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Cohomotopy groups and shape in the sense of Fox

by

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In [2] K. Borsuk introduced the relations of *fundamental domination* and *fundamental equivalence* in the class of compact metrizable spaces and proved that: (i) homotopy domination (equivalence) implies fundamental domination (equivalence), (ii) in the class of absolute neighbourhood retracts fundamental domination (equivalence) implies homotopy domination (equivalence). In [3] K. Borsuk introduced the notion of the *shape* of a compactum X; it is the collection of all compacta fundamentally equivalent to X. In [4] R. H. Fox extends the notion of shape to arbitrary metrizable spaces such that for compacta the extended notion coincides with Borsuk's original notion of shape and the properties (i) and (ii) are preserved.

In [5] and [6] I proved that in the class of compacta cohomotopy groups are invariances of shape and that if a compactum X fundamentally dominates a compactum Y and there exists an nth cohomotopy group $\pi^n(X)$ of the compactum X, then there exists an nth cohomotopy group $\pi^n(Y)$ of the compactum Y and $\pi^n(Y)$ is a divisor of $\pi^n(X)$.

The aim of this paper is to extend my results mentioned above to arbitrary metrizable spaces.

§ 1. Basic notions. In this section we recall the notions introduced by R. H. Fox in [4].

Consider an arbitrary category E and let $\sim$ be a compositive equivalence relation on the collection $\mathrm{Mor}E$ of morphisms of E. Two morphisms of E are *concurrent* if they have the same domain and the same range. If $u_1, u_2 \in \mathrm{Mor}E$ are concurrent and if $u \in \mathrm{Mor}E$ is a morphism such that $u_1 u \sim u_2 u$, then u is an *equalizer* of u_1 and u_2. An object $U \in \mathrm{Ob}E$ is a *predecessor* of an object $U' \in \mathrm{Ob}E$ in E if there exists a morphism $u \in \mathrm{Mor}E$ with domain U and range U', $u\colon U \to U'$.

A subcategory $\boldsymbol{U}$ of E is called an *inverse system* if

(1.1) any two objects of $\boldsymbol{U}$ have a common predecessor in $\boldsymbol{U}$

and

(1.2) any two concurrent morphisms of $\boldsymbol{U}$ have an equalizer in $\boldsymbol{U}$.

If $\boldsymbol{U}$ and $\boldsymbol{V}$ are inverse systems in E, then a *mutation* $\boldsymbol{f}\colon \boldsymbol{U}\to\boldsymbol{V}$ in E is called a collection of morphisms $f\colon U\to V$, where $f\in \operatorname{Mor} E$, $U\in \operatorname{Ob}\boldsymbol{U}$, $V\in\operatorname{Ob}\boldsymbol{V}$, such that

(1.3) if $u\in\operatorname{Mor}\boldsymbol{U}$, $f\in\boldsymbol{f}$, $v\in\operatorname{Mor}\boldsymbol{V}$ and vfu is defined, then $vfu\in\boldsymbol{f}$,

(1.4) every object of $\boldsymbol{V}$ is the range of a morphism belonging to $\boldsymbol{f}$,

(1.5) any two concurrent morphisms belonging to $\boldsymbol{f}$ have an equalizer in $\boldsymbol{U}$.

Morphisms belonging to a mutation $\boldsymbol{f}$ are called *constituents* of $\boldsymbol{f}$.

Consider two mutations $\boldsymbol{f}\colon\boldsymbol{U}\to\boldsymbol{V}$ and $\boldsymbol{g}\colon\boldsymbol{V}\to\boldsymbol{W}$ in E. The *composition* $\boldsymbol{gf}$ of the mutations $\boldsymbol{f}$ and $\boldsymbol{g}$ is the collection of all compositions gf such that $f\in\boldsymbol{f}$ and $g\in\boldsymbol{g}$ and gf is defined. The composition of mutations is a mutation.

The collection $\boldsymbol{u}=\operatorname{Mor}\boldsymbol{U}$ of all morphisms belonging to an inverse system $\boldsymbol{U}$ is a mutation from $\boldsymbol{U}$ to itself, $\boldsymbol{u}\colon\boldsymbol{U}\to\boldsymbol{U}$, and $\boldsymbol{fu}=\boldsymbol{f}$ and $\boldsymbol{ug}=\boldsymbol{g}$ whenever these compositions are defined.

Two mutations $\boldsymbol{f},\boldsymbol{g}\colon\boldsymbol{U}\to\boldsymbol{V}$ are *similar* (notation $\boldsymbol{f}\sim\boldsymbol{g}$) if

(1.6) concurrent morphisms $f\in\boldsymbol{f}$ and $g\in\boldsymbol{g}$ always have an equalizer in $\boldsymbol{U}$.

Similarity of mutations is a reflexive, symmetric, transitive and compositive relation.

Two inverse systems $\boldsymbol{U}$ and $\boldsymbol{V}$ in E are of the same *similarity type* (notation $\boldsymbol{U}\sim\boldsymbol{V}$) if there exist mutations $\boldsymbol{f}\colon\boldsymbol{U}\to\boldsymbol{V}$ and $\boldsymbol{g}\colon\boldsymbol{V}\to\boldsymbol{U}$ such that $\boldsymbol{gf}\sim\boldsymbol{u}=\operatorname{Mor}\boldsymbol{U}$ and $\boldsymbol{fg}\sim\boldsymbol{v}=\operatorname{Mor}\boldsymbol{V}$. This relation is reflexive, symmetric and transitive.

Consider the category $\operatorname{ANR}(\mathfrak{M})$ of metrizable absolute neighbourhood retracts with continuous mappings and the relation of homotopy between mappings denoted by $\simeq$. If the mutations $\boldsymbol{f}$ and $\boldsymbol{g}$ are similar in the category $\operatorname{ANR}(\mathfrak{M})$ with the relation $\simeq$, then $\boldsymbol{f}$ and $\boldsymbol{g}$ are called *homotopic* mutations (notation $\boldsymbol{f}\simeq\boldsymbol{g}$). The similarity type of an inverse system $\boldsymbol{U}$ in this category is called the *homotopy type* of $\boldsymbol{U}$. By the Kuratowski–Wojdysławski theorem ([1], p. 79) any metrizable space X can be considered as a closed subset of a space $P\in\operatorname{Ob}\operatorname{ANR}(\mathfrak{M})$. By the first theorem of Hanner ([1], p. 96) every open neighbourhood of X in P belongs to $\operatorname{Ob}\operatorname{ANR}(\mathfrak{M})$. Therefore the set of all open neighbourhoods of X in P with inclusions is an inverse system in the category $\operatorname{ANR}(\mathfrak{M})$. It is called the *complete neighbourhood system* of X in P and denoted by $\boldsymbol{U}(X,P)$.

Let X and Y be closed subsets of $\operatorname{ANR}(\mathfrak{M})$-spaces P and Q, respectively, and let $f\colon X\to Y$ be a continuous mapping. Then there exist a $U\in\operatorname{Ob}\boldsymbol{U}(X,P)$ and a continuous mapping $\hat f\colon U\to Q$ such that $\hat f(x)=f(x)$ for $x\in X$. The mapping $\hat f$ determines uniquely a mutation $\boldsymbol{f}\colon$ $\boldsymbol{U}(X,P)\to\boldsymbol{V}(Y,Q)$ from the complete neighbourhood system $\boldsymbol{U}(X,P)$ of X in P to the complete neighbourhood system $\boldsymbol{V}(Y,Q)$ of Y in Q; the constituents of $\boldsymbol{f}$ are mappings $g\colon \hat f^{-1}(V)\cap U\to V$ defined by $g(x)=\hat f(x)$, where $U\in\operatorname{Ob}\boldsymbol{U}(X,P)$ and $V\in\operatorname{Ob}\boldsymbol{V}(Y,Q)$. Such a mutation $\boldsymbol{f}$ is called an *extension* of the mapping f. It is easy to see that a composition $\boldsymbol{gf}$ of extensions $\boldsymbol{f}$ and $\boldsymbol{g}$ of the mappings f and g, respectively, is an extension of the composition gf of the mappings f and g.

By Theorem (3.2) of [4] the homotopy type of $\boldsymbol{U}(X,P)$ does not depend either on P or on the manner in which X is imbedded as a closed subset in P. The homotopy type of $\boldsymbol{U}(X,P)$ is called the *shape* of X and denoted by $\operatorname{Sh}X$.

Two compacta X and Y are fundamentally equivalent ([2], p. 233) if and only if $\operatorname{Sh}X=\operatorname{Sh}Y$ ([4], Theorem (4.3)).

We shall say that the shape of X is *dominated* by the shape of Y (notation $\operatorname{Sh}X\leqslant\operatorname{Sh}Y$) if there exist mutations $\boldsymbol{f}\colon\boldsymbol{U}(X,P)\to\boldsymbol{V}(Y,Q)$ and $\boldsymbol{g}\colon\boldsymbol{V}(Y,Q)\to\boldsymbol{U}(X,P)$ such that $\boldsymbol{gf}\simeq\boldsymbol{u}=\operatorname{Mor}\boldsymbol{U}$. The domination of shapes is well defined (i.e. it does not depend on the choice of P and Q) by Theorem (4.3) of [4].

It follows by (4.6) and (4.7) of [4] that if X and Y are compacta, then $\operatorname{Sh}X\leqslant\operatorname{Sh}Y$ if and only if X is fundamentally dominated by Y ([2], p. 233).

§ 2. Mappings into $\operatorname{ANR}(\mathfrak{M})$-spaces and mutations of complete neighbourhood systems. Consider a metrizable space X and an $\operatorname{ANR}(\mathfrak{M})$-space Z. Suppose that X is a closed subset of an $\operatorname{ANR}(\mathfrak{M})$-space P and consider the complete neighbourhood systems $\boldsymbol{U}(X,P)$ and $\boldsymbol{W}(Z,Z)$. The system $\boldsymbol{W}(Z,Z)$ consists of only one object Z and only one morphism which is an identity mapping on Z (it is a so-called rudimentary system, cf. [4]).

Let us prove that

(2.1) *Every mutation $\boldsymbol{f}\colon\boldsymbol{U}(X,P)\to\boldsymbol{W}(Z,Z)$ is homotopic to a mutation $\boldsymbol{g}\colon\boldsymbol{U}(X,P)\to\boldsymbol{W}(Z,Z)$, which is an extension of a mapping $g\colon X\to Z$.*

Proof. Take an arbitrary mapping $f_0\in\boldsymbol{f}$; $f_0\colon U_0\to Z$, $U_0\in\operatorname{Ob}\boldsymbol{U}(X,P)$. Let $\boldsymbol{g}\colon\boldsymbol{U}(X,P)\to\boldsymbol{W}(Z,Z)$ be the mutation consisting of all restrictions $f_0|U$ where $U\subset U_0$ and $U\in\operatorname{Ob}\boldsymbol{U}(X,P)$. The mutation $\boldsymbol{g}$ is an extension of the mapping $g=f_0|X\colon X\to Z$. It remains to prove that $\boldsymbol{g}\simeq\boldsymbol{f}$. Take arbitrary concurrent morphisms $f_1\in\boldsymbol{f}$ and $g_1\in\boldsymbol{g}$; $f_1,g_1\colon U_1\to Z$, $U_1\in\operatorname{Ob}\boldsymbol{U}(X,P)$. Obviously $U_1\subset U_0$ and $g_1=f_0|U_1$. Let $u_1\colon U_1\to U_0$ be an inclusion. Then $g_1=f_0u_1$. Since $f_0\in\boldsymbol{f}$ and $u_1\in\operatorname{Mor}\boldsymbol{U}(X,P)$, we have by (1.3) $f_0u_1\in\boldsymbol{f}$. The constituents f_0u_1 and f_1 of $\boldsymbol{f}$ are concurrent. By (1.5) they have an equalizer $u_2\in\operatorname{Mor}\boldsymbol{U}(X,P)$. Therefore there exist a $U\in\operatorname{Ob}\boldsymbol{U}(X,P)$ and an inclusion map $u_2\colon U_2\to U_1$ such that $f_0u_1u_2\simeq f_1u_2$.

Hence $g_1 u_2 \simeq f_1 u_2$. Therefore the morphisms g_1 and f_1 have an equalizer in $\boldsymbol{U}(X,P)$; thus $\boldsymbol{g} \simeq \boldsymbol{f}$ (see (1.6)) and the proof is finished.

(2.2) *Let $f, g\colon X \to Z$ be continuous mappings of a metrizable space X into an* ANR$(\mathfrak{M})$*-space Z and let $\boldsymbol{f}, \boldsymbol{g}\colon \boldsymbol{U}(X,P) \to \boldsymbol{W}(Z,Z)$ be extensions of the mappings f and g, respectively. If $\boldsymbol{f} \simeq \boldsymbol{g}$ then $f \simeq g$.*

Proof. The mutation $\boldsymbol{f}$ consists of maps of the form $\hat{f}\,|\,U$, where $U \subset U_1$, $U, U_1 \in \mathrm{Ob}\boldsymbol{U}(X,P)$ and $\hat{f}\colon U_1 \to Z$ is an extension of f. The mutation $\boldsymbol{g}$ consists of maps of the form $\hat{g}\,|\,U$, where $U \subset U_2$, $U, U_2 \in \mathrm{Ob}\boldsymbol{U}(X,P)$ and $\hat{g}\colon U_2 \to Z$ is an extension of g. Take $U_3 \in \mathrm{Ob}\boldsymbol{U}(X,P)$ such that $U_3 \subset U_1 \cap U_2$. Then $\hat{f}\,|\,U_3$ and $\hat{g}\,|\,U_3$ are concurrent constituents of the mutations $\boldsymbol{f}$ and $\boldsymbol{g}$, respectively. Since $\boldsymbol{f} \simeq \boldsymbol{g}$, then the morphisms $\hat{f}\,|\,U_3$ and $\hat{g}\,|\,U_3$ have an equalizer in $\boldsymbol{U}(X,P)$. Therefore there exists a $U_4 \in \mathrm{Ob}\boldsymbol{U}(X,P)$ such that $U_4 \subset U_3$ and $\hat{f}\,|\,U_4 \simeq \hat{g}\,|\,U_4$. Hence $\hat{f}\,|\,X \simeq \hat{g}\,|\,X$, thus $f \simeq g$.

Consider closed subsets X and Y of ANR$(\mathfrak{M})$-spaces P and Q, respectively, and an ANR$(\mathfrak{M})$-space Z. Let $\boldsymbol{f}\colon \boldsymbol{U}(X,P) \to \boldsymbol{V}(Y,Q)$ be a mutation from the complete neighbourhood system $\boldsymbol{U}(X,P)$ to the complete neighbourhood system $\boldsymbol{V}(Y,Q)$. Take an arbitrary continuous mapping $\varphi\colon Y \to Z$. Let $\boldsymbol{\varphi}\colon \boldsymbol{V}(Y,Q) \to \boldsymbol{W}(Z,Z)$ be an extension of φ. Consider the mutation $\boldsymbol{\varphi f}\colon \boldsymbol{U}(X,P) \to \boldsymbol{W}(Z,Z)$. By (2.1) it is homotopic to a mutation $\bar{\boldsymbol{\varphi}}\colon \boldsymbol{U}(X,P) \to \boldsymbol{W}(Z,Z)$, which is an extension of a mapping $\bar{\varphi}\colon X \to Z$. Therefore to each mapping $\varphi\colon Y \to Z$ we assign a certain mapping $\bar{\varphi}\colon X \to Z$. This assignment is not unique, but the homotopy class $[\bar{\varphi}]$ of the mapping $\bar{\varphi}$ depends only on the homotopy class $[\varphi]$ of the mapping φ, i.e.

(2.3) *If $\varphi \simeq \psi\colon Y \to Z$ then $\bar{\varphi} \simeq \bar{\psi}\colon X \to Z$.*

Proof. Since $\varphi \simeq \psi$, we have, by Theorem (3.1) of [4], $\boldsymbol{\varphi} \simeq \boldsymbol{\psi}$. Therefore $\boldsymbol{\varphi f} \simeq \boldsymbol{\psi f}$. Hence $\bar{\boldsymbol{\varphi}} \simeq \bar{\boldsymbol{\psi}}$ and by (2.2) $\bar{\varphi} \simeq \bar{\psi}$.

Let us denote by $[Z^X]$ the set of homotopy classes of mappings of X into Z. By (2.3) we can assign to an arbitrary mutation $\boldsymbol{f}\colon \boldsymbol{U}(X,P) \to \boldsymbol{V}(Y,Q)$ and an arbitrary ANR$(\mathfrak{M})$-space Z a function

$$\boldsymbol{f}^{\#}\colon [Z^Y] \to [Z^X]$$

defined by the formula $\boldsymbol{f}^{\#}([\varphi]) = [\bar{\varphi}]$. It will be called the function *induced* by the mutation $\boldsymbol{f}$.

It follows at once from the definition that

(2.4) *If $\boldsymbol{f} \simeq \boldsymbol{g}$ then $\boldsymbol{f}^{\#} = \boldsymbol{g}^{\#}$.*

(2.5) *If a composition $\boldsymbol{gf}$ is defined, then $(\boldsymbol{gf})^{\#} = \boldsymbol{f}^{\#}\boldsymbol{g}^{\#}$.*

(2.6) *If $\boldsymbol{u} = \mathrm{Mor}\,\boldsymbol{U}(X,P)\colon \boldsymbol{U}(X,P) \to \boldsymbol{U}(X,P)$, then $\boldsymbol{u}^{\#}\colon [Z^X] \to [Z^X]$ is an identity function.*

Let us observe that

(2.7) *If $\boldsymbol{f}\colon \boldsymbol{U}(X,P) \to \boldsymbol{V}(Y,Q)$ and $\boldsymbol{g}\colon \boldsymbol{V}(Y,Q) \to \boldsymbol{U}(X,P)$ are mutations such that $\boldsymbol{gf} \simeq \boldsymbol{u} = \mathrm{Mor}\boldsymbol{U}(X,P)$, then for an arbitrary* ANR$(\mathfrak{M})$*-space Z the function $\boldsymbol{f}^{\#}\colon [Z^Y] \to [Z^X]$ is onto and $\boldsymbol{g}^{\#}\colon [Z^X] \to [Z^Y]$ is a single-valued function.*

Indeed, by (2.4), (2.5) and (2.6) $\boldsymbol{f}^{\#}\boldsymbol{g}^{\#}$ is an identity function and hence we obtain (2.7).

§ 3. Homomorphisms of cohomotopy groups induced by mutations of complete neighbourhood systems. First we recall the definition of the nth cohomotopy group of a space X.

Let $S = S^n$ be the n-dimensional sphere. Let us choose a point $s_0 \in S$ and consider the subset

$$S \vee S = \big(S \times (s_0)\big) \cup \big((s_0) \times S\big)$$

of the Cartesian product $S \times S$. Let us define the mapping $\Omega\colon S \vee S \to S$ by the formula

$$\Omega(s, s_0) = \Omega(s_0, s) = s \quad \text{for} \quad s \in S.$$

Take two arbitrary continuous mappings $\varphi, \psi\colon X \to S$. A continuous mapping

$$\Phi\colon X \times \langle 0, 1\rangle \to S \times S$$

such that

$$\Phi(x, 0) = \big(\varphi(x), \psi(x)\big) \quad \text{and} \quad \Phi(x, 1) \in S \vee S \quad \text{for} \quad x \in X$$

is called a *normalizing homotopy* for the mappings φ and ψ. Then the mapping $\chi\colon X \to S \vee S$ defined by the formula $\chi(x) = \Phi(x, 1)$ is said to be a *normalization* of the mappings φ and ψ ([7], p. 210).

Let us suppose that a space X satisfies the following three conditions:

(3.1) For every two continuous mappings $\varphi, \psi\colon X \to S$ there exists a normalizing homotopy.

(3.2) If χ is a normalization of mappings φ and ψ, then the homotopy class $[\Omega\chi]$ of the mapping $\Omega\chi\colon X \to S$ depends only on the homotopy classes $[\varphi]$ and $[\psi]$.

(3.3) Addition in the set $[S^X]$ defined by the formula $[\varphi] + [\psi] = [\Omega\chi]$, where χ is a normalization of the mappings φ and ψ, makes the set $[S^X]$ an Abelian group.

This group is called the *n-th cohomotopy group* of X and denoted by $\pi^n(X)$. The addition defined in (3.3) is called the *n-th cohomotopy addition.* It may be defined if the conditions (3.1) and (3.2) are satisfied. In this case we say that the space X *admits the n-th cohomotopy addition*. Moreover, if the condition (3.3) is also satisfied, then we say that the space X *admits the existence of the n-th cohomotopy group*.

Since $S \in \mathrm{ANR}$, for arbitrary metrizable spaces X and Y any mutation $f\colon \boldsymbol{U}(X,P) \to \boldsymbol{V}(Y,Q)$ from a complete neighbourhood system $\boldsymbol{U}(X,P)$ to a complete neighbourhood system $\boldsymbol{V}(Y,Q)$ induces the function $f^{\#}\colon [S^Y] \to [S^X]$.

Let us prove the following

(3.4) LEMMA. *Suppose that the metrizable spaces X and Y both admit the n-th cohomotopy addition and let $f\colon \boldsymbol{U}(X,P) \to \boldsymbol{V}(Y,Q)$ be a mutation from a complete neighbourhood system $\boldsymbol{U}(X,P)$ to a complete neighbourhood system $\boldsymbol{V}(Y,Q)$. Then for every two continuous mappings $\varphi,\psi\colon Y \to S$ we have $f^{\#}([\varphi]+[\psi]) = f^{\#}([\varphi]) + f^{\#}([\psi])$.*

Proof. Let $\Phi\colon Y \times \langle 0,1\rangle \to S \times S$ be a normalizing homotopy for the mappings φ and ψ. Then $\Phi(y,0) = (\varphi(y),\psi(y))$ and $\Phi(y,1) \in S \vee S$ for $y \in Y$. The mapping $\chi\colon Y \to S \vee S$ defined by the formula $\chi(y) = \Phi(y,1)$ is a normalization of the mappings φ and ψ. Therefore

(3.5) $$[\varphi]+[\psi] = [\Omega\chi].$$

Since $S, S\vee S, S\times S \in \mathrm{ANR}$, there exists a neighbourhood $V_0 \in \mathrm{Ob}\,\boldsymbol{V}(Y,Q)$ such that the following two conditions are satisfied:

(3.6) There exist extensions $\varphi_0\colon V_0 \to S$, $\psi_0\colon V_0 \to S$ $\chi_0\colon V_0 \to S\vee S$ of the mappings φ,ψ,χ, respectively.

(3.7) There exists an extension $\Phi_0\colon V_0 \times \langle 0,1\rangle \to S\times S$ of the homotopy Φ such that
$\Phi_0(y,0) = (\varphi_0(y),\psi_0(y))$ and $\Phi_0(y,1) = \chi_0(y)$ for $y \in V_0$.

Since $f\colon \boldsymbol{U}(X,P) \to \boldsymbol{V}(Y,Q)$ is a mutation, there exists a $U_0 \in \mathrm{Ob}\,\boldsymbol{U}(X,P)$ and $f_0 \in f$ such that $f_0\colon U_0 \to V_0$. Hence

$$\varphi_0 f_0\colon U_0 \to S, \quad \psi_0 f_0\colon U_0 \to S, \quad \chi_0 f_0\colon U_0 \to S\vee S.$$

Let

(3.8) $$\begin{aligned} \varphi' &= \varphi_0 f_0|X\colon X \to S,\\ \psi' &= \psi_0 f_0|X\colon X \to S,\\ \chi' &= \chi_0 f_0|X\colon X \to S\vee S. \end{aligned}$$

We shall prove that

(3.9) $$f^{\#}([\varphi]) = [\varphi'], \quad f^{\#}([\psi]) = [\psi'], \quad f^{\#}([\Omega\chi]) = [\Omega\chi'].$$

Let $\phi\colon \boldsymbol{V}(Y,Q) \to \boldsymbol{W}(S,S)$ be the mutation consisting of all mappings of the form $\varphi_0|V$, where $V \in \mathrm{Ob}\,\boldsymbol{V}(Y,Q)$ and $V \subset V_0$. The mutation ϕ is

an extension of the mapping φ. Let $\phi'\colon \boldsymbol{U}(X,P) \to \boldsymbol{W}(S,S)$ be the mutation consisting of all mappings of the form $\varphi_0 f_0|U$, where $U \in \mathrm{Ob}\,\boldsymbol{U}(X,P)$ and $U \subset U_0$. Consider the mutation $\phi f\colon \boldsymbol{U}(X,P) \to \boldsymbol{W}(S,S)$. Let us observe that $\varphi_0 f_0 \in \phi f$ and $\varphi_0 f_0 \in \phi'$. We show that $\phi f \simeq \phi'$. Take two arbitrary concurrent constituents $\widetilde{\varphi} f\colon U \to S$ and $\varphi_0 f_0|U\colon U \to S$ of the mutations ϕf and ϕ', respectively. The mappings $f\colon U \to V$ and $f_0|U\colon U \to V$ are constituents of the mutation f. Therefore by (1.5) there exists a $U' \in \mathrm{Ob}\,\boldsymbol{U}(X,P)$ such that $f|U' \simeq f_0|U'$. Since $\widetilde{\varphi} = \varphi_0|V$, where V is the range of f, we have $\widetilde{\varphi} f|U' \simeq \varphi_0 f_0|U'$. Therefore the morphisms $\widetilde{\varphi} f$ and $\varphi_0 f_0$ have an equalizer. Thus $\phi f \simeq \phi'$. Hence by the definition of $f^{\#}$ we obtain $f^{\#}([\varphi]) = [\varphi']$. Analogously one can prove the remaining two conditions (3.9).

Let us define the mapping

$$\Phi'\colon X \times \langle 0,1\rangle \to S \times S$$

by the formula

$$\Phi'(x,t) = \Phi_0(f_0(x),t) \quad \text{for} \quad x \in X \quad \text{and} \quad 0 \leqslant t \leqslant 1.$$

Then by (3.7) and (3.8)

$$\Phi'(x,0) = \Phi_0(f_0(x),0) = (\varphi_0 f_0(x), \psi_0 f_0(x)) = (\varphi'(x), \psi'(x)),$$

$$\Phi'(x,1) = \Phi_0(f_0(x),1) = \chi_0 f_0(x) = \chi'(x) \quad \text{for} \quad x \in X.$$

Hence

(3.10) $$[\varphi'] + [\psi'] = [\Omega\chi'].$$

From (3.5), (3.9) and (3.10) we obtain Lemma (3.4).

Lemma (3.4) implies at once the following

(3.11) THEOREM. *If metrizable spaces X and Y admit the existence of the n-th cohomotopy groups $\pi^n(X)$ and $\pi^n(Y)$ and $f\colon \boldsymbol{U}(X,P) \to \boldsymbol{V}(Y,Q)$ is a mutation from a complete neighbourhood system $\boldsymbol{U}(X,P)$ to a complete neighbourhood system $\boldsymbol{V}(Y,Q)$, then the induced function $f^{\#}\colon \pi^n(Y) \to \pi^n(X)$ is a homomorphism.*

§ 4. Cohomotopy groups and the shape of metrizable spaces. Let us prove the following

(4.1) THEOREM. *Suppose X and Y are metrizable spaces such that $\mathrm{Sh}\,X \leqslant \mathrm{Sh}\,Y$. If the space Y admits the existence of the n-th cohomotopy group $\pi^n(Y)$, then the space X admits the existence of the n-th cohomotopy group $\pi^n(X)$ and $\pi^n(X) \underset{r}{\leqslant} \pi^n(Y)$, i.e. the group $\pi^n(X)$ is a divisor of the group $\pi^n(Y)$.*

Proof. By hypothesis there exist mutations $f\colon \boldsymbol{U}(X,P) \to \boldsymbol{V}(Y,Q)$ and $g\colon \boldsymbol{V}(Y,Q) \to \boldsymbol{U}(X,P)$ such that $gf \simeq \boldsymbol{u} = \mathrm{Mor}\,\boldsymbol{U}(X,P)$.

Take two arbitrary continuous mappings $\varphi, \psi\colon X \to S = S^n$. Since $S \in \mathrm{ANR}$ then there exist $U_0 \in \mathrm{Ob}\,\boldsymbol{U}(X, P)$ and continuous extensions $\varphi_0, \psi_0\colon U_0 \to S$ of the mappings φ and ψ, respectively. Since $\boldsymbol{g}$ is a mutation, by (1.4) there exist $V_0 \in \mathrm{Ob}\,\boldsymbol{V}(Y, Q)$ and $g_0 \in \boldsymbol{g}$ such that $g_0\colon V_0 \to U_0$. Hence $\varphi_0 g_0\colon V_0 \to S$ and $\psi_0 g_0\colon V_0 \to S$. Let $\varphi' = \varphi_0 g_0|Y\colon Y \to S$ and $\psi' = \psi_0 g_0|Y\colon Y \to S$. By hypothesis for the mappings φ' and ψ' there exists a normalizing homotopy $\Phi\colon Y \times \langle 0, 1\rangle \to S \times S$.

Then for $y \in Y$

$$\widetilde{\Phi}(y, 0) = \big(\varphi'(y), \psi'(y)\big) = \big(\varphi_0 g_0(y), \psi_0 g_0(y)\big), \qquad \Phi(y, 1) \in S \vee S.$$

Since $S \vee S$, $S \times S \in \mathrm{ANR}$, there exists a continuous extension

$$\widetilde{\Phi}\colon V_1 \times \langle 0, 1\rangle \to S \times S, \qquad V_1 \subset V_0,\ V_1 \in \mathrm{Ob}\,\boldsymbol{V}(Y, Q)$$

of the homotopy Φ such that

$$(4.2)\qquad \begin{matrix} \widetilde{\Phi}(y, 0) = \big(\varphi_0 g_0(y), \psi_0 g_0(y)\big) \\ \widetilde{\Phi}(y, 1) \in S \vee S \end{matrix} \qquad \text{for} \quad y \in V_1.$$

Take a constituent $f_1 \in \boldsymbol{f}$ whose range is V_1; $f_1\colon U_1 \to V_1$, $U_1 \in \mathrm{Ob}\,\boldsymbol{U}(X, P)$. Let $v\colon V_1 \to V_0$ be the inclusion mapping and put $f_0 = v f_1\colon U_1 \to V_0$, $f_0 \in \boldsymbol{f}$. Then $f_0(X) \subset V_1$, $g_0 f_0\colon U_1 \to U_0$, $g_0 f_0 \in \boldsymbol{g}\boldsymbol{f}$. Since $\boldsymbol{g}\boldsymbol{f} \simeq \boldsymbol{u}$, by (1.6) there exists a $U_2 \in \mathrm{Ob}\,\boldsymbol{U}(X, P)$ such that $U_2 \subset U_0 \cap U_1$ and $g_0 f_0|U_2 \simeq u\colon U_2 \to U_0$, $u \in \mathrm{Mor}\,\boldsymbol{U}(X, P)$. Hence $g_0 f_0|X \simeq u|X\colon X \to U_0$. Therefore there exists a homotopy

$$H\colon X \times \langle 0, 1\rangle \to U_0$$

such that

$$H(x, 0) = x \quad \text{and} \quad H(x, 1) = g_0 f_0(x).$$

Let us define the continuous mapping

$$G\colon X \times \langle 0, 1\rangle \to S \times S$$

by the formula

$$G(x, t) = \big(\varphi_0 H(x, t,) \psi_0 H(x, t)\big) \quad \text{for} \quad x \in X \text{ and } 0 \leqslant t \leqslant 1.$$

Then for $x \in X$

$$(4.3)\qquad \begin{aligned} G(x, 0) &= \big(\varphi_0 H(x, 0), \psi_0 H(x, 0)\big) = \big(\varphi_0(x), \psi_0(x)\big) = \big(\varphi(x), \psi(x)\big), \\ G(x, 1) &= \big(\varphi_0 H(x, 1), \psi_0 H(x, 1)\big) = \big(\varphi_0 g_0 f_0(x), \psi_0 g_0 f_0(x)\big). \end{aligned}$$

Let us define the mapping

$$\Psi\colon X \times \langle 0, 1\rangle \to S \times S$$

by the formula

$$\Psi(x, t) = \begin{cases} G(x, 2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ \widetilde{\Phi}\big(f_0(x), 2t-1\big) & \text{for } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

By (4.2) and (4.3) the mapping Ψ is well defined and continuous and it is a normalizing homotopy for the mappings φ and ψ. Therefore

(4.4) For every two continuous mappings $\varphi, \psi\colon X \to S$ there exists a normalizing homotopy.

We shall prove that

(4.5) If $\chi\colon X \to S \vee S$ is a normalization of the mappings $\varphi, \psi\colon X \to S$, then the homotopy class $[\Omega\chi]$ depends only on the homotopy classes $[\varphi]$ and $[\psi]$.

By (2.7) it follows that

(4.6) $\boldsymbol{g}^{\#}\colon [S^X] \to \pi^n(Y)$ is a single-valued function.

Let $\bar{\varphi}, \bar{\psi}\colon Y \to S$ be continuous mappings such that

(4.7) $\boldsymbol{g}^{\#}([\varphi]) = [\bar{\varphi}]$ and $\boldsymbol{g}^{\#}([\psi]) = [\bar{\psi}]$.

By the hypotheses there exists a normalization $\bar{\chi}\colon Y \to S \vee S$ of the mappings $\bar{\varphi}$ and $\bar{\psi}$ and the homotopy class $[\Omega\bar{\chi}]$ depends only on the homotopy classes $[\bar{\varphi}]$ and $[\bar{\psi}]$. In order to prove (4.5) it suffices by (4.6) and (4.7) to show that

$$(4.8)\qquad \boldsymbol{g}^{\#}([\Omega\chi]) = [\Omega\bar{\chi}].$$

Since $\chi\colon X \to S \vee S$ is a normalization of the mappings φ and ψ, then there exists a normalizing homotopy

$$\Phi\colon X \times \langle 0, 1\rangle \to S \times S$$

such that

$$\Phi(x, 0) = \big(\varphi(x), \psi(x)\big) \quad \text{and} \quad \Phi(x, 1) = \chi(x) \quad \text{for} \quad x \in X.$$

Since S, $S \vee S$, $S \times S \in \mathrm{ANR}$, then there exists a neighbourhood $U \in \mathrm{Ob}\,\boldsymbol{U}(X, P)$ such that the following two conditions are satisfied:

(4.9) There exist continuous extensions $\widetilde{\varphi}\colon U \to S$, $\widetilde{\psi}\colon U \to S$, $\widetilde{\chi}\colon U \to S \vee S$ of the mappings φ, ψ, χ, respectively.

(4.10) There exists a continuous extension $\widetilde{\Phi}\colon U \times \langle 0, 1\rangle \to S \times S$ of the homotopy Φ such that

$$\widetilde{\Phi}(x, 0) = \big(\widetilde{\varphi}(x), \widetilde{\psi}(x)\big) \text{ and } \widetilde{\Phi}(x, 1) = \widetilde{\chi}(x) \text{ for } x \in U.$$

Since $\boldsymbol{g}\colon \boldsymbol{V}(Y, Q) \to \boldsymbol{U}(X, P)$ is a mutation, there exist $V \in \mathrm{Ob}\,\boldsymbol{V}(Y, Q)$ and $g \in \boldsymbol{g}$ such that $g\colon V \to U$. Then $\widetilde{\varphi}g, \widetilde{\psi}g\colon V \to S$, $\widetilde{\chi}g\colon V \to S \vee S$.

Let us put

$$(4.11)\qquad \varphi' = \widetilde{\varphi}g|Y, \qquad \psi' = \widetilde{\psi}g|Y, \qquad \chi' = \widetilde{\chi}g|Y.$$

Then we have

(4.12) $$g^{\#}([\varphi]) = [\varphi'], \quad g^{\#}([\psi]) = [\psi'], \quad g^{\#}([\Omega\chi]) = [\Omega\chi'].$$

The proof of (4.12) is precisely the same as the proof of the (3.9).

Let us define the continuous mapping

$$\Psi\colon\ Y\times\langle 0,1\rangle \to S\times S$$

by the formula

$$\Psi(y,t) = \tilde{\Phi}(g(y),t) \quad \text{for} \quad y\in Y \text{ and } 0\leqslant t\leqslant 1.$$

Then by (4.10) and (4.11) we obtain

$$\Psi(y,0) = \tilde{\Phi}(g(y),0) = (\tilde{\varphi}g(y), \tilde{\psi}g(y)) = (\varphi'(y), \psi'(y)),$$

$$\Psi(y,1) = \tilde{\Phi}(g(y),1) = \tilde{\chi}g(y) = \chi'(y).$$

It follows that χ' is a normalization of the mappings φ' and ψ'.

By (4.7) and (4.12) we have

(4.13) $$[\bar{\varphi}] = [\varphi'] \quad \text{and} \quad [\bar{\psi}] = [\psi'].$$

Since the homotopy class $[\Omega\bar{\chi}]$ depends only on the homotopy classes $[\bar{\varphi}]$ and $[\bar{\psi}]$ and the homotopy class $[\Omega\chi']$ depends only on the homotopy classes $[\varphi']$ and $[\psi']$, by (4.13) we obtain $[\Omega\chi'] = [\Omega\bar{\chi}]$. Hence by (4.12) we obtain (4.8) and the proof of (4.5) is completed.

By (4.4) and (4.5) the space X admits the nth cohomotopy addition. Therefore by Lemma (3.4) for arbitrary continuous mappings $\varphi, \psi\colon X\to S$ and $\bar{\varphi}, \bar{\psi}\colon Y\to S$ we have

(4.14) $$g^{\#}([\varphi]+[\psi]) = g^{\#}([\varphi])+g^{\#}([\psi]),$$

(4.15) $$f^{\#}([\bar{\varphi}]+[\bar{\psi}]) = f^{\#}([\bar{\varphi}])+f^{\#}([\bar{\psi}]).$$

Since $g^{\#}$ is a single-valued function and the nth cohomotopy addition in the group $\pi^n(Y)$ is associative and commutative, by (4.14) it follows that

(4.16) The nth cohomotopy addition in the set $[S^X]$ is associative and commutative.

Let $[\bar{\varphi}_0]$ be the zero of the group $\pi^n(Y)$ and let $[\varphi_0] = f^{\#}([\bar{\varphi}_0])$. Then

(4.17) $[\varphi_0]+[\psi] = [\psi]$ for an arbitrary continuous mapping $\psi\colon X\to S$.

Indeed, let $[\bar{\psi}] = g^{\#}([\psi])$. Then $[\bar{\varphi}_0]+[\bar{\psi}] = [\bar{\psi}]$. Hence by (4.15) $f^{\#}([\bar{\varphi}_0])+$ $+f^{\#}([\bar{\psi}]) = f^{\#}([\bar{\psi}])$. But $f^{\#}([\bar{\varphi}_0]) = [\varphi_0]$ and $f^{\#}([\bar{\psi}]) = f^{\#}g^{\#}([\psi]) = [\psi]$. Hence we obtain (4.17).

It is easy to see that

(4.18) For an arbitrary continuous mapping $\varphi\colon X\to S$ there exists a continuous mapping $\psi\colon X\to S$ such that $[\varphi]+[\psi] = [\varphi_0]$.

Indeed, if $\bar{\psi}\colon Y\to S$ is a mapping such that $g^{\#}([\varphi])+[\bar{\psi}] = [\bar{\varphi}_0]$, then by (4.15) an arbitrary mapping $\psi\in f^{\#}([\bar{\psi}])$ satisfies the required condition.

From (4.16), (4.17) and (4.18) it follows that the space X admits the existence of the nth cohomotopy group $\pi^n(X)$.

By Theorem (3.11) the functions $f^{\#}\colon \pi^n(Y)\to\pi^n(X)$ and $g^{\#}\colon \pi^n(X)$ $\to\pi^n(Y)$ are homomorphisms and since $gf\simeq u$, by (2.4), (2.5) and (2.6) $f^{\#}g^{\#}\colon \pi^n(X)\to\pi^n(X)$ is an identity function. Therefore $\pi^n(X)\underset{r}{\leqslant}\pi^n(Y)$.

Thus, the proof of Theorem (4.1) is completed.

Theorems (3.11) and (4.1) and also (2.4), (2.5) and (2.6) imply at once the following

(4.19) THEOREM. *Suppose X and Y are metrizable spaces with* $\mathrm{Sh}\,X = \mathrm{Sh}\,Y$. *If the space X admits the existence of the n-th cohomotopy group $\pi^n(X)$, then the space Y admits the existence of the n-th cohomotopy group $\pi^n(Y)$ and the groups $\pi^n(X)$ and $\pi^n(Y)$ are isomorphic, i.e. cohomotopy groups are invariances of shape.*

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