# Boolean algebras with ordered bases

by

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A Boolean algebra with an ordered base is one which has a set of generators which is totally ordered under the natural partial ordering of the algebra. These algebras were first introduced by Mostowski and Tarski in [10] who established many of their basic algebraic properties. Their work was continued by Mayer and Pierce [9] who showed, among other things, that the Stone spaces of these algebras are precisely the orderable Boolean spaces. The simplicity of the condition imposed on their generators and the fact that every countable Boolean algebra is such an algebra make these algebras natural objects to investigate. In spite of this naturalness however they seem difficult to study. In part this difficulty stems from the absence of useful general criteria for a topological space to be orderable and a fortiori for a Boolean space to be orderable. Matters are not helped by the fact that not every subalgebra of a Boolean algebra with ordered base is such an algebra; in topological terms not every continuous Boolean image of an ordered Boolean space is orderable. Thus any attempt to characterise these algebras, for example, within the category of all Boolean algebras, must either separate off the non-ordered base subalgebras first or else characterise ordered base algebras together with all their subalgebras. A characterisation of the second kind is attempted here, in terms of retracts, in § 1. In § 2 we show that uncountable free Boolean algebras can not be embedded in ones with ordered bases; we also prove certain refinements of this result. In § 3 we establish the natural generalisation in terms of $\eta_k$-sets of Mostowski and Tarski's result that if one base for an algebra is scattered then all bases are scattered.

**§ 0. Preliminaries.** Any Boolean algebra with an ordered base can be realised as a certain field of subsets, more exactly as the set of all finite unions of intervals, of a totally ordered set and conversely. So that we shall refer to algebras with ordered bases as *interval algebras* and use the following notation. If $T$ is an ordered (= totally ordered) set then by $\mathrm{In}(T)$ we shall mean the set of all finite unions of intervals of $T$ of the form $(\ ,b]$, $(a,b]$, $(a,\ ]$ for $a,b\in T$ where $(\ ,b]=\{x\in T\colon\ x\leqslant b\}$ and $(a,\ ]=\{x\in T\colon\ x>a\}$, together with $\emptyset$, $T$. Clearly $\mathrm{In}(T)$ is a Boolean algebra with respect to set operations, and we shall call it the *interval algebra on* $T$. For a set $X$ we shall denote by $|X|$ the cardinal number of $X$ and we use $\omega$ to denote both the first infinite ordinal as well as the first infinite cardinal; similarly with $\omega_k$ for $k\geqslant 1$. We shall rely on the context to avoid any possible confusion between ordinals and cardinals. Finally we assume familiarity with the Stone duality theory for Boolean algebras.

**§ 1. Retracts.** In the category of posets (= partially ordered sets) with morphisms as all ordered preserving maps, chains are characterised by the following property

(1.1) *$C$ is a chain iff for any morphism $f\colon C\to D$ to an arbitrary poset $D$ there exists a morphism $g\colon D\to C$ such that $fg=1_D$.*

On the basis of the analogy that interval algebras are to Boolean algebras as chains are to posets it is not unreasonable to expect that a result similar to (1.1) might hold for interval algebras within the category of Boolean algebras. It is the purpose of the present section to explore this idea.

We first introduce some terminology. As usual (see [5]) we say that a Boolean algebra $D$ is a *retract* of an algebra $C$ if there exist morphisms $f\colon C\to D$, $g\colon D\to C$ such that $fg=1_D$ (note that $f$ is necessarily epi and $g$ is necessarily mono); further we say that in this case the morphism $f$ is a *retraction* onto $D$. We shall call a Boolean algebra $C$ *retractive* iff every epi $f\colon C\to D$ is a retraction. Dual (in the category sense) to this we say that a Boolean algebra $C$ is *co-retractive* iff for every mono $i\colon B\to C$, there exists a morphism $j\colon C\to B$ (necessarily epi) such that $ji=1_B$. Thus a retractive algebra is one for which every morphism from it is a retraction and a co-retractive algebra is one which has a retraction onto each of its subalgebras. If $S(A)$ denotes the Stone space of the algebra $A$ then it is easily seen that

(1.2) *$D$ is a retract of $C$ iff $S(C)$ is a retract of $S(D)$*

so that

(1.3) *$C$ is retractive iff $S(C)$ is co-retractive*

and

(1.4) *$C$ is co-retractive iff $S(C)$ is retractive*

where in (1.2), (1.3), (1.4) the term retract on the right refers to the usual topological idea of retract (see [2]) with respect to the category of all Boolean spaces and continuous maps between them; so that a space is retractive iff for every continuous epi $f\colon X\to Y$ there exists a continuous $g\colon Y\to X$ such that $fg=1_Y$ and a space is co-retractive iff it retracts onto each of its closed subspaces.



The idea of retracts of Boolean algebras is not new. In 1935 von Neumann and Stone [11] investigated the following problem. If $A$ is a Boolean algebra, then under what conditions on the ideal $J$ will there exist a subset $S$ of $A$ which meets every equivalence class modulo $J$ in exactly one point and which is a subalgebra of $A$? Such a subalgebra of $A$ would be isomorphic to $A/J$ so that in the above terminology the question amounts to conditions on $J$ for the canonical epimorphism $A\to A/J$ to be a retraction. The restriction to the canonical epimorphism can be removed by the following observation. If the canonical epimorphism $f\colon A\to A/J$ is a retraction so that there exists $j\colon A/J\to A$ satisfying $fj=1_{A/J}$ and if $g\colon A\to A/J$ is an arbitrary epimorphism, then $gj$ is an automorphism of $A/J$ and therefore $gj(gj)^{-1}=1_{A/J}$, so that $g$ is a retraction.

THEOREM 1.5. *Interval algebras are retractive.*

Proof. This is given essentially by Mostowski and Tarski who show [10], Satz 2.2 and 2.3, that the von Neumann–Stone problem is solved for all ideals $J$ in the case of interval algebras. This together with the above observation is what is required.

The converse of Theorem 1.5 is false since, as we shall now show, there exist retractive Boolean algebras which are not interval algebras.

THEOREM 1.6. *The finite co-finite algebra $FC(T)$ on set $T$ (i.e. the field of all finite or co-finite subsets of $T$) is retractive.*

Proof. This is easier to see topologically. Let $X$ be the one-point compactification of a set (i.e. discrete space) of power $|T|$ with respect to $a\in X$. Then, as is known, $X$ is the Stone space of $FC(T)$. We must show that $X$ is co-retractive, i.e. $X$ retracts onto any closed subset $F$ of $X$.

Case 1. $a\notin F$ so that $F$ is finite.

Define $f\colon X\to F$ by $f(x)=x_0$ for all $x\in X-F$ and $f(x)=x$ otherwise, then $f$ is continuous. To see this let $P\subseteq X$ be open. If $x_0\notin P$, then $f^{-1}(P)=P\cap F$ is a finite subset of $X$ not containing $a$, and so is open. If $x_0\in P$, then $f^{-1}(P)$ is cofinite and so is open.

Case 2. $a\in F$.

Define $f\colon X\to F$ by $f(x)=a$ for all $x\in X-F$ and $f(x)=x$ otherwise, then $f$ is continuous. To see this let $P\subseteq X$ be open. If $a\notin P$ then $f^{-1}(P)$ does not contain $a$ and so is open. If $a\in P$ then $X-P$ is finite and so $(X-P)\cap F$ is finite, which means that $f^{-1}(P)=X-(X-P)\cap F$ is cofinite and so is open. Thus in both cases $f$ is retraction onto $F$, as required.

For $|T|=\omega$ Theorem 1.6 gives nothing more than Theorem 1.5 since $FC(\omega)$ is countable and hence an interval algebra. For $|T|>\omega$,

$FC(T)$ cannot be an interval algebra since it is an uncountable algebra in which every chain is countable. Thus the converse to Theorem 1.5 is false. However $FC(T)$ is isomorphic to a subalgebra of an interval algebra [1]. Thus let $T$ be the set of all ordinals $\leqslant \omega_1$ in their natural ordering and consider the subalgebra $A$ of $\mathrm{In}(T)$ generated by all elements of the form $(i, i+1]$ where $i$ is an ordinal $< \omega_1$. Then $A$ is isomorphic to $FC(\omega_1)$ since it is an atomic algebra every one of whose elements is a finite union of atoms or the complement of such a union. Thus unlike that of being a chain the property of being an interval algebra is not hereditary. Theorem 1.6 and the remarks just made suggest that the following conjecture might be a way of retrieving the analogy between chains (posets and interval algebras) Boolean algebras.

Conjecture (A). *Retractive Boolean algebras are precisely the subalgebras of interval algebras.*

Before this however one would have to settle the following

Conjecture (B). *Every subalgebra of an interval algebra is retractive.*

The difficulty in settling this question lies in the lack of an adequate characterisation of subalgebras of interval algebras; equivalently in the lack of a characterisation of when continuous images of ordered spaces are orderable. Certainly there are subalgebras of interval algebras which are not interval algebras other than those isomorphic to $FC(T)$. Thus for example if $A$ is the subalgebra of $\mathrm{In}(T)$ constructed after Theorem 1.6 and $B$ is the subalgebra of $\mathrm{In}(T)$ generated by $A$ together with any additional countable set of elements of $\mathrm{In}(T)$ then $B$ is neither an interval algebra nor isomorphic to $FC(\omega_1)$. As a very weak support of the second conjecture above we shall show that the simplest such algebra $B$ is retractive.

Lemma 1.7. *If $D$ is the algebra of all subsets of a set $T$, $C$ the algebra of all finite and cofinite subsets of $T$ and $B$ the subalgebra of $D$ generated by $C$ and some $p \in D$, $p \notin C$ then the Stone space of $B$ is the disjoint union $X \cup Y$ of clopen $X$ and $Y$ where $X$ is the one-point compactification of a set of power $|T|$ and $Y$ is the one-point compactification of a set of power $\min(|p|, |-p|)$.*

Proof. We shall just consider the case $|T| = \omega_1$, $|p| = \omega$; the arguments for the other cases being essentially the same. Thus $X, Y$ are the one-point compactifications of sets of power $\omega_1, \omega$ respectively with respect to points $\alpha \in X$, $\beta \in Y$. Clopen subsets of $X \cup Y$ are unions of clopen subsets of $X$ and of $Y$ i.e. unions of finite or cofinite subsets of $X - \{\alpha\}$ and of $Y - \{\beta\}$. So that if $P$ is a clopen subset of $X \cup Y$, then $P$ is either a finite or cofinite subset of $X \cup Y - \{\alpha\} - \{\beta\}$ or $P$ differs

[1] I am grateful to R. S. Pierce for pointing this fact out.



by finitely many points of $X \cup Y$ from either $X$ or $Y$. The function which maps the singletons of $T$ onto those of $X \cup Y - \{\alpha\} - \{\beta\}$ and which maps $p$ onto $Y - \{\beta\}$ obviously extends to an isomorphism of the algebra $B$ onto the clopen field of $X \cup Y$ which is what is required.

Suppose now that $A$ is the subalgebra of $\mathrm{In}(T)$ (where $T$ is the set of ordinals $\leqslant \omega_1$) which is generated by all elements of the form $(i, i+1]$ as before; and $B$ is the subalgebra of $\mathrm{In}(T)$ generated by $A$ and an element of the form $p = (\ , i]$ for some $i < \omega_1$. Then

(1.8) *$B$ is a subalgebra of an interval algebra which is retractive and which is neither isomorphic to a finite-cofinite algebra nor to an interval algebra.*

By the Lemma, the Stone space $S(B)$ can be taken as the disjoint union $X \cup Y$ of clopen sets $X$ and $Y$ which are the one-point compactifications of sets of power $\omega_1, \omega$ respectively. We show that $S(B)$ is co-retractive. Let $F$ be a closed subset of $S(B)$. If $F \cap X \neq \emptyset$ and $F \cap Y \neq \emptyset$, then by the retractivity of $FC(\omega_1)$, $FC(\omega)$ respectively there exist retractions $f\colon X \to F \cap X$, $g\colon Y \to F \cap Y$ and so $F$ is a retract of $S(B)$. If $F \cap X = \emptyset$ then the retraction $g$ exists as before and to obtain a continuous $f\colon X \to F$ we observe that for any ordinal $k$ a closed subset of $FC(\omega_k)$ is homeomorphic to $FC(\omega_j)$ for some $j \leqslant k$ and that there exists a continuous epi from $FC(\omega_k)$ to $FC(\omega_j)$. Finally if $F \cap Y = \emptyset$ then there exists a retraction $f\colon X \to F$ by the retractivity of $FC(\omega_1)$ and to obtain a continuous $g\colon Y \to F$ we just observe that there exists a closed subset of $F$ homeomorphic to $FC(i)$ for some $i \leqslant \omega$ and then argue as before. Thus (1.8) holds.

We turn now to retractive Boolean spaces i.e. to co-retractive algebras which we can characterise completely.

Theorem 1.9. *A Boolean algebra is co-retractive iff it is isomorphic to a finite-cofinite algebra.*

Proof. Let …e a finite-cofinite algebra. The Stone space of $A$ is the one-point co…tification of a discrete space $X$ with respect to $\alpha \in X$. We show that … co-retractive by showing that $X$ is retractive. Thus let $f\colon X \to Y$ be … …ntinuous epi onto the Boolean space $Y$. The map $f$ induces a disjoint partition $X = \bigcup_{y \in Y} f^{-1}(y)$ into closed subsets. Let $F \subseteq X$ be obtained by selecting one point from each partition set subject only to the restriction that the point selected from $f^{-1}(f(\alpha))$ is $\alpha$. Then $F$ is closed since it contains $\alpha$. By Stone duality the insertion map of $F$ into $X$ is the dual of the required retraction onto the subalgebra of $A$ inserted into $A$ by the dual of $f$. Conversely suppose $A$ is co-retractive. We assume for the moment that $A$ is countable and show that (a) $A$ is isomorphic to $FC(\omega)$. If (a) fails to hold then $A$ must contain a subalgebra $B$ which is generated by a copy $C$ of $FC(\omega)$ together with an element $p$ which is not the union of finitely many atoms of $C$. It is clear that we can take $B$ to be the algebra $B$ in Lemma 1.7 i.e. the subalgebra of the power set of a denumerable set $T$ generated by the finite or cofinite subsets of $T$ together with a subset $p \subseteq T$ such that $|p| = |-p| = \omega$. We show that $B$ is not co-retractive. Suppose that $h\colon B \to C$ is a retraction. Let $h(p) = q \in C$. Since $p$ is infinite we know that $p > k_n$ $(n = 1, 2, \ldots)$ where the $k_n$ are distinct finite subsets of $T$. Thus $q \geqslant h(k_n) = k_n$ for $n = 1, 2, \ldots$, which means that $q$ can not be finite. A similar argument starting from $-p$ being infinite shows that $q$ can not be cofinite. Thus $B$ has no retraction onto its subalgebra $C$ and is therefore not co-retractive. But the property of being co-retractive is inherited by subalgebras so that no algebra which embeds $B$ is co-retractive, which proves (a). We have therefore shown that if $A$ is a co-retractive algebra, then every countable subalgebra of $A$ is isomorphic to $FC(\omega)$. From this it easily follows that $A$ is isomorphic to $FC(\omega_k)$ for some ordinal $k$.

We conclude this section by raising the dual form of the von Neumann–Stone question for interval algebras; if $A$ is a subalgebra of an interval algebra $B$, can we find conditions on $A$ for $A$ to be a retract of $B$? Clearly, since morphisms preserve the property of having an ordered base, $A$ must be an interval algebra. If we make the restriction that the base for $A$ is a subset of that for $B$ then we can give a simple answer.

THEOREM 1.10. *If $C_1, C_2$ are ordered sets and $C_1 \subset C_2$ then* $\mathrm{In}(C_1)$ *is a retract of* $\mathrm{In}(C_1)$ *iff $C_1$ is a retract of $C_2$.*

If $h\colon \mathrm{In}(C_2) \to \mathrm{In}(C_1)$ is a retraction and $C_i^* = \{p \in \mathrm{In}(C_i)\colon p = (\ , x]$ for $x \in T_i\}$ for $i = 1, 2$ then the restriction $h^*\colon C_2^* \to C_1^*$ of $h$ to $C_2^*$ yields the desired retraction of ordered sets. The converse follows from the fact that if $f\colon C_2 \to C_1$ is an order preserving map then $f$ extends to a Boolean morphism $f^*\colon \mathrm{In}(C_2) \to \mathrm{In}(C_1)$.

Theorem 1.10 becomes more interesting in those cases where we have detailed knowledge about the retracts of the base $C$. We shall not pursue this question here except to mention the following result whose proof we omit (¹).

THEOREM 1.11 (GCH). *If $C$ is the $\eta_\alpha$-set of power $\omega_\alpha$ and $D$ is a retract of $C$ with $|D| < \omega_\alpha$ then $D$ is Dedekind complete* (²). *Conversely if $D$ is a Dedekind complete ordered set and $|D| \leqslant \omega_\alpha$ then $D$ can be embedded into $C$ as a retract of $C$.*

(¹) Theorem 1.11 is a special case of more general result which holds for systems with homogeneous universal objects ($\eta_\alpha$ sets here) and appropriate injective objects (Dedekind complete sets here). Mączyński [7] proved it first for the category of Boolean algebras. A generalisation which includes both his result and Theorem 1.11 can be found in [1], Theorem 5.2.

(²) Our use of the term Dedekind complete implies the existence of end-points.



**§ 2. Independent elements.** In [9] Mayer and Pierce showed that an infinite $\sigma$-complete Boolean algebra could not be an interval algebra. An extension of their result would be that an infinite $\sigma$-complete algebra can not be embedded into an interval algebra. Now any infinite $\sigma$-complete algebra contains a copy of the power set algebra on $\omega$ which contains $2^\omega$ independent, i.e. free (see below) elements; so that a further extension would be the result that no interval algebra contains $\omega_1$ independent elements. We shall give two different proofs of this last result in this section. First we recall the definition of independence.

If $T = \{t_i\colon i \in I\}$ is a subset of a Boolean algebra $A$, then the elements of $T$ are said to be *independent* if

(2.1) $$\varepsilon_1 t_{i(1)} \cap \ldots \cap \varepsilon_n t_{i(n)} \neq 0$$

where $\varepsilon_i = \pm 1$ and $i(1), \ldots, i(n)$ is any finite sequence of distinct members of $I$.

It is known ([14], p. 43) that the notions of independent element and free generator are equivalent in the sense that if the elements $\{t_i\colon i \in I\}$ above generate $A$, then they are free generators iff they are independent.

The first proof of the result mentioned above will give it as an immediate consequence of the following topological theorem of L. B. Treybig and A. J. Ward [15].

THEOREM 2.2 (TREYBIG-WARD). *If $X, Y$ are infinite compact Hausdorff spaces and $f\colon L \to X \times Y$ is a continuous epi, where $L$ is a compact ordered space then both $X, Y$ are metrizable.*

THEOREM 2.3. *If $A$ is an interval algebra and $T$ is a set of independent elements of $A$, then $|T| \leqslant \omega$.*

Proof. Suppose that $|T| \geqslant \omega_1$. Then the free Boolean algebra $F$ on $\omega_1$ generators is a subalgebra of $A$. By Stone duality this means that the Stone space $\{0, 1\}^{\omega_1}$ of $F$ is a continuous image of $S(A)$. But $S(A)$ is a compact space which by [9] is ordered, and $\{0, 1\}^{\omega_1}$ is homeomorphic to the product $\{0, 1\}^\omega \times \{0, 1\}^{\omega_1}$, thus by Theorem 2.2 the space $\{0, 1\}^{\omega_1}$ is metrizable which is known not to be the case. Thus our supposition that $|T| \geqslant \omega_1$ leads to a contradiction.

Without some restriction on $A$ the cardinality bound in Theorem 2.3 can not be improved since the free Boolean algebra on $\omega$ generators is isomorphic to the interval algebra on the rationals. This suggests the obvious restriction to interval algebras which do not embed a dense chain.

THEOREM 2.4. *If $A = \mathrm{In}(T)$ and $T$ is a scattered ordered set (i.e. has no dense subset) then any set of independent elements of $A$ is finite.*

Proof. The construction of the Stone space of $\mathrm{In}(T)$ given in [9] shows that it is homeomorphic to the order topology on a scattered ordered set. No such space can have the Cantor set $\{0, 1\}^\omega$ as a continuous image (see [14], p.35) so that the free Boolean algebra on $\omega$ generators can not be a subalgebra of $A$, which was to be proved.

The notion of independence used here is part of a more general one which applies to a wide variety of systems, details of which can be found in [3], [8]. In particular, the definition given in (2.1) suitably modified to omit reference to complements applies to distributive lattices, and by a result of Marczewski [8] (v), p.142, it follows that Theorem 2.3 and Theorem 2.4 are true if the algebra occurring in them is treated purely as a distributive lattice. Now since the apparatus of Stone duality applied to free Boolean algebras etc. can not be transferred intact to distributive lattices, it is of interest to give a proof of Theorem 2.3 which does not make use of the Treybig-Ward Theorem. We shall now do this starting from Theorem 2.4 and noting first that a straightforward algebraic proof (effectively a translation of the one given above) can be given of Theorem 2.4 so that the eventual second proof of Theorem 2.3 will be a purely algebraic one.

Second Proof of Theorem 2.3. Let $A = \mathrm{In}(T)$ be the interval algebra on an ordered set $T$, and suppose $A$ has $\omega_1$ independent elements. Since each element is the union of finitely many intervals of $T$, it follows that $A$ has $\omega_1$ independent elements each of which is the union of $k$ intervals of $T$ for some *fixed* $k < \omega$. Let $p_n$ $(n = 1, 2, \ldots)$ be $\omega$ of these independent elements and let $V$ be the union of the end points of the $p_n$ (where here the end points of an interval of the form $(\ , x]$ are $-\infty, x$ and those of $(x,\ ]$ are $x, \infty$). $V$, as a subset of $T$ with additional end points $-\infty, \infty$ is totally ordered by the ordering of $T$. Let $q_n = p_n \cap V$, then $q_n \in \mathrm{In}(V)$ and are independent and so, by Theorem 2.4, $V$ can not be scattered. Now $V = E_1 \cup \ldots \cup E_{2k}$ where $E_{2i-1}, E_{2i}$ are the sets of left hand (respectively right hand) end points of the $i$th interval for $1 \leqslant i \leqslant k$. Suppose that $X$ is an infinite, scattered subset of $E_1$. Then $Y = \{m\colon q_m \cap X \neq \emptyset\}$ is infinite and so the union of all the end points of $q_m$ for $m \in Y$ (not including the left hand end points of the first intervals) must, by Theorem 2.4, contain a dense set and so is not empty. We can repeat this argument for the right hand end points of all the $q_m$ with $m \in Y$. If the argument is carried out for $2k$ times in all (each time using the density given by Theorem 2.4 to guarantee non-emptiness) we arrive at an infinite set of independent elements the union of whose end points is scattered, which contradicts Theorem 2.4, and so the proof is finished.

Digression. The following very weak notion of independence can be considered in a general setting and in particular for Boolean algebras. For a Boolean algebra $A$ and a subset $V$ of $A$, denote by $[V]$ the subalgebra of $A$ generated by $V$. Call $V$ *irredundant if* $p \notin [V - \{p\}]$ for all $p \in V$. It is easy to show that any chain not containing $0, 1$ is irredundant so that every interval algebra (and in particular every countable Boolean algebra) has an irredundant base; and in contrast to Theorem 2.4 every interval algebra $A$ has $|A|$ irredundant elements. However this phenomenon is not peculiar to interval algebras since if $V$ is independent in the usual sense then $V$ is irredundant. We do not know in general which Boolean algebras have irredundant bases.



Returning to Theorem 2.3, we observe that neither of the methods used prove it seem capable of settling the following conjecture, a positive answer to which (since projective algebras are retracts of free ones) would extend Theorem 2.3.

Conjecture (C). *If $A$ is an uncountable projective Boolean algebra then $A$ can not be embedded in any interval algebra.*

Finally in this section we note that the Treybig-Ward Theorem yields, as an immediate consequence, an extension of some results of Mayer and Pierce [9], Th. 5.3, 5.5. If $C, A, B$ are Boolean algebras let us write $C = A + B$ if the Stone space $S(C)$ of $C$ is (upto homeomorphism) the product $S(A) \times S(B)$ of the Stone spaces of $A, B$. (Thus $C$ is the co-product of $A, B$ in the category of Boolean algebras.)

Theorem 2.5. *If $A$ is any uncountable Boolean algebra and $B$ any infinite Boolean algebra then $A + B$ is neither an interval algebra nor a subalgebra of an interval algebra.*

Proof. Immediate from Theorem 2.2 in view of the fact that the Stone space of an algebra $A$ is metrizable iff $A$ is countable.

**§ 3. Different ordered bases.** In [10] Mostowski and Tarski proved that if an interval algebra $A$ has a scattered base, then all bases for $A$ are scattered. We shall prove below (Theorem 3.2) the natural generalisation of this result to $\eta_k$ sets for $k \geqslant 1$. First a lemma in which we use the following notation. Fo[r] ... order type $\theta$ we denote by $\theta^*$ the reverse type (i.e. the order type of a ... obtained from one of type $\theta$ by inverting the order relation). If the ... red set $H$ of type $\theta$ is the union of disjoint sets $H_i$ of type $\theta_i$ we sa... at $\theta$ is a *shuffle* of the type $\theta_i$ and write $\theta = \bigcup_i \theta_i$. (Of course many different shuffles may result from a given set of types so that the symbol $\bigcup_i \theta_i$ is not uniquely defined.)

Lemma 3.1. *If $P, Q$ are ordered bases for an interval algebra $A$ of order-types $\alpha, \beta$ respectively, then there exist shuffles $\sigma, \tau$ such that*

$$\alpha \leqslant \sigma = \bigcup_{k<\omega} (\beta^*\beta)^k \quad \textit{and} \quad \beta \leqslant \tau = \bigcup_{k<\omega} (\alpha^*\alpha)^k.$$

Proof. Consider first a chain $C_1$ (ordered by inclusion) of single intervals of a set $P$ with type $P = \alpha$. By correlating the intervals $(x, y]$

of $P$ with the ordered pairs $(x, y)$ we can set up an order isomorphism between $C_1$ and a subset of $P\times P$ ordered by

$$(x, y) \leqslant (x_1, y_1) \text{ iff } x > x_1 \text{ or } x = x_1 \text{ and } y \leqslant y_1 .$$

Thus type $C_1 \leqslant a^*a$. Now for fixed $k<\omega$, let $C_k$ be a chain whose members are unions of $k$ disjoint intervals of the set $P$. Then $C_k$ can be embedded by an order isomorphism into the set of all $k$-tuples of a set of type $a^*a$ ordered by first differences, so that type $C_k \leqslant (a^*a)^k$. If now $P$ and $Q$ are as given in the statement of the lemma then each member of $Q$ is a finite union of intervals of $P$ which may be assumed to be disjoint. Thus $Q$ is the union of disjoint sets $D_k$ $(k = 1, 2, \ldots)$ where $D_k$ is the set of all members of $Q$ which are the union of precisely $k$ disjoint intervals and so type $D_k \leqslant$ type $C_k$. Thus type $Q \leqslant \tau$ where $\tau$ is a shuffle of the types $(a^*a)^k$ for $k = 1, 2, \ldots$ Similarly type $P \leqslant \sigma$ where $\sigma$ is constructed by a symmetric argument.

THEOREM 3.2. *If for a given ordinal $k \geqslant 1$ an ordered base for the interval algebra $A$ contains an $\eta_k$ set of power $\omega_k$ then any ordered base for $A$ contains an $\eta_k$ set.*

Proof. The following facts about $\eta_k$ sets of power $\omega_k$ are known (see e.g. [12], [6]):

(i) $(\eta_k)^j = \eta_k = \eta_k^*$ for $j<\omega$ and arbitrary $k$,

(ii) $\eta_k \to (\eta_k)^1_\omega$ for $k \geqslant 1$,

where in (ii) the symbol $\theta \to (\theta)^1_\omega$ means that if a set of type $\theta$ is the union of $\omega$ subsets, then at least one of these subsets contains a set of type $\theta$ (see [12] for further details.) Suppose now that $a, \beta$ are types of two bases for $A$ and $a \geqslant \eta_k$. By the lemma there exists a shuffle $\sigma$ such that

$$a \leqslant \sigma = \bigcup_{j<\omega} (\beta^*\beta)^j$$

and so by (ii) we have $\eta_k \leqslant (\beta^*\beta)^j$ for some $j<\omega$, which by (i) means that $(\eta_k^*\eta_k)^j \leqslant (\beta^*\beta)^j$ from which, by a further application of (i), we deduce that $\eta_k \leqslant \beta$.

The bound given in Lemma 3.1 is very crude and is moreover only useful when applied, as in Theorem 3.2, to types which satisfy the relation $\theta \to (\theta)^1_\omega$. For certain types e.g. the type $\lambda$ of the continuum and its natural extensions $\lambda_k$ for which the relation $\theta \to (\theta)^1_\omega$ fails (see [13; p. 318]) different arguments can be used which show that the bound in Lemma 3.1 is too wide. Thus if $A$ is the interval algebra on the reals and $P$ is any ordered base for $A$ then it is not difficult to show that type $P \leqslant \lambda$, and similarly for the types $\lambda_k$. Finally we observe that if an order theoretic property (P) of a base is equivalent to an algebraic one then,



of course, all bases will have property (P) if one base does; so that for example if one base for an interval algebra is dense then all bases will be dense, since an ordered base is dense iff the algebra is atomless.

### References


[1] P. Bacsich, *Injectivity in model theory* (to appear).
[2] K. Borsuk, *Theory of retracts*, Monografie Matematyczne 44, Warszawa 1967.
[3] S. Fajtlowicz and E. Marczewski, *On some properties of the family of independent sets in abstract algebras*, Coll. Math. 20 (1969), pp. 189–195.
[4] P. Erdös and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. 62 (1957), pp. 427–489.
[5] P. Halmos, *Lectures on Boolean algebras*, Van Nostrand, 1963.
[6] E. Harzheim, *Kombinatorische Batrachtungen über die struktur der Potenzmenge*, Math. Nachrichten 34 (1967), pp. 123–141.
[7] M. Mączyński, *A property of retracts of $a^+$-homogeneous, $a^+$-universal Boolean algebras*, Bull. Acad. Pol. Sci. 14 (1966), pp. 607–608.
[8] E. Marczewski, *Independence in algebras of sets and Boolean algebras*, Fund. Math. 48 (1960), pp. 135–145.
[9] R. D. Mayer and R. S. Pierce, *Boolean algebras with ordered bases*, Pacif. Jour. Math. 10 (1960), pp. 925–942.
[10] A. Mostowski and A. Tarski, *Boolesche Ringe mit geordneter Basis*, Fund. Math. 32 (1939), pp. 69–86.
[11] J. von Neumann and M. H. Stone, *The determination of representative elements in the residual classes of a Boolean algebra*, Fund. Math. 25 (1935), pp. 353–376.
[12] K. Padmavally, *Generalisation of the order-type of rational numbers*, Rev. Mat. Hisp. Am. 14 (1954), pp. 1–24.
[13] B. Rotman, *On the comparison of order types*, Acta Math. Acad. Sci. Hung. 19 (1968), pp. 311–27.
[14] R. Sikorski, *Boolean algebras*, (2nd edition), Springer–Verlag, 1964.
[15] L. B. Treybig, *Concerning continuous images of compact ordered spaces*, Proc. Amer. Math. Soc. 15 (1964), pp. 866–871.


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## ERRATA

| Page, ligne | Au lieu de | Lire |
|---|---|---|
| $107^9$ | $F_{1,0}$ | $F_{l,0}$ |
| $112_5$ | $q^ix$ | $\lvert q^ix \rvert$ |
| $133^9$ | 1871 | 1971 |
| $170^6$ | denote | denotes |

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# Errata to the paper "On some problems of Borsuk"

Fundamenta Mathematicae 73 (1972), pp. 271–274

by

**E. Barton** (Urbana, Ill.)

Page 272, line 10: replace $X \times 0$ by $X \times I$
and replace $h\widehat{|[t,1]}$ by $h|_{[t,1]}$.
Page 272, line 16: replace $h((0,1])$ by $\overline{h((0,1])}$.
Page 273, line 20: replace $h((0,1])$ by $\overline{h((0,1])}$.
Page 274, line 12: replace that by than.
Page 274, line 14: replace $1 \leqslant m < n$ by $1 < m < n$.