this connection it should be noted that a metrizable space is basically complete if and only if it has a topology-preserving metric in which it is complete (¹). This article presents certain definitions and some set-theoretical and topological lemmas, which are both fundamental for the theory and have wider applicability, and proves certain characterization theorems.

One of the principal results proved here is that *a Hausdorff space is an open continuous image of a complete metric space if and only if it is a basically complete space.* It should be emphasized that regularity is not assumed. This theorem leads to the result that *the class of basically complete spaces is the intersection of all classes $\mathcal{C}$ of Hausdorff spaces such that* 1) $\mathcal{C}$ *includes all metrically topologically complete spaces and* 2) $\mathcal{C}$ *is closed with respect to the application of open continuous mappings with Hausdorff images.*

For the purposes of indicating the scope of the present results and of providing a basis for further discussion in subsequent papers we list here certain criteria for topological completeness. These are formulated in terms of two classes $\mathcal{B}$ and $\mathcal{C}$ of topological spaces. The members of $\mathcal{B}$ are subject to some uniformization condition (²) and $\mathcal{C}$ is a subclass of $\mathcal{B}$

* This work was supported by the United States Atomic Energy Commission. A preliminary version was given as a colloquium lecture at the University of New Mexico in May 1968.

(¹) Such metrizable spaces will be called *metrically topologically complete* in accord with a standard usage.

(²) *Uniformization condition* and *topological uniformization*, as employed here, do not necessarily connote for the spaces to which they are applied the presence of a topology-preserving uniformity in the sense of A. Weil's definition.

the members of which satisfy a completeness requirement related to the uniformization. The class $\mathcal{B}$ contains all metrizable spaces. A classical prototype is the case where $\mathcal{B}$ is the class of metrizable spaces and $\mathcal{C}$ is the class of metrically topologically complete spaces.

Criteria of completeness

1. Closed subspaces of members of $\mathcal{C}$ are in $\mathcal{C}$.

2. Inner limiting subspaces ($\equiv G_\delta$-subspaces) of members of $\mathcal{C}$ are in $\mathcal{C}$.

3. If $(E_n)_{n \in N}$ is a sequence of elements of $\mathcal{C}$ each of which is a dense subspace of a topological space S then $\bigcap \{E_n \colon n \in N\}$ is dense in S.

4. If $(E_n)_{n \in N}$ is a sequence of members of $\mathcal{C}$ which are all subspaces of a certain space S, then $\bigcap \{E_n \colon n \in N\} \in \mathcal{C}$.

5. If $S \in \mathcal{C}$ and there is an open continuous mapping of S onto a Hausdorff space $R \in \mathcal{B}$, then $R \in \mathcal{C}$.

6. If $S \in \mathcal{C}$ and there exists a perfect mapping of S onto $R \in \mathcal{B}$, then $R \in \mathcal{C}$.

7. A metrizable space is in $\mathcal{C}$ if and only if it is metrically topologically complete.

8. If a member of $\mathcal{B}$ is complete in the sense of Čech then it is in $\mathcal{C}$. In particular, locally compact Hausdorff members of $\mathcal{B}$ are in $\mathcal{C}$.

9. Any Hausdorff countably compact member of $\mathcal{B}$ is in $\mathcal{C}$.

10. The Cartesian product space of any countable family of members of $\mathcal{C}$ is in $\mathcal{C}$.

11. Non-degenerate connected and locally connected members of $\mathcal{C}$, which are in some appropriate sense uniformly first countable, are arcwise connected.

12. There is a fundamental set-theoretic characterization of subspaces of members of $\mathcal{C}$ which belong to $\mathcal{C}$.

Let us consider certain familiar situations in the light of these criteria. When $\mathcal{B}$ is the class of metrizable spaces and $\mathcal{C}$ is the class of metrically topologically complete spaces all the criteria are satisfied. When $\mathcal{B}$ is the class of Moore spaces (i.e., regular T_0-spaces having a development in the sense of R. H. Bing [3]) and $\mathcal{C}$ is the class of monotonically complete (³) Moore spaces the criteria are again satisfied. If in the preceding statement $\mathcal{C}$ is taken to be the class of complete Moore spaces (i.e., those regular T_0-spaces having a development satisfying part four of Axiom 1 of [11]) then number 5 is not satisfied. If $\mathcal{B}$ is the class of Tychonoff spaces and $\mathcal{C}$ is the class of spaces complete in the sense of Čech then criteria 5 and 9 are not satisfied. If $\mathcal{B}$ is the same class and $\mathcal{C}$ is the class

(³) i.e., the spaces have a base $\mathcal{B}$ such that the closures of the elements of any monotonic subcollection of $\mathcal{B}$ have a point in common ([15], p. 813).

of spaces in $\mathcal{B}$ that have a topology-preserving complete uniform structure then criteria 2, 3, 5, 7, 9, and 11 are not satisfied.

It will be shown in these articles that: *If $\mathcal{B}$ is the class of Hausdorff spaces having bases of countable order and $\mathcal{C}$ is the class of basically complete spaces then all of the criteria are satisfied.* It may be noted that this is a genuine extension of the various first countable cases mentioned in the preceding paragraph.

Certain analogues of the results stated here have been obtained for some non first countable situations where T_0 regularity has been assumed. Some of these have appeared [14]; others will appear elsewhere.

The notation and terminology used here conforms closely with that of Kelley [9]. The letter N denotes the set of positive integers and letters i, j, k, and n are used to signify members of N. The notation $(U_n)_{n \in N}$ is used for a sequence; this is often abbreviated to (U_n). A collection of sets is said to be *monotonic* if and only if for any two of its members one includes the other. A sequence $(U_n)_{n \in N}$ of sets is called *decreasing* if and only if for each $n \in N$, $U_{n+1} \subset U_n$. A *representative* of a sequence $(\mathcal{A}_n)_{n \in N}$ of sets is, by definition, a sequence (B_n) such that, for each n, $B_n \in \mathcal{A}_n$. A space is said to be *essentially* T_1 [16] if and only if for any points x, y of the space either $\overline{\{x\}} = \overline{\{y\}}$ or $\overline{\{x\}}$ does not intersect $\overline{\{y\}}$ (Davis introduced these spaces under the designation of R_0 spaces in [4]).

2. Set-theoretical and topological preliminaries. The concepts of monotonically contracting sequence and primitive sequence are introduced here and some fundamental relations connecting them are established. These are given in Lemmas 2.1 and 2.3 and are proven in sufficient generality so as to apply to situations outside the scope of this paper.

DEFINITION 2.1. Suppose S is a set and $M \subset S$. A sequence $(\mathcal{G}_n)_{n \in N}$ is called a *monotonically contracting sequence of M in S* if and only if, for each n, $\mathcal{G}_n$ is a collection of subsets of S covering M such that if $G \in \mathcal{G}_n$ and $x \in M \cap G$ there exists $G' \in \mathcal{G}_{n+1}$ such that $x \in G'$ and $G' \subset G$.

DEFINITION 2.2. A sequence $(\mathcal{H}_n)_{n \in N}$ is called a *primitive sequence of M in S* if and only if, for each $n \in N$, these conditions are satisfied: $(\mathrm{P1})_n$ $\mathcal{H}_n$ is a well ordered collection of subsets of S covering M. $(\mathrm{P2})_n$ Each $H \in \mathcal{H}_n$ contains a point not in any predecessor of H in $\mathcal{H}_n$. $(\mathrm{P3})_n$ If $x \in M$, $j < n$, and H and H' are the first elements of $\mathcal{H}_j$ and $\mathcal{H}_n$, respectively, that contain x then $H' \subset H$.

LEMMA 2.1. *Suppose M is a subset of a set S. If $(\mathcal{G}_n)_{n \in N}$ is a monotonically contracting sequence of M in S there exists a primitive sequence $(\mathcal{H}_n)_{n \in N}$ of M in S such that for each $n \in N$: $(1)_n$ $\mathcal{H}_n \subset \mathcal{G}_n$. $(2)_n$ If $j < n$, $x \in M$, and H and H' are the first elements of $\mathcal{H}_j$ and $\mathcal{H}_n$, respectively, that contain x then if x is in a proper subset of H belonging to $\mathcal{G}_n$ then H' is a proper subset of H.*

Proof. Let $<_0$ denote a well ordering of the set M and for each n, let $\mathcal{G}'_n$ denote a well ordered collection whose elements are those of $\mathcal{G}_n$. Let $\mathcal{H}_1$ denote the collection of all elements $H \in \mathcal{G}'_1$ such that H contains a point not in any predecessor of H in $\mathcal{G}'_1$. Suppose collections $\mathcal{H}_1, \ldots, \mathcal{H}_k$ exist such that $(\mathrm{P1})_n$, $(\mathrm{P2})_n$, and $(\mathrm{P3})_n$ of Definition 2.2, $(1)_n$, and $(2)_n$ are satisfied for $1 \leqslant n \leqslant k$. Suppose $<_{k-1}$ is a well ordering on M. For $x, x' \in M$ define $x <_k x'$ if and only if for the first elements H and H' of $\mathcal{H}_k$ that contain x and x' respectively either (a) H precedes H' in $\mathcal{H}_k$, or (b) $H = H'$ and x lies in a proper subset of H belonging to $\mathcal{G}_{k+1}$ and x' does not, or (c) $H = H'$ and either both x and x' lie in proper subsets of H belonging to $\mathcal{G}_{k+1}$ or both do not and $x <_{k-1} x'$. It is straightforward to establish that $<_k$ well orders M. Let M_k denote the set M under the well ordering $<_k$. We shall define a sequence function of type M_k in $\mathcal{G}'_{k+1}$ in order to apply the transfinite recursion theorem ([6], p. 70). For $x \in M_k$ let $s(x)$ denote the initial segment determined by x. Suppose for $x \in M_k$, t is a function on $s(x)$ to $\mathcal{G}'_{k+1}$. If there exists a first $x' <_k x$ such that $x \in t(x')$, let $f(t)$ denote $t(x')$. Suppose no such x' exists and H is the first element of $\mathcal{H}_k$ that contains x. If x is in a proper subset G of H belonging to $\mathcal{G}'_{k+1}$ let $f(t)$ denote the first such G. If x is not in a proper subset of H belonging to $\mathcal{G}'_{k+1}$ let $f(t)$ denote H. By the transfinite recursion theorem there exists a function U on M_k to $\mathcal{G}'_{k+1}$ such that $U(x) = f(U|s(x))$ for all $x \in M_k$.

For every H in the range of U, let $p(H)$ denote the first element of M_k such that $U(p(H)) = H$. Then p is a one-to-one function on the range of U into M_k. This permits the definition of a well ordering $\precsim$ by $H \precsim H'$ if and only if $p(H) <_k p(H')$ for H, H' in the range of U. Call the resulting well ordered set $\mathcal{H}_{k+1}$.

1. *If $x \in M_k$ the first element of $\mathcal{H}_{k+1}$ that contains x is $U(x)$.*

For $x \in U(x)$ since $f(U|s(x)) = U(x)$ contains x. If $H \precsim U(x)$ then $p(H) <_k p(U(x)) \leqslant_k x$ by the definition of p. Therefore if $x \in H$, there is a first $z <_k x$ such that $x \in U(z)$ and by the definition of f, $U(x) = U(z)$. Since $z \leqslant_k p(H)$, $U(z) \lesssim H \precsim U(x)$ which involves a contradiction.

Now $(\mathrm{P1})_{k+1}$ is clearly satisfied and $(\mathrm{P2})_{k+1}$ follows from 1. Furthermore $(1)_{k+1}$ is satisfied. Suppose $x \in M$, and H and H' are the first elements of $\mathcal{H}_k$ and $\mathcal{H}_{k+1}$, respectively, that contain x. Let G be the first element of $\mathcal{H}_k$ that contains $p(H')$. Then since $p(H') \leqslant_k x$, $G \lesssim H$. Since $U(p(H')) = H'$ is a subset of the first element of $\mathcal{H}_k$ that contains $p(H')$, it follows that $x \in G$ and therefore $G = H$. Hence $(\mathrm{P3})_{k+1}$ is satisfied since $(\mathrm{P3})_k$ is. Suppose that $j < k+1$, $x \in M$, and H and H' are the first elements of $\mathcal{H}_j$ and $\mathcal{H}_{k+1}$ respectively that contain x and x is in a proper subset of H belonging to $\mathcal{G}_{k+1}$. If x is in a proper subset of H belonging to $\mathcal{G}_k$ then the first element H'' of $\mathcal{H}_k$ that contains x is a proper subset of H by $(2)_k$. Since $H' \subset H''$ by $(\mathrm{P3})_{k+1}$, $(2)_{k+1}$ is valid. Suppose that x is not in a proper subset of H belonging to $\mathcal{G}_k$. Since $(\mathcal{G}_n)$ is monotonically contracting an inductive argument shows that $H \in \mathcal{G}_k$ and the first element of $\mathcal{H}_k$ containing x is H. Now $p(H') \leqslant_k x$. Since x is in a proper subset of H which belongs to $\mathcal{G}_{k+1}$ and since $p(H') \in H$ it follows from the definition of $<_k$, part (b), that $p(H')$ is in a proper subset of H belonging to $\mathcal{G}_{k+1}$. By the definition of f, $U(p(H')) = H'$ is a proper subset of H. Therefore $(2)_{k+1}$ is valid. By induction, there exists a sequence $(\mathcal{H}_n)_{n \in N}$ as described.

LEMMA 2.2. *Suppose $(\mathcal{H}_n)_{n \in N}$ is a primitive sequence of M in S and $(G_n)_{n \in N}$ is a decreasing sequence of sets such that for each n there exists $y_n \in G_n \cap M$ such that the first element of $\mathcal{H}_n$ that contains y_n includes G_n. Then there exists a decreasing representative $(H_n)_{n \in N}$ of $(\mathcal{H}_n)_{n \in N}$ such that, for each n, H_n is the first element of $\mathcal{H}_n$ that includes a term of $(G_n)_{n \in N}$.*

Proof. For each n there exists a first $H_n \in \mathcal{H}_n$ that includes a term of (G_n). For each n there exists $j > n+1$ such that $G_j \subset H_n \cap H_{n+1}$. Let H denote the first element of $\mathcal{H}_n$ that contains y_j. Since $(\mathcal{H}_n)$ is a primitive sequence, H includes the first element H' of $\mathcal{H}_j$ that contains y_j. Thus $H \supset H' \supset G_j$. Therefore H does not precede H_n. Since $y_j \in H_n$ it follows that $H = H_n$. Similarly H_{n+1} is the first element of $\mathcal{H}_{n+1}$ that contains y_j. Therefore $H_n \supset H_{n+1}$.

LEMMA 2.3. *Suppose M is a subset of a set S and $\mathcal{A}$ is a collection of subsets of S such that if $A, B \in \mathcal{A}$ and $A \cap B \neq \emptyset$, then $A \cap B \in \mathcal{A}$. Suppose $(\mathcal{H}_n)_{n \in N}$ is a primitive sequence of M in S whose terms are subcollections of $\mathcal{A}$. Then there exists a decreasing, monotonically contracting sequence $(\mathcal{G}_n)_{n \in N}$ of M in S whose terms are subcollections of $\mathcal{A}$ such that for every decreasing representative $(G_n)_{n \in N}$ of $(\mathcal{G}_n)_{n \in N}$ there exists a decreasing representative $(H_n)_{n \in N}$ of $(\mathcal{H}_n)_{n \in N}$ such that for each n, H_n is the first element of $\mathcal{H}_n$ that includes a term of $(G_n)_{n \in N}$.*

Proof. Define $\mathcal{G}'_1$ as $\mathcal{H}_1$, and for $n > 1$ let $\mathcal{G}'_n$ denote the collection of all sets $G \cap G'$ where $G \in \mathcal{H}_n$, $G' \in \mathcal{G}'_{n-1}$, and G' contains a point $x \in G \cap M$ not in any predecessor of G in $\mathcal{H}_n$. For each n, let $\mathcal{G}_n = \bigcup \{\mathcal{G}'_k\colon k \geqslant n\}$. Then $(\mathcal{G}_n)_{n \in N}$ is a monotonically contracting sequence of M in S such that each $\mathcal{G}_{n+1} \subset \mathcal{G}_n \subset \mathcal{A}$. If (G_n) is a decreasing representative of $(\mathcal{G}_n)$ then for each n there exists $k_n \geqslant n$ such that $G_n \in \mathcal{G}'_{k_n}$. Each G_n contains a point $x \in M$ such that the first element of $\mathcal{H}_{k_n}$ that contains x includes G_n. Therefore the first element of $\mathcal{H}_n$ that contains x includes G_n. By Lemma 2.2, there exists a decreasing representative (H_n) of $(\mathcal{H}_n)$ with the property stated in the conclusion.

LEMMA 2.4. *Suppose M is a subspace of a topological space X and $(\mathcal{H}_n)_{n \in N}$ is a primitive sequence of M in X whose terms are collections of sets open in X. Suppose that $(\mathcal{B}_n)_{n \in N}$ is a sequence of bases for X. Then*

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there exists a primitive sequence $(\mathfrak{W}_n)_{n\in N}$ *of* M *in* X *such that for each* n *the following conditions are satisfied:* $(3)_n$ $\mathfrak{W}_n \subset \mathfrak{B}_n$ *and if* $x \in M$ *and* W *and* H *are the first elements of* $\mathfrak{W}_n$ *and* $\mathfrak{H}_n$, *respectively, that contain* x, *then* $W \subset H$. *If* X *is regular and* x, W, *and* H *are as just stated, then* $\overline{W} \subset H$. $(4)_n$ *If* X *is regular and* $k < n$, $x \in M$, *and* W' *and* W *are the first elements of* $\mathfrak{W}_k$ *and* $\mathfrak{W}_n$, *respectively, that contain* x *then* $\overline{W} \subset W'$.

Proof. For each n, let $\mathfrak{B}_n'$ denote a well ordered collection whose elements are those of $\mathfrak{B}_n$. If $\mathfrak{H}$ is a well ordered collection and $H \in \mathfrak{H}$ let $\pi(H, \mathfrak{H}) = \{x \in H\colon x$ does not belong to any predecessor of H in $\mathfrak{H}\}$. Let M_0 denote M well ordered by a relation $<_0$ such that if H precedes H' in $\mathfrak{H}_1$ then all elements of $\pi(H, \mathfrak{H}_1) \cap M$ precede all elements of $\pi(H', \mathfrak{H}_1) \cap$ $\cap M$ in M_0. Suppose $x \in M_0$ and t is a function on $s(x)$ into $\mathfrak{B}_1'$. If there exists a first $x' <_0 x$ in M_0 such that $x \in t(x')$ let $f_0(t)$ denote $t(x')$. If no such x' exists let $f_0(t)$ denote the first element $B \in \mathfrak{B}_1'$ containing x such that B (or, if X is regular, $\overline{B}$) is a subset of the first element of $\mathfrak{H}_1$ that contains x. There exists a function U_0 on M_0 to $\mathfrak{B}_1'$ such that $f_0(U_0|s(x))$ $= U_0(x)$. If W is in the range of U_0 let $p(W)$ denote the first element $x \in M_0$ such that $U_0(x) = W$. The range of U_0 may be well ordered by the relation $\preccurlyeq$ defined by $W \preccurlyeq W'$ if and only if $p(W) <_0 p(W')$. Call the resulting well ordered set $\mathfrak{W}_1$. Then $\mathfrak{W}_1$ is a well ordered subcollection of $\mathfrak{B}_1$ covering M such that if $x \in M$ and W and H are the first elements of $\mathfrak{W}_1$ and $\mathfrak{H}_1$, respectively, that contain x then $W \subset H$ and, if X is regular $\overline{W} \subset H$. For, as in the proof of Lemma 2.1, no. 1, $W = U_0(x)$. Let y denote $p(U_0(x))$. Then $f_0(U_0|s(y)) = U_0(y) = U_0(x)$ is a subset of the first element $H' \in \mathfrak{H}_1$ that contains y. If H is the first element of $\mathfrak{H}_1$ containing x, H does not follow H'. If H precedes H' then $x <_0 y$ by the agreement concerning $<_0$. But this is impossible by the definition of p. Thus $H = H' \supset W$. Also if X is regular, $H \supset \overline{W}$.

Suppose $\mathfrak{W}_1, \ldots, \mathfrak{W}_k$ are collections satisfying conditions $(\mathrm{P1})_n$–$(\mathrm{P3})_n$, $(3)_n$ and, if X is regular, $(4)_n$ for all n such that $1 \leqslant n \leqslant k$. Suppose $<_{k-1}$ is a well ordering of M for $k \geqslant 1$. Define a relation $<_k$ on M by $x <_k x'$ for $x, x' \in M$ if and only if (a) the first elements H and H' of $\mathfrak{H}_{k+1}$ that contain x and x', respectively, are such that H precedes H', or (b) if $H = H'$, the first elements W and W' of $\mathfrak{W}_k$ that contain x and x', respectively, are such that W precedes W', or (c) if $W = W'$ and $H = H'$ then $x <_{k-1} x'$. The relation $<_k$ may be shown to be a well ordering. Let M_k denote M ordered by $<_k$. Suppose $x \in M_k$ and t is a function on $s(x)$ to $\mathfrak{B}_{k+1}'$. If there exists a first $x' <_k x$ such that $x \in t(x')$ let $f(t)$ denote $t(x')$. Suppose no such x' exists. If W and H are the first elements of $\mathfrak{W}_k$ and $\mathfrak{H}_{k+1}$, respectively, that contain x there exists a first $B \in \mathfrak{B}_{k+1}'$ such that $x \in B$ and $B \subset W \cap H$ (or if X is regular, $\overline{B} \subset W \cap H$). Let $f(t)$ denote B. By the transfinite recursion theorem there exists a function U on M_k to $\mathfrak{B}_{k+1}'$ such that $U(x) = f(U|s(x))$ for all $x \in M_k$. Let $\mathfrak{R} = \{U(x)\colon x \in M_k\}$. For each $W \in \mathfrak{R}$ let $p(W)$ be the first element of M_k such that $U(p(W)) = W$. The collection $\mathfrak{R}$ may be well ordered by W precedes W' if and only if $p(W) <_k p(W')$ for $W, W' \in \mathfrak{R}$. Let $\mathfrak{W}_{k+1}$ denote $\mathfrak{R}$ ordered by this ordering. Using arguments similar to those used in the proof of Lemma 2.1, it may be seen that $\mathfrak{W}_1, \ldots, \mathfrak{W}_{k+1}$ satisfy $(\mathrm{P1})_n$–$(\mathrm{P3})_n$, $(3)_n$, and $(4)_n$ (if X is regular) for all n such that $1 \leqslant n \leqslant k+1$. Therefore a sequence $(\mathfrak{W}_n)_{n\in N}$ may be obtained which satisfies those conditions for all $n \in N$.

3. Basically complete spaces. Recall that a collection of sets is called *perfectly decreasing* if and only if it contains a proper subset of each of its elements [16]. A collection $\mathfrak{B}$ of subsets of a space X is called a *base of countable order* if and only if $\mathfrak{B}$ is a base for X such that any perfectly decreasing subcollection of $\mathfrak{B}$ is a base at any point common to its members. This concept was introduced by Arhangel'skiĭ [1] who proved that a Hausdorff space is metrizable if and only if it is paracompact and has a base of countable order. Metrizable spaces and Moore spaces have bases of countable order while the space Ω of countable ordinals with the order topology is an example of a space having such a base which is neither a Moore space nor metrizable. Further information about such spaces is in [16].

We shall say that a collection (sequence) of subsets of a space X *converges* to a point $x \in X$ if and only if every open set containing x includes a member of the collection (term of the sequence). The following two concepts were introduced in [15]. A collection $\mathfrak{B}$ of subsets of a space X is called a *λ-base* if and only if $\mathfrak{B}$ is a base of countable order and every nonempty perfectly decreasing monotonic subcollection of $\mathfrak{B}$ convegers to some point of X. A space X is said to have *λ-bases locally* if and only if it has a base $\mathfrak{B}$ such that each $B \in \mathfrak{B}$ has a λ-base.

DEFINITION 3.1. A space X is called *basically complete* if and only if it is Hausdorff and has λ-bases locally.

The following are examples of basically complete spaces. (Proofs will be given later in some instances.)

1. The space Ω of countable ordinals with the order topology.
2. The "long line" [for terminology, cf. 8].
3. Metrically topologically complete spaces.
4. Let S denote the set of real numbers and τ the topology generated by sets D satisfying one of these conditions: (1) $D = \{x\}$ where x is an irrational number. (2) For some $a, b, r \in S$ such that r is rational and $a < r < b$, $D = \{r\} \cup \{x \in S\colon x$ is irrational and $a < x < b\}$. Then (S, τ) is not regular but it is basically complete.

An example of a space which has λ-bases locally but is not T_2 is given by the set S of positive integers with a topology determined by a base consisting of all sets $\{n\} \cup S_p$ where $S_p = \{k \in S: p \leqslant k\}$ and $n \in S$. This space does not have the Baire category property whereas all basically complete spaces do. The domain of example 2 of [15], p. 266 is a screenable metacompact Hausdorff space which has a monotonically complete base of countable order but does not have a λ-base although it does satisfy the Baire category property.

THEOREM 3.1. *If a topological space X is the union of open subspaces which have λ-bases then X has a λ-base.*

Proof. This theorem may be given a proof similar to that of Theorem 1 of [16].

COROLLARY 3.1. *If a space has λ-bases locally then each of its open subspaces has a λ-base.*

THEOREM 3.2. *If an essentially T_1 space X has a λ-base $\mathcal{B}$ there exists a sequence $(\mathcal{G}_n)_{n \in N}$ of bases for X such that each $\mathcal{G}_n \subset \mathcal{B}$ and which satisfies:*

(Λ) *Any decreasing representative $(G_n)_{n \in N}$ of $(\mathcal{G}_n)_{n \in N}$ such that each $G_n \neq \emptyset$ converges to some $x \in X$ and also to every element of $\bigcap \{G_n: n \in N\}$.*

Proof. By Lemma 2.1 there exists a primitive sequence $(\mathcal{H}_n)_{n \in N}$ of X in X such that each $\mathcal{H}_n \subset \mathcal{B}$ with the property described in the lemma. For each n let $\mathcal{G}_n = \bigcup \{\mathcal{H}_k: k \geqslant n\}$. Suppose x belongs to an open set D in X. For each n let H_n be the first element of $\mathcal{H}_n$ that contains x. If $\{H_n: n \in N\}$ is perfectly decreasing it is a base at x and some $H_n \subset D$. If it is not perfectly decreasing then for some n, $H_j = H_n$ for all $j \geqslant n$. If $y \in H_n$ and $y \neq x$ then $y \in \overline{\{x\}}$. For if $y \notin \overline{\{x\}}$, then $x \notin \overline{\{y\}}$ and thus $H_n \setminus \overline{\{y\}}$ is a proper open subset of H_n containing x and thus $H_{n+1} \neq H_n$ which involves a contradiction. Thus $H_n \subset \overline{\{x\}}$. Since X is essentially T_1, any open set containing x includes $\overline{\{x\}}$. Thus $H_n = \overline{\{x\}} \subset D$. Therefore each $\mathcal{G}_n$ is a base for X. Suppose (G_n) is a decreasing representative of $(\mathcal{G}_n)$ such that no $G_n = \emptyset$. For each n there exists a first $H_n \in \mathcal{H}_n$ that includes a term of (G_n). By Lemma 2.2 the sequence $(H_n)_{n \in N}$ is monotonically decreasing and if some $H_m = H_{m+1}$ then $H_j = \overline{\{x\}}$ for some $x \in X$ and all $j \geqslant m$ by an argument similar to one used above and $\{H_n: n \in N\}$ converges to every point of $\overline{\{x\}} = \bigcap \{H_n: n \in N\}$. If no $H_m = H_{m+1}$ then $\{H_n: n \in N\}$ is a perfectly decreasing monotonic subcollection of $\mathcal{B}$ and thus converges to some $x \in X$ and to all $y \in \bigcap \{H_n: n \in N\}$. Since each H_n includes all but finitely many G_n's it follows that (G_n) satisfies the conclusion of condition (Λ).

THEOREM 3.3. *If X is a topological space and there exists a monotonically contracting sequence $(\mathcal{G}_n)$ of collections of open subsets of X covering X which satisfies (Λ) of Theorem 3.2, then X is essentially T_1 and has a λ-base.*

Proof. By Lemma 2.1 there exists a primitive sequence $(\mathcal{H}_n)_{n \in N}$ of X in X such that each $\mathcal{H}_n \subset \mathcal{G}_n$. Let $\mathcal{B}$ denote $\bigcup \{\mathcal{H}_n: n \in N\}$. Using condition ($\Lambda$) and the fact that $\mathcal{H}_n \subset \mathcal{G}_n$ it may be shown that $\mathcal{B}$ is a base for X. Suppose $\mathcal{C}$ is a perfectly decreasing monotonic subcollection of $\mathcal{B}$. Suppose M is a finite subset of N such that for each $i \in M$, H_i is the first element of $\mathcal{H}_i$ that belongs to $\mathcal{C}$. Then, for some $k \in M$, $H_k \subset H_i$ for all $i \in M$ because $\mathcal{C}$ is monotonic. Since $\mathcal{C}$ is perfectly decreasing it contains a proper subset H of H_k. Suppose $H \in \mathcal{H}_j$ for some $j \in M$. Then $H \subset H_j$. But since H_j precedes H and H contains an element of X not in any predecessor, a contradiction is involved. It follows that for each $n \in N$ there exists a first $H_n \in \mathcal{H}_n$ that includes an element of $\mathcal{C}$. By Lemma 2.2, (H_n) is decreasing and therefore converges to some element of X and to each element of $\bigcap \{H_n: n \in N\}$. Therefore $\mathcal{B}$ is a λ-base.

Suppose $x, y \in X$ and $z \in \overline{\{x\}} \cap \overline{\{y\}}$. There exists a decreasing representative (G_n) of $(\mathcal{G}_n)$ such that $\{G_n: n \in N\}$ is a base at z. Therefore $x, y \in G_n$ for all n. It follows readily from this that $\overline{\{x\}} = \overline{\{y\}}$ so that X is essentially T_1.

Remark. It does not follow that if a space has a λ-base, it is essentially T_1. This may be seen in reference to a space having exactly two points and exactly two open sets. But analogously with developability, if S is a space having a sequence $(\mathcal{G}_n)$ of bases such that if P is a point common to all terms of a representative (G_n) of $(\mathcal{G}_n)$, then (G_n) is a base at P, it follows that S is essentially T_1.

The following theorem refines Theorem 9 of [15] in that it explicity includes λ-bases in its scope and does not employ the hypothesis that the spaces are essentially T_1.

THEOREM 3.4. *If X has a base of countable order (respectively, a λ-base) then any base for X has a subcollection which is a base of countable order (respectively, a λ-base).*

Proof. Suppose $\mathcal{G}$ is a base of countable order for X and $\mathcal{W}$ is a base for X. By Lemma 2.1, $\mathcal{G}_n$ being $\mathcal{G}$ for all n, there exists a primitive sequence $(\mathcal{H}_n)_{n \in N}$ of X in itself satisfying the conditions of the lemma. Taking $\mathcal{B}_n = \mathcal{W}$ for all n in Lemma 2.4, one obtains a primitive sequence $(\mathcal{W}_n)_{n \in N}$ of X in itself satisfying condition $(3)_n$ with respect to $(\mathcal{H}_n)_{n \in N}$ for all n. Let $\mathcal{B}$ denote $\bigcup \{\mathcal{W}_n: n \in N\}$. Suppose x is in an open set U of X. For each n, let W_n be the first element of $\mathcal{W}_n$ that contains x. Since W_n is a subset of the first element H_n of $\mathcal{H}_n$ that contains x, Lemma 2.2 implies that $(H_n)_{n \in N}$ is a decreasing representative of $(\mathcal{H}_n)_{n \in N}$. If $H_n \neq H_{n+1}$ for any n, then $\{H_n: n \in N\}$ is a base at x since $\mathcal{G}$ is a base of countable order. If $H_n = H_{n+1}$ for some n, then, by condition $(2)_n$, $H_n \cap U = H_n$. Thus some $H_n \subset U$ and since $W_n \subset H_n$, $\mathcal{B}$ is a base.

Suppose $\mathcal{K}$ is a perfectly decreasing monotonic subcollection of $\mathcal{B}$, $J \subset N$ is finite, and for each $n \in J$, $\mathcal{W}_n$ has a first element $W_n \in \mathcal{K}$. There exists $k \in J$ such that W_k is a subset of all W_n for $n \in J$. There exist W and $i \in N$ such that W is a proper subset of W_k and $W \in \mathcal{W}_i \cap \mathcal{K}$. If $i \in J$, either $W = W_i$ or W_i precedes W. Since W is a proper subset of W_k, $W \neq W_i$; and since $W \subset W_i$, W_i cannot precede W, by condition $(P2)_n$ of the definition of primitive sequence. Thus $i \notin J$. It follows that for each $n \in N$ there exists a first element W_n of $\mathcal{W}_n$ that belongs to $\mathcal{K}$. It may be seen that for each n, $W_{n+1} \subset W_n$. By Lemmas 2.2 and 2.4, there exists a decreasing representative $(H_n)_{n\in N}$ of $(\mathcal{H}_n)_{n\in N}$ such that for each n, H_n is the first element of $(\mathcal{H}_n)_{n\in N}$ that includes a term of $(W_n)_{n\in N}$. If x is common to the members of $\mathcal{K}$ it follows from the condition $(2)_n$ of Lemma 2.1 that $\{H_n: n \in N\}$ is a base at x. Since each H_n includes a member of $\mathcal{K}$, $\mathcal{K}$ is a base at x. Thus $\mathcal{B}$ is a base of countable order. If $\mathcal{G}$ is a λ-base, then $(H_n)_{n\in N}$ and, therefore, $\mathcal{K}$, converges to some point of X. Therefore $\mathcal{B}$ is also a λ-base.

THEOREM 3.5. *A metrizable space is metrically topologically complete if and only if it has a λ-base.*

Proof. A space having a λ-base has a base $\mathcal{B}$ such that the closures of the elements of any monotonic subcollection of $\mathcal{B}$ have a point in common. Vedenisov [12] showed that any metric space with such a base has a topology-preserving metric in which it is complete.

On the other hand suppose ϱ is a metric with respect to which a space X is complete. For each n, let $\mathcal{G}_n$ denote the collection of all open sets of ϱ-diameter $\leqslant 1/n$. For each n, $\mathcal{G}_n$ is a base for X and Cantor's theorem [10] implies that $(\mathcal{G}_n)_{n\in N}$ satisfies the condition (Λ) of Theorem 3.2.

N. Aronszajn [2] in seeking a general class of spaces for which an arc theorem holds discovered a class of spaces closely related to the spaces considered here. He did not give a theory of such spaces, however. In view of the sequential characterization of bases of countable order given in [16], Theorem 2 and Aronszajn's formulation (which is equivalent to that given in the second part of the following definition) it seems appropriate to introduce the following terminology, which bears an analogy to that of *Moore space* and *complete Moore space* (cf. the introduction for this terminology).

DEFINITION 3.2. A space is called an *Aronszajn space* if and only if it is a regular T_0-space having a base of countable order. A space is called a *complete Aronszajn space* if and only if there exists a sequence $(\mathcal{G}_n)_{n\in N}$ of bases for X such that if $(G_n)_{n\in N}$ is a decreasing representative of $(\mathcal{G}_n)_{n\in N}$ then $(\overline{G}_n)_{n\in N}$ converges to a unique point of X.

THEOREM 3.6. *A space is a complete Aronszajn space if and only if it is a regular T_0-space having a λ-base.*

Proof. The proof of necessity is straightforward with the use of Theorem 3.3. If X is regular T_0 and has a λ-base there exists a sequence $(\mathcal{G}_n)$ as in Theorem 3.2 and a primitive sequence $(\mathcal{H}_n)$ of X in X such that each $\mathcal{H}_n \subset \mathcal{G}_n$. By Lemma 2.4 there exists a primitive sequence $(\mathcal{W}_n)$ of X in X such that if W and H are the first elements of $\mathcal{W}_n$ and $\mathcal{H}_n$ that contain $x \in X$, then $\overline{W} \subset H$. For each n, let $\mathcal{V}_n$ denote $\bigcup\{\mathcal{W}_k: k \geqslant n\}$. It may readily be seen that each $\mathcal{V}_n$ is a base. Suppose (V_n) is a decreasing representative of $(\mathcal{V}_n)$. By Lemma 2.2 there exist decreasing representatives (W_n) of $(\mathcal{W}_n)$ and (H_n) of $(\mathcal{H}_n)$ such that for each n, W_n is the first element of $\mathcal{W}_n$ that includes a term of (V_n) and H_n is the first element of $\mathcal{H}_n$ that includes a term of $(\overline{W}_n)$. Since each $\mathcal{H}_n \subset \mathcal{G}_n$, (H_n) converges to some $x \in X$ and therefore $(\overline{V}_n)$ does also. Since X is T_2, x is unique.

THEOREM 3.7. *A regular space having a λ-base has λ-bases locally.*

Proof. Suppose X is a regular space with a λ-base. Since X is essentially T_1 there exists a sequence $(\mathcal{G}_n)_{n\in N}$ of bases for X satisfying the condition (Λ) of Theorem 3.2. Suppose U is open in X. For each $n \in N$, let $\mathcal{W}_n$ be the collection of all $W \in \mathcal{G}_n$ such that $\overline{W} \subset U$. Then $\mathcal{W}_n$ covers U. If $x \in W \in \mathcal{W}_n$ there exists $G \in \mathcal{G}_{n+1}$ such that $x \in G \subset W$. Since $\overline{G} \subset \overline{W} \subset U$, $G \in \mathcal{W}_{n+1}$. Therefore the sequence $(\mathcal{W}_n)_{n\in N}$ is a monotonically contracting sequence of U in itself. Suppose $(W_n)_{n\in N}$ is a decreasing representative of $(\mathcal{W}_n)_{n\in N}$ such that each $W_n \neq \emptyset$. Since $W_n \in \mathcal{G}_n$ there exists $x \in X$ such that $(\overline{W}_n)$ converges to x and to every element of $\bigcap\{\overline{W}_n: n \in N\}$. Since $x \in \bigcap\{\overline{W}_n: n \in N\}$ and each $\overline{W}_n \subset U$, it follows that $x \in U$. By Theorem 3.3, U has a λ-base. Thus every open subset of X has a λ-base.

The following theorem provides a broad class of examples of basically complete spaces as well as information about T_1 first countable scattered spaces. Recall that a subspace of a topological space is called *scattered* if and only if it has no subspace which is dense in itself. Note that essentially T_1 scattered spaces are T_1.

THEOREM 3.8. *A T_1 first countable scattered space has λ-bases hereditarily.*

Proof. Any scattered space X may be well ordered so that for each $x \in X$ there exists an open set U_x containing x but no successor of x. Suppose X is also T_1 and first countable. Then for each $x \in X$ there exists a base $\{D_{nx}: n \in N\}$ such that $D_{n+1\,x} \subset D_{nx} \subset U_x$ for all $n \in N$. For each n, let $\mathcal{H}_n = \{D_{nx}: x \in X\}$ well ordered by the relation $\precneqq$ where $D_{nx} \precneqq D_{nx'}$ if and only if x precedes x' in X. It may be verified that $(\mathcal{H}_n)_{n\in N}$ is a primitive sequence of X in X whose terms are collections of open sets. For each n, let $\mathcal{W}_n = \bigcup\{\mathcal{H}_k: k \geqslant n\}$. Then each $\mathcal{W}_n$ is a base for X. Suppose (W_n) is a decreasing representative of $(\mathcal{W}_n)$. By Lemma 2.2 there exists a decreasing representative (H_n) of $(\mathcal{H}_n)$ such that for each n, H_n is the first element of $\mathcal{H}_n$ that includes a term of (W_n). For some $j > n+1$, W_j

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$\subset H_n \cap H_{n+1}$. There exists $x \in X$ such that $W_j = D_{kx}$ for some $k \geqslant j$. The first element of $\mathcal{H}_n$ that contains x is D_{nx}. Since $x \in H_n$ and $D_{nx} \supset D_{kx}$ it follows that $H_n = D_{nx}$. Similarly $H_{n+1} = D_{n+1x}$. If $m > n+1$, $H_m = D_{mx'}$ by the same argument and $H_n = D_{nx'}$. Thus $x = x'$ and $H_n = D_{nx}$ for all n. Thus $x \in \bigcap \{W_n: n \in N\}$ and $\{W_n: n \in N\}$ is a base at x. Theorem 3.2 implies that X has a λ-base. The property of being a scattered space is hereditary.

Comment 3.1. Note that the space Ω of countable ordinals with the order topology is thus basically complete hereditarily, as is any first-countable space of ordinals with the order topology. This may be contrasted to the situation in uniform space theory in which Ω has a unique structure which is not complete [5].

4. The characterization theorem.

Theorem 4.1. *A Hausdorff space is an open continuous image of a complete metric space if and only if it is basically complete.*

This theorem is a consequence of the following two theorems, the first of which was proved in [15].

Theorem 4.2. *Suppose Y is an essentially T_1 space which is the range of an open continuous mapping φ. If the domain of φ has a λ-base so does Y. If the domain of φ has λ-bases locally so does Y.*

Theorem 4.3. *Suppose X is a basically complete space. Then X is an open continuous image of a metrically topologically complete space which has the same weight as X and which is a closed subspace of a Baire space* [4].

Proof. Let $\mathcal{V}$ denote a base of minimal cardinality for X. We may assume $|\mathcal{V}| \geqslant \aleph_0$. By Theorem 3.4 and Corollary 3.1 some subcollection $\mathcal{B}$ of $\mathcal{V}$ is a λ-base for X. By Theorem 3.2 there exists a sequence $(\mathcal{G}_n)$ of bases for X such that each $\mathcal{G}_n \subset \mathcal{B}$ and which satisfies (Λ). Corollary 3.1 implies that each member of each $\mathcal{G}_n$ has a λ-base. Let B denote a function on $\bigcup \{\mathcal{G}_n: n \in N\}$ such that, for each $G \in \bigcup\{\mathcal{G}_n: n \in N\}$, $B(G)$ is a λ-base for G.

Consider each $\mathcal{G}_n$ as having the discrete topology and let M denote the product space of the family $(\mathcal{G}_n)_{n \in N}$. Then the weight of M does not exceed the weight of X. Let W denote the subspace of M such that $(G_n) \in W$ if and only if (G_n) is a decreasing representative of $(\mathcal{G}_n)_{n \in N}$ and either: (a) For all n, there exists a sequence $G_{1n}, \ldots, G_{nn}$ such that: (1) $G_{1n} \subset G_n$. (2) For each $j \leqslant n-1$, $G_{j+1,n}$ is a proper subset of G_{jn}. (3) For all $j \leqslant n$, $G_{jn} \in B(G_j)$. (4) $G_{n+1} \subset G_{nn}$; or: (b) For some k, G_k is a degenerate point set, i.e., a singleton, but for all $n \leqslant k-1$ there exist sequences $G_{1n}, \ldots, G_{nn}$ with the properties (1)–(4) of (a).

[4] i.e., a product space of a countable family of discrete spaces.

I. We show that for each $x \in X$ there exists a sequence (G_n) of type (a) or of type (b) such that $x \in \bigcap \{G_n: n \in N\}$. If $x \in X$ and $\{x\}$ is open, then $\{x\} \in \mathcal{G}_n$ for all n since each $\mathcal{G}_n$ is a base. Therefore the sequence (G_n) where each $G_n = \{x\}$ is of type (b). Suppose $\{x\}$ is not open. Then there exists $G_1 \in \mathcal{G}_1$ containing x such that $G_1 \neq \{x\}$. Suppose $G_1, \ldots, G_m$ exist such that for each n $(1 \leqslant n \leqslant m)$ $G_n \in \mathcal{G}_n$, $x \in G_n$ and for all n $(1 \leqslant n \leqslant m-1)$ there exists a sequence $G_{1n}, \ldots, G_{nn}$ with the properties specified under (a) such that $x \in G_{kn}$ for all $k \leqslant n$. Since $x \in G_m$, $G_m \subset G_1$, and $\{x\} \neq G_m$ it follows that there is an element $G_{1m} \in B(G_1)$ which contains x and is a proper subset of G_m. Suppose $G_{1m}, \ldots, G_{km}$ have been defined where $k < m$ such that for all $j \leqslant k$, $x \in G_{jm} \in B(G_j)$, and $G_{j+1,m}$ is a proper subset of G_{jm} for $j \leqslant k-1$. Since $\{x\} \neq G_{km}$ and $G_{km} \subset G_{k+1}$ there exists a $G_{k+1,m} \in B(G_{k+1})$ which contains x and is a proper subset of G_{km}. Thus a sequence $G_{1m}, \ldots, G_{mm}$ with properties (1)–(4) of (a) may be defined. Since $\{x\} \neq G_{mm}$ and $\mathcal{G}_{m+1}$ is a base there exists $G_{m+1} \in \mathcal{G}_{m+1}$ which contains x and is a proper subset of G_{mm}. By induction, there exists a sequence (G_n) of type (a) as desired.

II. For each $(G_n) \in W$ there exists a unique point x common to the terms of (G_n) and $\{G_n: n \in N\}$ is a base at x. If (G_n) is of type (b) this is obvious. If (G_n) is of type (a) then, for each n, the sequence $(G_{nj})_{j \geqslant n}$ is is a perfectly decreasing monotonic subcollection of $B(G_n)$. Hence there exists a point $x_n \in G_n$ such that (G_{nj}) converges to x_n and since X is T_2 there is no other point x in X to which (G_{nj}) converges. For $n \leqslant j$ we have $G_{nj} \subset G_{1j}$, thus each sequence (G_{nj}) converges to x_1 and therefore $x_1 = x_n \in G_n$ for $n = 1, 2, \ldots$ Hence $x_1 \in \bigcap \{G_n: n \in N\}$ and $x_1 \in \bigcap \{G_{1n}: n \in N\}$. Since (G_{1n}) converges to x_1, (G_{1n}) is a base at x_1 and because $G_{n+1} \subset G_{1n}$, (G_n) is also a base at x_1.

Let φ denote the mapping of W onto X such that $\varphi((G_n))$ is the unique element common to the terms of (G_n). Let $(G_n) \in W$ and let $S(G|n)$ denote the set of all $(V_n) \in W$ such that $V_k = G_k$ for $1 \leqslant k \leqslant n$. Then $\{S(G|n): n \in N$ and $(G_k) \in W\}$ is a base for W. Using I, it may be seen that $\varphi(S(G|n)) = G_n$. It follows that φ is an open continuous mapping of W onto X and the weight of W equals the weight of X.

Suppose W is given the usual Baire metric ϱ [7], and (x_n) is a Cauchy sequence in W with respect to ϱ. There exists an increasing sequence (m_n) of positive integers such that $\varrho(x_k, x_j) < 1/n$ for all $k, j \geqslant m_n$. By an argument used in [13], it may be shown that if G_n denotes the nth term of x_{m_n}, then if $k \geqslant m_n$ the first n terms of x_k are $G_1, \ldots, G_n$. Since $x_{m_n} \in W$, $G_1, \ldots, G_n$ satisfy the conditions on the first n terms of a sequence in W for all n. Hence $(G_n) \in W$. It may be seen that (x_n) converges to (G_n), so that W is complete with respect to ϱ.

Comment 4.1. The proof of this theorem is similar to the proof, given in [13], of the existence of an open mapping of a complete metric space

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onto a regular T_0-space having a λ-base. Both of these theorems may be proved from a unified point of view which encompasses certain non first countable situations. This is carried out in [14]. Here it seems preferable to give a direct proof with appropriate references to [13] rather than use the general mapping lemma of [14].

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OHIO UNIVERSITY
Athens, Ohio

SANDIA LABORATORIES
Albuquerque, New Mexico

Reçu par la Rédaction le 5. 1. 1971

Models of second order arithmetic with definable Skolem functions

by

Andrzej Mostowski (Warszawa)

Let A_2 be the axiomatic system of second order arithmetic as described in [2].

In the study of the problem whether the standard part of a model of A_2 is itself a model of A_2 we introduced the following model theoretic concept: Let A be a structure of type σ and P a singular predicate of σ. Let B be another structure of type σ' such that A is a reduct of B. We say that B is an *S-structure* for A and P if

1° all the Skolem functions of B are definable in B;

2° each subset of P^A (the interpretation of P in A) which is parametrically definable in B is so definable in A. (See [3].)

Using Lévy's model for A_2 (see e.g. [4], pp. 241–247) we can easily exhibit an ω-model A in which all the axioms of A_2 with the exception of the axiom of choice are valid such that no S-structure exists for A and the predicate $N(\cdot)$. For ω-models of the full system A_2 the situation is different: we shall prove the following

THEOREM. *If M is a denumerable ω-model for A_2 then there exists an S-structure for M and the predicate $N(\cdot)$.*

Proof of this theorem will occupy the rest of this paper. We shall use a very primitive form of the forcing argument. Our proof was influenced mainly by the result of Felgner [1].

LEMMA 1. *The following scheme is provable in A_2* (*cf.* (iii) *below for the meaning of* $z^{(n)}$):

$$S(w)\ \&\ C(w)\ \&\ (n)_N(c)_S\{C(c)\to(\mathrm{E}\bar{c})_S[C(\bar{c})\ \&\ B(c,\bar{c})\ \&\ D(n,\bar{c})]\}$$
$$\to(\mathrm{E}z)_S\{(z^{(0)}=w)\ \&\ (n)_N[C(z^{(n)})\ \&\ B(z^{(n)},z^{(n+1)})\ \&\ D(n,z^{(n+1)})]\}.$$

Read "c is a vertex" for $C(c)$ and "$\bar{c}$ is an nth extension of c" for $B(c,\bar{c})\ \&\ D(n,\bar{c})$.

The scheme can then be expressed as follows. If for every integer n every vertex has an nth extension which is also a vertex then for every