Cohomotopy groups and shape in the sense of Fox
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In [2] K. Borsuk introduced the relations of *fundamental domination* and *fundamental equivalence* in the class of compact metrizable spaces and proved that: (i) homotopy domination (equivalence) implies fundamental domination (equivalence), (ii) in the class of absolute neighbourhood retracts fundamental domination (equivalence) implies homotopy domination (equivalence). In [3] K. Borsuk introduced the notion of the *shape* of a compactum $X$; it is the collection of all compacta fundamentally equivalent to $X$. In [4] R. H. Fox extends the notion of shape to arbitrary metrizable spaces such that for compacta the extended notion coincides with Borsuk’s original notion of shape and the properties (i) and (ii) are preserved.

In [5] and [6] I proved that in the class of compacta cohomotopy groups are invariants of shape and that if a compactum $X$ fundamentally dominates a compactum $Y$ and there exists an $n$th cohomotopy group $\pi^n(X)$ of the compactum $X$, then there exists an $n$th cohomotopy group $\pi^n(Y)$ of the compactum $Y$ and $\pi^n(Y)$ is a divisor of $\pi^n(X)$.

The aim of this paper is to extend my results mentioned above to arbitrary metrizable spaces.

§ 1. Basic notions. In this section we recall the notions introduced by R. H. Fox in [4].

Consider an arbitrary category $E$ and let $\sim$ be a compositive equivalence relation on the collection Mor$E$ of morphisms of $E$. Two morphisms of $E$ are *concurrent* if they have the same domain and the same range. If $u_1, u_2 \in$ Mor$E$ are concurrent and if $u \in$ Mor$E$ is a morphism such that $u_1, u \sim u_2$, then $u$ is an *equalizer* of $u_1$ and $u_2$. An object $U \in$ Ob$E$ is a *predecessor* of an object $U' \in$ Ob$E$ in $E$ if there exists a morphism $u \in$ Mor$E$ with domain $U$ and range $U'$, $u: U \to U'$.

A subcategory $U$ of $E$ is called an *inverse system* if

(1.1) any two objects of $U$ have a common predecessor in $U$

and

(1.2) any two concurrent morphisms of $U$ have an equalizer in $U$. 

If $U$ and $V$ are inverse systems in $E$, then a mutation $f: U \to V$ in $E$ is called a collection of morphisms $f: U \to V$, where $f \in \text{Mor} E$, $U \in \text{Ob} U$, $V \in \text{Ob} V$, such that

(1.3) if $u \in \text{Mor} U$, $f \in f_1$, $v \in \text{Mor} V$ and $uf$ is defined, then $uf \in f_1$,

(1.4) every object of $V$ is the range of a morphism belonging to $f$,

(1.5) any two concurrent morphisms belonging to $f$ have an equalizer in $U$.

Morphisms belonging to a mutation $f$ are called constituents of $f$.

Consider two mutations $f: U \to V$ and $g: V \to W$ in $E$. The composition $gf$ of the mutations $f$ and $g$ is the collection of all compositions $gf$ such that $f \in f_1$ and $g \in g_1$ and $gf$ is defined. The composition of mutations is a mutation.

The collection $u = \text{Mor} U$ of all morphisms belonging to an inverse system $U$ is a mutation from $U$ to itself, $u: U \to U$, and $fu = f$ and $ug = g$ whenever these compositions are defined.

Two mutations $f, g: U \to V$ are similar (notation $f \sim g$) if

(1.6) concurrent morphisms $f \in f_1$ and $g \in g_1$ always have an equalizer in $U$.

Similarity of mutations is a reflexive, symmetric, transitive and composition relation.

Two inverse systems $U$ and $V$ in $E$ are of the same similarity type (notation $U \sim V$) if there exist mutations $f: U \to V$ and $g: V \to U$ such that $gf \sim u = \text{Mor} U$ and $fg \sim v = \text{Mor} V$. This relation is reflexive, symmetric and transitive.

Consider the category $\text{ANR}(3)$ of metrizable absolute neighbourhood retracts with continuous mappings and the relation of homotopy between mappings denoted by $\simeq$. If the mutations $f$ and $g$ are similar in the category $\text{ANR}(3)$ with the relation $\simeq$, then $f$ and $g$ are called homotopic mutations (notation $f \simeq g$). The similarity type of an inverse system $U$ in this category is called the homotopy type of $U$. By the Kuratowski–Wojdyslawski theorem (1), p. 79) any metrizable space $X$ can be considered as a closed subset of a space $P \in \text{Ob ANR}(3)$. By the first theorem of Hanner (11), p. 96) every open neighbourhood of $X$ in $P$ belongs to $\text{Ob ANR}(3)$. Therefore the set of all open neighbourhoods of $X$ in $P$ with inclusions is an inverse system in the category $\text{ANR}(3)$. It is called the complete neighbourhood system of $X$ in $P$ and denoted by $U(X, P)$.

Let $X$ and $Y$ be closed subsets of $\text{ANR}(3)$-spaces $P$ and $Q$, respectively, and let $f: X \to Y$ be a continuous mapping. Then there exist a $U \in \text{Ob} U(X, P)$ and a continuous mapping $f: U \to Q$ such that $f(x) = f(z)$ for $x \in X$. The mapping $f$ determines uniquely a mutation $f$.

$U(X, P) \to V(Y, Q)$ from the complete neighbourhood system $U(X, P)$ of $X$ in $P$ to the complete neighbourhood system $V(Y, Q)$ of $Y$ in $Q$; the constituents of $f$ are all mappings $g: f^{-1}(Y) \cap U \to V$ defined by $g(x) = f(x)$, where $U \in \text{Ob} U(X, P)$ and $V \in \text{Ob} V(Y, Q)$. Such a mutation $f$ is called an extension of the mapping $f$. It is easy to see that a composition $gf$ of extensions $f$ and $g$ of the mappings $f$ and $g$, respectively, is an extension of the composition $gf$ of the mappings $f$ and $g$.

By Theorem (3.2) of [4] the homotopy type of $U(X, P)$ does not depend on $f$ or on the manner in which $X$ is imbedded as a closed subset in $P$. The homotopy type of $U(X, P)$ is called the shape of $X$ and denoted by $\text{Sh} X$.

Two compacta $X$ and $Y$ are fundamentally equivalent ([2], p. 233) if and only if $\text{Sh} X = \text{Sh} Y$ ([4], Theorem (4.3)).

We shall say that the shape of $X$ is dominated by the shape of $Y$ (notation $\text{Sh} X \lesssim \text{Sh} Y$) if there exist mutations $f: U(X, P) \to V(Y, Q)$ and $g: V(Y, Q) \to U(X, P)$ such that $gf \lesssim u = \text{Mor} U$. The domination of shapes is well defined (i.e. it does not depend on the choice of $P$ and $Q$) by Theorem (4.3) of [4].

It follows by (1.6) and (1.7) of [4] that if $X$ and $Y$ are compacta, then $\text{Sh} X \lesssim \text{Sh} Y$ if and only if $X$ is fundamentally dominated by $Y$ ([2], p. 233).

§ 2. Mappings into $\text{ANR}(3)$-spaces and mutations of complete neighbourhood systems. Consider a metrizable space $X$ and an $\text{ANR}(3)$-space $Z$.

Suppose that $X$ is a closed subset of an $\text{ANR}(3)$-space $P$ considering the complete neighbourhood systems $U(X, P)$ and $W(Z, Z)$. The system $W(Z, Z)$ consists of only one object $Z$ and only one morphism which is an identity mapping on $Z$ (it is a so-called rudimentary system, cf. [4]).

Let us prove that

(2.1) Every mutation $f: U(X, P) \to W(Z, Z)$ is homotopic to a mutation $g: U(X, P) \to W(Z, Z)$, which is an extension of a mapping $g: X \to Z$.

Proof. Take an arbitrary mapping $f_0 \in f_1, f_2, U_0 \to Z, U_0 \in \text{Ob} U(X, P), L_0: U_0 \to W(Z, Z)$ be the mutation consisting of all restrictions $f_0 U$ where $U \subseteq U_0$ and $U \in \text{Ob} U(X, P)$. The mutation $g$ is an extension of the mapping $g = f_0 U: X \to Z$. It remains to prove that $g \simeq f$. Take arbitrary concurrent morphisms $f_1 \in f_2$ and $g_1 \in g_2, f_1, g_1: U_1 \to Z, U_1 \in \text{Ob} U(X, P)$. Obviously $U_1 \subseteq U_0$ and $g_1 = f_0 U_1$. Let $w_1: U_1 \to U_1$ be an inclusion. Then $g_1 = w_1$. Since $f_1 \in f_2$ and $u_1 \in \text{Mor} U(X, P)$, we have by (1.3) $f_0 u_1 f_2$. The constituents $f_0 u_1 f_2$ of $f$ are concurrent. By (1.3) they have an equalizer $u_2 \in \text{Mor} U(X, P)$. Therefore there exist a $U \in \text{Ob} U(X, P)$ and an inclusion map $w_2: U \to U_1$ such that $f_0 u_2 w_2 f_2$. Then
Hence \( \eta_{i}\eta_{n} \simeq f_{i}\eta_{n} \). Therefore the morphisms \( \eta_{i} \) and \( f_{i} \) have an equalizer in \( U(X, P) \); thus \( g \simeq f \) (see (1.6)) and the proof is finished.

(2.2) Let \( f, g : X \to Z \) be continuous mappings of a metrizable space \( X \) into an ANR(\( \mathfrak{M} \))-space \( Z \) and let \( f, g : U(X, P) \to W(Z, Z) \) be extensions of the mappings \( f \) and \( g \), respectively. If \( f \simeq g \) then \( f \simeq g \).

Proof. The mutation \( f \) consists of maps of the form \( \beta(U) \), where \( U \subseteq U_{1}, U_{2} \subseteq \text{Ob}(U(X, P)) \) and \( f : U_{1} \to Z \) is an extension of \( f \). The mutation \( g \) consists of maps of the form \( \gamma(U) \), where \( U \subseteq U_{2}, U_{3} \subseteq \text{Ob}(U(X, P)) \) and \( g : U_{3} \to Z \) is an extension of \( g \). Take \( U := \text{Ob}(U(X, P)) \) such that \( U \subseteq U_{1} \cap U_{2} \). Then \( \beta(U) \) and \( \gamma(U) \) are concurrent constituents of the mutations \( f \) and \( g \), respectively. Since \( f \simeq g \), then the morphisms \( f \mid U_{1} \) and \( g \mid U_{2} \) have an equalizer in \( U(X, P) \). Therefore there exists a \( U_{4} \subseteq \text{Ob}(U(X, P)) \) such that \( U_{4} \subseteq U_{1} \cap U_{2} \). Hence \( \beta(U) \simeq g(U) \), and thus \( f \simeq g \).

Consider closed subsets \( X \) and \( Y \) of ANR(\( \mathfrak{M} \))-spaces \( P \) and \( Q \), respectively, and an ANR(\( \mathfrak{M} \))-space \( Z \). Let \( f : U(X, P) \to V(Y, Q) \) be a mutation from the complete neighbourhood system \( U(X, P) \) to the complete neighbourhood system \( V(Y, Q) \). Let \( \varphi \) be a continuous mapping \( \varphi : Y \to Z \). Then \( \varphi \) is a continuous mapping \( \varphi : U(X, P) \to W(Z, Z) \). By (2.1) it is homotopic to a mutation \( \varphi' : U(X, P) \to W(Z, Z) \) which is an extension of a mapping \( \varphi' : X \to Z \). Therefore each mapping \( \varphi' : Y \to Z \) assigns a continuous mapping \( \varphi' : X \to Z \). This assignment is not unique, but the homotopy class \( [\varphi] \) of the mapping \( \varphi \) depends only on the homotopy class \( [\varphi] \) of the mapping \( \varphi \). Equivalently, \( [\varphi] \) depends only on the homotopy class \( [\varphi] \) of the mapping \( \varphi \).

(2.3) If \( \varphi \simeq \psi \) then \( \varphi \simeq \psi \) : \( X \to Z \).

Proof. Since \( \varphi \simeq \psi \), we have, by Theorem (3.1) of [4], \( \varphi \simeq \psi \). Therefore \( \varphi \simeq \psi \) and by (2.2) \( \varphi \simeq \psi \).

Let us denote by \( [\varphi] \) the set of homotopy classes of mappings \( \varphi \). By (2.3) we can assign to an arbitrary mutation \( f : U(X, P) \to V(Y, Q) \) an arbitrary ANR(\( \mathfrak{M} \))-space \( Z \) a function

\[
f_{\varphi} : [Z^{2}] \to [Z^{2}]
\]

defined by the formula \( f_{\varphi}(\varphi) = [\varphi] \). It will be called the function induced by the mutation \( f \).

It follows at once from the definition that

(2.4) If \( f \simeq g \), then \( f_{\varphi} \simeq g_{\varphi} \).

(2.5) If \( f \) is a composition of \( \varphi \), then \( \varphi f_{\varphi} = f_{\varphi} \).

(2.6) If \( \varphi = \text{Mor}(U(X, P)) \), then \( f_{\varphi} : [Z^{2}] \to [Z^{2}] \) is an identity function.

Let us observe that

(2.7) If \( f : U(X, P) \to V(Y, Q) \) and \( g : V(Y, Q) \to U(X, P) \) are mutations such that \( f \simeq g \) \( \text{Mor}(U(X, P)) \), then \( f_{\varphi} \simeq g_{\varphi} \) is a single-valued function.

Indeed, by (2.4), (2.5) and (2.6) \( f_{\varphi} \) is an identity function and hence we obtain (2.7).

§ 3. Homomorphisms of cohomotopy groups induced by mutations of complete neighbourhood systems. First we recall the definition of the \( n \)-th cohomotopy group of a space \( X \).

Let \( S = S^{n} \) be the \( n \)-dimensional sphere. Let us choose a point \( s_{0} \in S \) and consider the subset

\[
S \times S = (S \times \{s_{0}\}) \cup \{s_{0} \times S\}
\]

of the Cartesian product \( S \times S \). Let us define the mapping \( \Omega : S \times S \to S \) by the formula

\[
\Omega(s, s_{0}) = \Omega(s_{0}, s) = s \quad \text{for} \quad s \in S.
\]

Take two arbitrary continuous mappings \( \varphi, \psi : X \to S \). A continuous mapping

\[
\Phi : X \times \{0, 1\} \to S \times S
\]

such that

\[
\Phi(x, 0) = (\varphi(x), \psi(x)) \quad \text{and} \quad \Phi(x, 1) \in S \times S \quad \text{for} \quad x \in X
\]

is called a normalizing homotopy for the mappings \( \varphi \) and \( \psi \). Then the mapping \( \chi : X \to S \) defined by the formula \( \chi(x) = \Phi(x, 1) \) is said to be a normalizing of the mappings \( \varphi \) and \( \psi \) ([7], p. 210).

Let us suppose that a space \( X \) satisfies the following three conditions:

(3.1) For every two continuous mappings \( \varphi, \psi : X \to S \) there exists a normalizing homotopy.

(3.2) If \( \gamma \) is a normalizing of mappings \( \varphi \) and \( \psi \), then the homotopy class \( [\gamma] \) of the mapping \( \gamma(x) = \Phi(x, 1) \) depends only on the homotopy classes \( [\varphi] \) and \( [\psi] \).

(3.3) Addition in the set \( [S^{n}] \) defined by the formula \( [\varphi] + [\psi] = [\gamma] \), where \( \gamma \) is a normalization of the mappings \( \varphi \) and \( \psi \) makes the set \( [S^{n}] \) an Abelian group.

This group is called the \( n \)-th cohomotopy group of \( X \) and denoted by \( \pi_{n}(X) \). The addition defined in (3.3) is called the \( n \)-th cohomotopy addition. It
may be defined if the conditions (3.1) and (3.2) are satisfied. In this case we say that the space $X$ admits the $n$-th cohomotopy addition. Moreover, if the condition (3.3) is also satisfied, then we say that the space $X$ admits the existence of the $n$-th cohomotopy group.

Since $S \in ANR$, for arbitrary metrisable spaces $X$ and $Y$ any mapping $f: U(X, P) \rightarrow V(Y, Q)$ from a complete neighbourghood system $U(X, P)$ to a complete neighbourghood system $V(Y, Q)$ induces the function $f^\Theta: [S^2] \rightarrow [S^2]$.

Let us prove the following.

(3.4) **Lemma.** Suppose that the metrisable spaces $X$ and $Y$ both admit the $n$-th cohomotopy addition and let $f: U(X, P) \rightarrow V(Y, Q)$ be a mutation from a complete neighbourghood system $U(X, P)$ to a complete neighbourghood system $V(Y, Q)$. Then for every two continuous mappings $\varphi, \psi: Y \rightarrow S$ we have $f^\Theta([\varphi]) + f^\Theta([\psi]) = f^\Theta([\varphi + \psi])$.

Proof. Let $\Phi: X \times [0, 1] \rightarrow S \times S$ be a normalizing homotopy for the mappings $\varphi$ and $\psi$. Then $\Phi(y, 0) = [\varphi(y), \psi(y)]$ and $\Phi(y, 1) = [\varphi(y) + \psi(y)]$ for $y \in Y$. The mapping $x: Y \rightarrow S \times S$ defined by the formula $x(y) = \Phi(y, 1)$ is a normalizing of the mappings $\varphi$ and $\psi$. Therefore

$$[\varphi] + [\psi] = [x(y)] = [x_1(y)].$$

Since $S, S \times S, S \times S \in ANR$, there exists a neighbourghood $V_0 \in Ob V(Y, Q)$ such that the following two conditions are satisfied:

(3.6) There exist extensions $\varphi_0: V_0 \rightarrow S$, $\varphi_1: V_0 \rightarrow S \times S$ of the mappings $\varphi, \psi, x$, respectively.

(3.7) There exists an extension $\Phi_0: V_0 \times [0, 1] \rightarrow S \times S$ of the homotopy $\Phi$ such that

$$\Phi_0(x, 0) = (\varphi_0(x), \psi_0(x)) \text{ and } \Phi_0(x, 1) = (\varphi_1(x), \psi_1(x))$$

for $y \in V_0$.

Since $f: U(X, P) \rightarrow V(Y, Q)$ is a mutation, there exists a $U_0 \in Ob U(X, P)$ and $f_0 \in f$ such that $f_0: U_0 \rightarrow V_0$. Hence

$$\varphi_0 U_0 \rightarrow S,$$

$$\psi_0 U_0 \rightarrow S \times S.$$

Let

$$\varphi' = \varphi_0 U_0: X \rightarrow S,$$

$$\psi' = \varphi_0 U_0: X \rightarrow S \times S,$$

$$\gamma' = \gamma_0 U_0: X \rightarrow S \times S.$$

We shall prove that

(3.9) $f^\Theta([\varphi']) = [\varphi'], f^\Theta([\psi']) = [\psi'], f^\Theta([\gamma']) = [\gamma']$.

Let $\psi: V(Y, Q) \rightarrow W(S, B)$ be the mutation consisting of all mappings of the form $\varphi_0 U$, where $\psi \in Ob V(Y, Q) \text{ and } V \subset \psi V_0$. The mutation $\psi$ is an extension of the mapping $\psi$. Let $\varphi': U(X, P) \rightarrow W(S, B)$ be the mutation consisting of all mappings of the form $\varphi_0 U$, where $U \in Ob U(X, P)$ and $V \subset \psi V_0$. Consider the mutation $\varphi: U(X, P) \rightarrow W(S, B)$. Let us observe that $\varphi_0 U \in \psi_0 U$ and $\psi_0 U \in \varphi'$. We show that $\varphi \equiv \varphi'$. Take two arbitrary concurrent constituents $\varphi_0 U \rightarrow S$ and $\varphi_0 U \rightarrow S$ of the mutations $\varphi$ and $\varphi'$, respectively. The mappings $f: U \rightarrow V$ and $f_0 U: U \rightarrow V$ are constituents of the mutation $f$. Therefore by (1.3) there exists a $U \in Ob U(X, P)$ such that $f_0 U = f_0 U$. Since $\varphi = \varphi_0 V$, where $V$ is the range of $f$, we have $\varphi_0 U = \varphi_0 U$. Therefore the morphisms $f_0 U$ and $\varphi_0 U$ have an equalizer. Thus $\varphi \equiv \varphi'$. Hence by the definition of $f^\Theta$ we obtain $f^\Theta([\varphi]) = [\varphi']$. Analogously one can prove the remaining two conditions (3.9).

Let us define the mapping

$$\Phi': X \times [0, 1] \rightarrow S \times S$$

by the formula

$$\Phi'(x, t) = \varphi_0 f(x, t) \text{ for } x \in X \text{ and } 0 \leq t \leq 1.$$

Then by (3.7) and (3.8)

$$\Phi'(x, 0) = \varphi_0 f_0(x, 0) = \varphi_0 f_0(x),$$

$$\Phi'(x, 1) = \varphi_0 f_0(x, 1) = \gamma_0 f_0(x) = \gamma(x) \text{ for } x \in X.$$

Hence

(3.10) $[\varphi'] + [\psi'] = [\gamma']$.

From (3.5), (3.9) and (3.10) we obtain Lemma (3.4).

Lemma (3.4) implies at once the following

(3.11) **Theorem.** If metrisable spaces $X$ and $Y$ admit the existence of the $n$-th cohomotopy group $\pi^n(X)$ and $\pi^n(Y)$ and $f: U(X, P) \rightarrow V(Y, Q)$ is a mutation from a complete neighbourghood system $U(X, P)$ to a complete neighbourghood system $V(Y, Q)$, then the induced function $f^\Theta: \pi^n(Y) \rightarrow \pi^n(X)$ is a homomorphism.

§ 4. Cohomotopy groups and the shape of metrisable spaces. Let us prove the following

(4.1) **Theorem.** Suppose $X$ and $Y$ are metrisable spaces such that $Sh X \subseteq Sh Y$. If the space $X$ admits the existence of the $n$-th cohomotopy group $\pi^n(Y)$, then the space $X$ admits the existence of the $n$-th cohomotopy group $\pi^n(X)$ and $\pi^n(X) \subseteq \pi^n(Y)$, i.e. the group $\pi^n(X)$ is a divisor of the group $\pi^n(Y)$.

Proof. By hypothesis there exist mutations $f: U(X, P) \rightarrow V(Y, Q)$ and $g: V(Y, Q) \rightarrow U(X, P)$ such that $gf \equiv u = Mor U(X, P)$.
Take two arbitrary continuous mappings \( \varphi, \psi: X \to S = S^\alpha \). Since \( S \in \text{ANR} \) then there exist \( U_\alpha \subseteq \text{Ob}(U(X, P)) \) and continuous extensions \( \varphi_\alpha, \psi_\alpha: U_\alpha \to S \) of the mappings \( \varphi \) and \( \psi \), respectively. Since \( g \) is a mutation, by (1.4) there exist \( V_\alpha \subseteq \text{Ob}(V(Y, Q)) \) and \( g_\alpha \in g \) such that \( g_\alpha: V_\alpha \to U_\alpha \). Hence \( \varphi_\alpha g_\alpha: V_\alpha \to S \) and \( \psi_\alpha g_\alpha: V_\alpha \to S \). Let \( \varphi' = \varphi_\alpha g_\alpha: X \to S \) and \( \psi' = \psi_\alpha g_\alpha: X \to S \). By hypothesis for the mappings \( \varphi' \) and \( \psi' \) there exists a normalizing homotopy
\[
\Phi: Y \times (0, 1) \to S \times S.
\]
Then for \( y \in Y \)
\[
\Theta(y, 0) = (\varphi'(y), \psi'(y)) = (\varphi_\alpha g_\alpha(y), \psi_\alpha g_\alpha(y)), \quad (y, 1) \in S \times S.
\]
Since \( S \times S \subseteq \text{ANR} \), there exists a continuous extension
\[
\Theta: V_1 \times (0, 1) \to S \times S, \quad V_1 \subseteq V_\alpha, \quad V_1 \subseteq \text{Ob}(V(Y, Q))
\]
of the homotopy \( \Phi \) such that
\[
\Theta(y, 0) = (\varphi_\alpha g_\alpha(y), \psi_\alpha g_\alpha(y)) \quad \text{for} \quad y \in V_1.
\]
(4.2)
Then for \( y \in Y \)
\[
\Theta(y, 1) \in S \times S
\]
\[
\Theta(y, 1) = (\varphi_\alpha g_\alpha(y), \psi_\alpha g_\alpha(y)) \quad \text{for} \quad y \in V_1.
\]
(4.3)
Take a constituent \( f_1 \in f \) whose range is \( V_1; f_1: U_1 \to V_1, \quad f_1 \subseteq \text{Ob}(U(X, P)) \). Let \( v_1: V_1 \to V_\alpha \) be the inclusion mapping and put \( f_1 = f_1: U_1 \to V_\alpha, f_1 \subseteq f \). Then \( f_1(X) \subseteq V_1, \varphi_\alpha g_\alpha: V_1 \to U_\alpha, g_\alpha \subseteq g \). Since \( g \preceq u \), by (1.6) there exists a \( U_\alpha \subseteq \text{Ob}(U(X, P)) \) such that \( U_\alpha \subseteq U_\alpha \subseteq V_1 \) and \( g_\alpha \subseteq g_\alpha \subseteq U_\alpha \). Hence \( \varphi_\alpha g_\alpha \circ u \chi : X \to U_\alpha \). Therefore there exists a homotopy
\[
H: X \times (0, 1) \to U_\alpha
\]
such that
\[
H(x, 0) = x \quad \text{and} \quad H(x, 1) = \varphi_\alpha g_\alpha(x).
\]
Let us define the continuous mapping
\[
G: X \times (0, 1) \to \text{Ob}(U(X, P))
\]
by the formula
\[
G(x, t) = (\varphi_\alpha H(x, t), \psi_\alpha H(x, t)) \quad \text{for} \quad x \in X \text{ and } 0 \leq t \leq 1.
\]
Then for \( x \in X \)
\[
G(x, 0) = (\varphi_\alpha H(x, 0), \psi_\alpha H(x, 0)) = (\varphi(x), \psi(x)),
\]
\[
G(x, 1) = (\varphi_\alpha H(x, 1), \psi_\alpha H(x, 1)) = (\varphi_\alpha g_\alpha(x), \psi_\alpha g_\alpha(x)).
\]
Let us define the mapping
\[
\Psi: X \times (0, 1) \to \text{Ob}(U(X, P))
\]
by the formula
\[
\Psi(x, t) = \begin{cases} G(x, 2t) & \text{for} \quad 0 \leq t \leq \frac{1}{2}, \\ G(x, 2t-1) & \text{for} \quad \frac{1}{2} \leq t \leq 1. \end{cases}
\]
By (4.2) and (4.3) the mapping \( \Psi \) is well defined and continuous and it is a normalizing homotopy for the mappings \( \varphi \) and \( \psi \). Therefore
\[
(4.1) \quad \text{For every two continuous mappings} \varphi, \psi: X \to S \text{ there exists a normalizing homotopy.}
\]
We shall prove that
\[
(4.5) \quad \text{If} \quad \varphi: X \to S \subseteq S \text{ is a normalization of the mappings} \varphi, \psi: X \to S, \text{ then the homotopy class} \{\varphi_\gamma\} \text{ depends only on the homotopy classes} \{\varphi\} \text{ and} \{\psi\}.
\]
By (2.7) it follows that
\[
(4.6) \quad g^{\psi_\alpha}(x) = [\varphi_\alpha] \text{ and } g^{\psi_\alpha}(x) = [\varphi].
\]
Let \( \varphi, \psi: Y \to S \) be continuous mappings such that
\[
(4.7) \quad g^{\psi_\gamma}(x) = [\varphi_\gamma] \text{ and } g^{\psi_\gamma}(x) = [\varphi].
\]
By the hypotheses there exists a normalization \( \varphi_\gamma: X \to S \subseteq S \) of the mappings \( \varphi \) and \( \psi \) and the homotopy class \{\varphi_\gamma\} depends only on the homotopy classes \{\varphi\} and \{\psi\}. In order to prove (4.5) it suffices by (4.6) and (4.7) to show that
\[
(4.8) \quad g^{\psi_\gamma}(x) = [\varphi_\gamma] \text{ and } g^{\psi_\gamma}(x) = [\varphi].
\]
Since \( \varphi_\gamma: X \to S \subseteq S \) is a normalization of the mappings \( \varphi \) and \( \psi \), then there exists a normalizing homotopy
\[
\Phi: X \times (0, 1) \to S \times S
\]
such that
\[
\Phi(x, 0) = (\varphi(x), \psi(x)) \quad \text{and} \quad \Phi(x, 1) = \varphi_\gamma(x) \quad \text{for} \quad x \in X.
\]
Since \( S \subseteq S \subseteq S \subseteq \text{ANR} \), then there exists a neighbourhood \( U \subseteq \text{Ob}(U(X, P)) \) such that the following two conditions are satisfied:
\[
(4.9) \quad \text{There exist continuous extensions} \quad \varphi: U \to S, \psi: U \to S, \quad \varphi_\gamma: U \to S \subseteq S \quad \text{of the mappings} \varphi, \psi, \varphi_\gamma, \text{ respectively.}
\]
\[
(4.10) \quad \text{There exists a continuous extension} \quad \varphi: U \times (0, 1) \to S \times S \quad \text{of the homotopy} \Phi \quad \text{such that}
\]
\[
\varphi(x, 0) = (\varphi(x), \varphi_\gamma(x)) \quad \text{and} \quad \varphi(x, 1) = \varphi_\gamma(x) \quad \text{for} \quad x \in U.
\]
Since \( g: V(Y, Q) \to U(X, P) \) is a mutation, there exist \( v \in \text{Ob}(V(Y, Q)) \) and \( g \in g \) such that \( v \in U \). Then \( g_\gamma, g_\gamma: V \to S, g_\gamma: V \to S \subseteq S \). Let us put
\[
(1.11) \quad \varphi' = \varphi_\gamma, \psi' = \varphi_\gamma, \chi' = \varphi_\gamma.
\]
Then we have

\[(4.12) \quad g^R([\varphi]) = [\varphi], \quad g^R([\varphi]) = [\varphi'], \quad g^R([\varphi]) = [\varphi'].
\]

The proof of (4.12) is precisely the same as the proof of (3.9).

Let us define the continuous mapping

\[\psi: Y \times (0,1) \rightarrow S \times S\]

by the formula

\[\psi(y, t) = \rho(y(t), t) \quad \text{for} \quad y \in Y \text{ and } 0 < t < 1.
\]

Then by (4.10) and (4.11) we obtain

\[\psi(y, 0) = \rho(y(0), 0) = \rho(y(y), y(y)) = \rho(y', y(y)),
\]

\[\psi(y, 1) = \rho(y(y), 1) = \rho(y) = \rho(y).
\]

It follows that \( \chi' \) is a normalization of the mappings \( \varphi' \) and \( \varphi'. \)

By (4.7) and (4.12) we have

\[(4.13) \quad [\varphi] = [\varphi] \quad \text{and} \quad [\varphi] = [\varphi].
\]

Since the homotopy class \([g^R]\) depends only on the homotopy classes \([\varphi]\) and \([\varphi]\) and the homotopy class \([\varphi']\) depends only on the homotopy classes \([\varphi']\) and \([\varphi']\), by (4.13) we obtain \([g^R] = [\varphi]. \)

Hence by (4.12) we obtain (4.8) and the proof of (4.5) is completed.

By (4.4) and (4.5) the space \(X\) admits the \(n\)th cohomotopy addition. Therefore by Lemma (3.4) for arbitrary continuous mappings \(\varphi, \psi: X \rightarrow S\) and \(\varphi, \psi: Y \rightarrow S\) we have

\[(4.14) \quad g^R([\varphi] + [\psi]) = g^R([\varphi]) + g^R([\psi]),
\]

\[(4.15) \quad f^R([\varphi] + [\psi]) = f^R([\varphi]) + f^R([\psi]).
\]

Since \(g^R\) is a single-valued function and the \(n\)th cohomotopy addition in the group \(\pi^n(X)\) is associative and commutative, by (4.14) it follows that

\[(4.16) \quad \text{The} \ n\text{th cohomotopy addition in the set} [S^X] \text{ is associative and commutative.}
\]

Let \(\alpha\) be the zero of the group \(\pi^n(Y)\) and let \(\alpha = f^R([\alpha]).\) Then

\[(4.17) \quad g^R([\alpha] + [\psi]) = [\psi] \quad \text{for an arbitrary continuous mapping} \psi: X \rightarrow S.
\]

Indeed, let \(\psi = g^R([\psi]).\) Then \(\alpha = [\psi].\) Hence by (4.15) \(f^R([\alpha] + [\psi]) = f^R([\psi]).\) But \(f^R([\alpha]) = [\alpha]\) and \(f^R([\psi]) = f^R([\psi]) = [\psi].\) Hence we obtain (4.17).

It is easy to see that

\[(4.18) \quad \text{For an arbitrary continuous mapping} \psi: X \rightarrow S \text{ there exists a continuous mapping} \psi: X \rightarrow S \text{ such that} [\psi] + [\psi] = [\alpha].
\]

Indeed, if \(\psi: Y \rightarrow S\) is a mapping such that \(g^R([\psi]) = [\psi]\), then by (4.15) an arbitrary mapping \(\psi: f^R([\psi]) = [\psi]\) satisfies the required condition.

From (4.16), (4.17) and (4.18) it follows that the space \(X\) admits the existence of the \(n\)th cohomotopy group \(\pi^n(X)\).

By Theorem (3.11) the functions \(f^R: \pi^n(X) \rightarrow \pi^n(X)\) and \(g^R: \pi^n(X) \rightarrow \pi^n(X)\) are homomorphisms and since \(gf \sim u\), by (2.4), (2.5) and (2.6) \(g^R g^R: \pi^n(X) \rightarrow \pi^n(X)\) is an identity function. Therefore \(\pi^n(X) \leq \pi^n(X)\).

Thus, the proof of Theorem (4.1) is completed.

Theorems (3.11) and (4.1) and also (2.4), (2.5) and (2.6) imply at once the following

\[(4.19) \quad \text{Theorem. Suppose} X \text{ and} Y \text{ are metrizable spaces with} \ Sh X = Sh Y. \text{ If the space} X \text{ admits the existence of the} n\text{-th cohomotopy group} \pi^n(X), \text{ then the space} Y \text{ admits the existence of the} n\text{-th cohomotopy group} \pi^n(Y) \text{ and the groups} \pi^n(X) \text{ and} \pi^n(Y) \text{ are isomorphic, i.e. cohomotopy groups are invariants of shape.}
\]

References


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Recu par la Redaction le 22. 2. 1971

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