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Some Wallman compactifications of locally compact spaces

by

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In 1938, Wallman [14] introduced the compactification of a T_1 space known by his name. In 1964, Frink [4] used a modification of Wallman's procedure to obtain a certain class of compactifications of Tychonoff spaces; these compactifications have been called Wallman compactifications. Using the concept of normality of a base $\mathcal{F}$ for the closed sets in a Hausdorff space X, Frink constructed a compactification of X, denoted by $w(\mathcal{F})$, as the space of all ultrafilters of sets of $\mathcal{F}$. By choosing different bases $\mathcal{F}$ for a non-compact Tychonoff space X, different Hausdorff compactifications of X may be obtained in the form of Wallman compactifications $w(\mathcal{F})$. Frink asked whether every Hausdorff compactification can be obtained by this construction. This question remains unanswered; however, many partial results have been obtained: [1], [2], [3], [5], [6], [7], [9], [10], [11], [12], and [13]. Thus, there has been some interest in methods of manufacturing normal bases. This paper is devoted to the study of a general method of constructing such bases in locally compact spaces. Our method is an amplification of the construction of the Alexandroff compactification. As an application of our results, we are able to prove that certain compactifications not previously known to be Wallman are indeed so. For example, we prove that if the remainder in a compactification Y is a certain type of retract of Y then Y is a Wallman compactification.

Throughout this paper, all topological spaces are assumed to be Hausdorff and locally compact; also, we let X denote a fixed locally compact Hausdorff space.

I. The construction of normal bases. Let $\mathcal{F}$ be a family of closed subsets of the space X. $\mathcal{F}$ is a *ring* iff $\mathcal{F}$ is closed under finite unions and intersections. $\mathcal{F}$ is *disjunctive* iff given any closed set E and any point $x \notin E$, there exists $A \in \mathcal{F}$ such that $x \in A$ and $A \cap E = \emptyset$. $\mathcal{F}$ is *normal* iff given any two disjoint sets $A_1, A_2 \in \mathcal{F}$ there exist $C_1, C_2 \in \mathcal{F}$ such that $A_1 \cap C_1 = \emptyset$, $A_2 \cap C_2 = \emptyset$, and $C_1 \cup C_2 = X$. $\mathcal{F}$ is a *normal base* iff $\mathcal{F}$ is normal,

$\mathcal{F}$ is a ring, $\mathcal{F}$ is disjunctive, and $\mathcal{F}$ is a base for closed sets. Thus, according to Frink, if $\mathcal{F}$ is a normal base, then $w(\mathcal{F})$ is a Hausdorff compactification of X.

An important concept for us will be that of a co-compact set. A subset L of X is *co-compact* iff $\mathrm{Cl}_X(X-L)$ is compact.

Certain rather nice relations are valid for co-compact sets. We state these now for future reference but leave the easy proofs to the reader.

THEOREM 1. *Let Y be a compactification of X, let $K = Y-X$, and let L be co-compact in X. Then $K \subset \mathrm{Int}_Y \mathrm{Cl}_Y(L)$ and if $A \subset X$ then $K \cap \mathrm{Cl}_Y(A) = K \cap \mathrm{Cl}_Y(A \cap L)$.*

THEOREM 2. *Let $\mathcal{F}$ be a family of closed subsets of X such that every compact subset of X is in $\mathcal{F}$ and every co-compact subset of X is in $\mathcal{F}$. Then $\mathcal{F}$ is disjunctive and is a base for closed sets.*

Proof. Let $x \in X$ and let S be a closed subset of X such that $x \notin S$. By local compactness of X, there exists a compact neighborhood B of x such that $S \cap B = \emptyset$. Let $L = \mathrm{Cl}_X(X-B)$. Then L is co-compact. Finally, $x \in \{x\} \in \mathcal{F}$, $S \subset L \in \mathcal{F}$ and $\{x\} \cap L = \emptyset$. This completes the proof.

As justification for restricting our attention to those normal bases which contain all compact sets and all co-compact sets we cite Theorem 1, Theorem 3 (below), Corollary 2 of Theorem 4 (below), and Theorem 8 (below). Thus, we turn our attention to the task of constructing normal bases which contain all compact sets and all co-compact sets. The solution lies in directing our attention to an auxilliary family which, loosely speaking, we can consider to consist of sets which are "unbounded". This idea is embodied in the following definition: If $\mathcal{K}$ is a family of closed subsets of X, then we let $\mathrm{CM}(\mathcal{K}) = \{(H \cap L) \cup B \mid H \in \mathcal{K},\ L$ is co-compact and B is compact$\}$. We shall refer to $\mathrm{CM}(\mathcal{K})$ as the compact modification of $\mathcal{K}$.

The proof of the next theorem is a straightforward application of the algebra of sets and will be omitted.

THEOREM 3. *Let $\mathcal{K}$ be a ring of closed sets such that $\emptyset, X \in \mathcal{K}$. Then $\mathrm{CM}(\mathcal{K})$ is a ring of closed sets which contains $\mathcal{K}$ as a subset and, moreover, contains all compact sets and all co-compact sets.*

Thus, if $\mathcal{K}$ is a ring of closed sets such that $\emptyset, X \in \mathcal{K}$ and such that $\mathrm{CM}(\mathcal{K})$ is a normal family, then Theorems 2 and 3 imply that $\mathrm{CM}(\mathcal{K})$ is a normal base. Our next question is: What conditions must $\mathcal{K}$ satisfy so that $\mathrm{CM}(\mathcal{K})$ is a normal family? In order to give an answer to this question, we make the following formal definition.

DEFINITION. Let $\mathcal{K}$ be a family of closed subsets of X. Then $\mathcal{K}$ is *normal at* ∞ iff there exists a co-compact set J such that for every co-compact set $L \subset J$, the trace of $\mathcal{K}$ on L (i.e. $\{H \cap L \mid H \in \mathcal{K}\}$) is a normal family of subsets of L.

Now we are ready to state and prove the following:

THEOREM 4. *Let $\mathcal{K}$ be a family of closed sets which is normal at ∞. Then $\mathrm{CM}(\mathcal{K})$ is a normal family.*

Proof. Let $A_1, A_2 \in \mathrm{CM}(\mathcal{K})$ such that $A_1 \cap A_2 = \emptyset$. Let $H_1, H_2 \in \mathcal{K}$, L_1, L_2 be co-compact, and B_1, B_2 be compact such that $A_1 = (H_1 \cap L_1) \cup B_1$ and $A_2 = (H_2 \cap L_2) \cup B_2$. Since B_1 and B_2 are disjoint compact subsets of the locally compact space X, there exist compact sets K_1, K_2 such that $B_1 \subset \mathrm{int}_X(K_1)$, $B_2 \subset \mathrm{int}_X(K_2)$, and $K_1 \cap K_2 = \emptyset$. Now, the assumption that $\mathcal{K}$ is normal at ∞ guarantees the existence of a co-compact set J such that $\{H \cap L \mid H \in \mathcal{K}\}$ is a normal family of subsets of L whenever L is co-compact and $L \subset J$. Let $L = L_1 \cap L_2 \cap \mathrm{Cl}_X(X-K_1) \cap \mathrm{Cl}_X(X-K_2) \cap J$. Then L is co-compact and $L \subset J$. Also, $(H_1 \cap L) \cap (H_2 \cap L) = \emptyset$. So, since $\{H \cap L \mid H \in \mathcal{K}\}$ is normal, there exist $V_1, V_2 \in \mathcal{K}$ such that

$$(H_1 \cap L) \cap (V_1 \cap L) = \emptyset\,, \quad (H_2 \cap L) \cap (V_2 \cap L) = \emptyset\,, \quad \text{and} \quad (V_1 \cap L) \cup (V_2 \cap L) = L\,.$$

Next, let $P = \mathrm{Cl}_X(X-L)$ and note that $(A_1 \cap P) \cup B_1$ and $(A_2 \cap P) \cup B_2$ are disjoint compact subsets of X. Therefore, there exist compact sets S and T such that $(A_1 \cap P) \cup B_1 \subset \mathrm{int}_X(S)$, $(A_2 \cap P) \cup B_2 \subset \mathrm{int}_X(T)$, and $S \cap T = \emptyset$. Finally, let $C_1 = (V_1 \cap L) \cup (T \cap P)$ and let $C_2 = (V_2 \cap L) \cup (P - \mathrm{int}_X(T))$. We leave it to the reader to check that

$$C_1, C_2 \in \mathrm{CM}(\mathcal{K})\,, \quad A_1 \cap C_1 = \emptyset\,, \quad A_2 \cap C_2 = \emptyset\,, \quad \text{and} \quad C_1 \cup C_2 = X\,.$$

COROLLARY 1. *Let $\mathcal{F}$ be a ring of closed sets which is normal at ∞ and suppose $\emptyset, X \in \mathcal{F}$. Then $\mathrm{CM}(\mathcal{F})$ is a normal base.*

COROLLARY 2. *Let $\mathcal{F}$ be a ring of closed sets which is normal at ∞ and which contains all compact and all co-compact sets. Then $\mathcal{F}$ is a normal base.*

Proof. Merely note that $\mathcal{F} = \mathrm{CM}(\mathcal{F})$ and apply Corollary 1.

LEMMA. *Let $\mathcal{F}$ be a normal base, let T be compact and let S be closed such that $T \cap S = \emptyset$. Then there exist $P, Q \in \mathcal{F}$ such that $T \subset P$, $S \subset Q$, P is compact, Q is co-compact, and $P \cap Q = \emptyset$.*

Proof. First, use the local compactness of X to manufacture a compact set B such that $T \subset \mathrm{int}_X(B) \subset B \subset X - S$. Let $L = \mathrm{Cl}_X(X-B)$ and note that L is co-compact, $T \cap L = \emptyset$, and $S \subset L$. Since $\mathcal{F}$ is a base for closed sets, for each $x \in T$, there exists a co-compact $M_x \in \mathcal{F}$ such that $x \notin M_x$ and $L \subset M_x$. Since $\mathcal{F}$ is disjunctive, for each $x \in T$, there exists $H_x \in \mathcal{F}$ such that $x \in H_x$ and $H_x \cap M_x = \emptyset$. Since $\mathcal{F}$ is a normal family, for each $x \in T$, there exist $C_x, D_x \in \mathcal{F}$ such that $H_x \cap C_x = \emptyset$, $M_x \cap D_x = \emptyset$, and $C_x \cup D_x = X$. Then $T \subset \bigcup_{x \in T} (X - C_x)$ so, since T is compact, there exists a finite number of points $x_1, \ldots, x_n \in T$ such that

$T \subset \bigcup_{i=1}^{n} (X - C_{x_i})$. Let $P = \bigcup_{i=1}^{n} D_{x_i}$, let $Q = \bigcap_{i=1}^{n} M_{x_i}$ and it follows that P and Q satisfy the conclusion of the lemma.

If $\mathcal{F}$ is a normal base and if $E \subset X$, then $\{F \cap E \mid F \in \mathcal{F}\}$ may fail to be a normal family of subsets of E. This observation was made by Frink. In view of this fact, the following theorem is of some interest.

THEOREM 5. *If $\mathcal{F}$ is a normal base, then $\mathcal{F}$ is normal at ∞.*

Proof. In the definition of "normal at ∞", we take $J = X$ and let L be any co-compact set. We must show $\{F \cap L \mid F \in \mathcal{F}\}$ is a normal family of subsets of L. Thus, let $F_1, F_2 \in \mathcal{F}$ such that $(F_1 \cap L) \cap (F_2 \cap L) = \emptyset$. Then L and $F_1 \cap F_2$ are disjoint closed subsets of X and $F_1 \cap F_2$ is compact (since it is contained in $\mathrm{Cl}_X(X - L)$). Apply the lemma to produce sets $P, Q \in \mathcal{F}$ such that $F_1 \cap F_2 \subset P$, $L \subset Q$, and $P \cap Q = \emptyset$. Then $(F_1 \cap Q), (F_2 \cap Q) \in \mathcal{F}$ and $(F_1 \cap Q) \cap (F_2 \cap Q) = \emptyset$. By the normality of $\mathcal{F}$, there exist sets $C_1, C_2 \in \mathcal{F}$ such that $(F_1 \cap Q) \cap C_1 = \emptyset$, $(F_2 \cap Q) \cap C_2 = \emptyset$, and $C_1 \cup C_2 = X$. We complete the proof by observing that

$$(F_1 \cap L) \cap (C_1 \cap L) = \emptyset, \quad (F_2 \cap L) \cap (C_2 \cap L) = \emptyset \quad \text{and} \quad (C_1 \cap L) \cup (C_2 \cap L) = L.$$

We close this section with the following theorem.

THEOREM 6. *Let $\mathcal{S}$ be a family of closed subsets of X such that $\mathcal{S}$ is closed under finite intersections. Let $\mathcal{K}$ be the family of finite unions of sets of $\mathcal{S}$ (i.e. $\mathcal{K}$ is the ring generated by $\mathcal{S}$). Then if $\mathcal{S}$ is normal at ∞, then $\mathcal{K}$ is normal at ∞.*

Proof. Produce a co-compact set J such that $\{P \cap L \mid P \in \mathcal{S}\}$ is a normal family of subsets of L whenever L is a co-compact set such that $L \subset J$. This same set J will work for normality of $\mathcal{K}$ at ∞. To see this, let L be a co-compact set such that $L \subset J$. Let $P_1, \ldots, P_n, Q_1, \ldots, Q_m \in \mathcal{S}$ such that

$$\Big[\Big(\bigcup_{i=1}^{n} P_i\Big) \cap L\Big] \cap \Big[\Big(\bigcup_{j=1}^{m} Q_j\Big) \cap L\Big] = \emptyset.$$

Then for each i, j, noting that $(P_i \cap L) \cap (Q_j \cap L) = \emptyset$, produce sets $V_{ij}, W_{ij} \in \mathcal{S}$ such that

$$(P_i \cap L) \cap (V_{ij} \cap L) = \emptyset, \quad (Q_j \cap L) \cap (W_{ij} \cap L) = \emptyset, \quad \text{and} \quad (V_{ij} \cap L) \cup (W_{ij} \cap L) = L.$$

Let

$$C_1 = \bigcap_{i=1}^{n} \bigcup_{j=1}^{m} V_{ij}, \quad C_2 = \bigcap_{j=1}^{m} \bigcup_{i=1}^{n} W_{ij}$$

and check that $C_1, C_2 \in \mathcal{K}$,

$$\Big[\Big(\bigcup_{i=1}^{n} P_i\Big) \cap L\Big] \cap (C_1 \cap L) = \emptyset, \quad \Big[\Big(\bigcup_{j=1}^{m} Q_j\Big) \cap L\Big] \cap (C_2 \cap L) = \emptyset \quad \text{and} \quad (C_1 \cap L) \cup (C_2 \cap L) = L.$$

We remark that trivial examples in the Euclidean plane can be used to show that the converse of Theorem 6 is not true.

II. Wallman compactifications determined by compact modifications. In this section we first present a few known results about compactifications and normal bases for later reference. We then use some of these results to complete the justification for restricting our attention to normal bases containing all compact and all co-compact sets (see Theorem 8).

Let Y be a compactification of X and let $\mathcal{F}$ be a normal base on X. We write $Y \cong w(\mathcal{F})$ whenever Y and $w(\mathcal{F})$ are equivalent compactifications, i.e. whenever there exists a homeomorphism of Y onto $w(\mathcal{F})$ which is the identity on X.

Steiner [7] calls a family $\mathcal{F}$ of closed subsets of a space Y *separating* on Y iff given any closed set E of Y and given any point $y \notin E$, there exist $F_1, F_2 \in \mathcal{F}$ such that $y \in F_1$, $E \subset F_2$ and $F_1 \cap F_2 = \emptyset$. He says that a family $\mathcal{F}$ of closed subsets of a space Y has the *trace property with respect to* a dense subset X iff whenever $F_1, \ldots, F_n \in \mathcal{F}$ are a finite number of sets such that $\bigcap_{i=1}^{n} F_i \neq \emptyset$, then $X \cap \bigcap_{i=1}^{n} F_i \neq \emptyset$. Steiner then proves the following:

THEOREM 7. *Let $\mathcal{F}$ be a normal base on X and let Y be a compactification of X. Let $\overline{\mathcal{F}} = \{\mathrm{Cl}_Y(F) \mid F \in \mathcal{F}\}$. Then $Y \cong w(\mathcal{F})$ if and only if $\overline{\mathcal{F}}$ is separating on Y and $\overline{\mathcal{F}}$ has the trace property with respect to X.*

We remark that it is an immediate consequence of the definition of $w(\mathcal{F})$ that whenever $F_1, \ldots, F_n \in \mathcal{F}$, $\mathrm{Cl}\big(\bigcap_{i=1}^{n} F_i\big) = \bigcap_{i=1}^{n} \mathrm{Cl}(F_i)$, where the closure is taken in the space $w(\mathcal{F})$.

We would now like to relate these ideas to our process of forming compact modifications.

THEOREM 8. *Let $\mathcal{F}$ be a normal base such that $\emptyset, X \in \mathcal{F}$. Then $w(\mathcal{F}) \cong w\big(\mathrm{CM}(\mathcal{F})\big)$.*

Proof. Let $Y = w(\mathcal{F})$, let $\mathcal{G} = \mathrm{CM}(\mathcal{F})$, let $\overline{\mathcal{G}} = \{\mathrm{Cl}_Y(G) \mid G \in \mathcal{G}\}$, and let $K = Y - X$. Then by Theorem 7, it suffices to show that $\overline{\mathcal{G}}$ is separating on Y and $\overline{\mathcal{G}}$ has the trace property with respect to X.

To show $\overline{\mathcal{G}}$ is separating, let $y \in Y$ and let E be closed in Y such that $y \notin E$. Let $\overline{\mathcal{F}} = \{\mathrm{Cl}_Y(F) \mid E \in \mathcal{F}\}$. Then by Theorem 7, $\overline{\mathcal{F}}$ is separating. Thus, there exists $F_1, F_2 \in \mathcal{F}$ such that $y \in \mathrm{Cl}_Y(F_1)$, $E \subset \mathrm{Cl}_Y(F_2)$, and

$\mathrm{Cl}_Y(F_1) \cap \mathrm{Cl}_Y(F_2) = \emptyset$. Since $\mathcal{F} \subset \mathcal{G}$, $\mathrm{Cl}_Y(F_1), \mathrm{Cl}_Y(F_2) \in \bar{\mathcal{G}}$ and hence, $\bar{\mathcal{G}}$ is separating.

To show $\bar{\mathcal{G}}$ has the trace property with respect to X, let $F_1, \ldots, F_n \in \mathcal{F}$, let $L_1, \ldots, L_n$ be co-compact in X, and let $B_1, \ldots, B_n$ be compact in X such that

$$\bigcap_{i=1}^{n} \mathrm{Cl}_Y[(F_i \cap L_i) \cup B_i] \neq \emptyset$$

and suppose, by way of contradiction, that

$$\bigcap_{i=1}^{n} [(F_i \cap L_i) \cup B_i] = \emptyset .$$

Let $y \in \bigcap_{i=1}^{n} \mathrm{Cl}_Y[(F_i \cap L_i) \cup B_i]$ and note that $y \in K$. Thus,

$$y \in \bigcap_{i=1}^{n} K \cap \mathrm{Cl}_Y[(F_i \cap L_i) \cup B_i] = \bigcap_{i=1}^{n} K \cap \mathrm{Cl}_Y(F_i \cap L_i) = \bigcap_{i=1}^{n} K \cap \mathrm{Cl}_Y(F_i) ,$$

the latter equality being a consequence of Theorem 1. Since $Y = w(\mathcal{F})$ and $F_1, \ldots, F_n \in \mathcal{F}$, $\bigcap_{i=1}^{n} \mathrm{Cl}_Y(F_i) = \mathrm{Cl}_Y(\bigcap_{i=1}^{n} F_i)$, and so $y \in \mathrm{Cl}_Y(\bigcap_{i=1}^{n} F_i)$. But since

$$\bigcap_{i=1}^{n} (F_i \cap L_i) = \emptyset \ , \quad \bigcap_{i=1}^{n} F_i \subset \bigcup_{i=1}^{n} (X - L_i) \subset \bigcup_{i=1}^{n} \mathrm{Cl}_X(X - L_i)$$

and therefore $\bigcap_{i=1}^{n} F_i$ is compact in X. Thus,

$$y \in \mathrm{Cl}_Y\left(\bigcap_{i=1}^{n} F_i\right) = \bigcap_{i=1}^{n} F_i \subset X$$

which is impossible since $y \in K = Y - X$.

As an example, we apply our methods to the problem of constructing the closed disk of the Euclidean plane out of the open disk. It was conjectured by Njastad [6] that the closed disk is not a Wallman compactification of the open disk. E. F. Steiner [7] proved that this conjecture was false.

In the following example we present an alternative proof of this fact.

EXAMPLE. Let X be the open disk in the plane and let Y be the closed disk. For each pair of real numbers (a, b), let $L_{(a,b)} = \{(r\cos\theta, r\sin\theta) \mid 0 \leqslant r < 1 \text{ and } a \leqslant \theta \leqslant b\}$ and let $\mathcal{L} = \{L_{(a,b)} \mid a \text{ and } b \text{ are real}\}$. Let $\mathcal{K}$ be the ring generated by $\mathcal{L}$. Then it is not difficult to show that $\mathcal{K}$ is normal at ∞ and using Theorem 7, to show that $w(\mathrm{CM}(\mathcal{K})) \cong Y$.

III. Retracts and Wallman compactifications. In this section, we consider compactifications Y of our space X such that the remainder $Y - X$ is a retract of Y. If the retract map sends every co-compact subset of X onto $Y - X$, we are able to prove that Y is a Wallman compactification of X. Obviously, many common compactifications are of this form. Nevertheless, it is clear that not all compactifications are of this form since a necessary condition for this is that the cardinal number of $Y - X$ not exceed the cardinal number of X.

DEFINITION. Let Y be a compactification of X, let $K = Y - X$ and let $f\colon Y \to K$ be a continuous map. Then we shall say that *f maps onto K at ∞* iff for every co-compact set $L \subset X$, $f[L] = K$.

LEMMA. *Let Y be a compactification of X, let $K = Y - X$, and let $f\colon Y \to K$ be a continuous map which is the identity on K. Then each of the following statements implies the other three:*

1. *f maps onto K at ∞;*
2. *for any closed $H \subset K$, $H = K \cap \mathrm{Cl}_Y(X \cap f^{-1}(H))$;*
3. *for any $y \in K$, $\{y\} = K \cap \mathrm{Cl}_Y(X \cap f^{-1}(\{y\}))$;*
4. *for any $y \in K$, $y \in \mathrm{Cl}_Y(X \cap f^{-1}(\{y\}))$.*

Proof. Check first that for any closed set $H \subset K$, the inclusion $K \cap \mathrm{Cl}_Y(X \cap f^{-1}(H)) \subset H$ holds since f is the identity on K. We shall show that $1 \to 2 \to 3 \to 4 \to 1$.

Suppose f maps onto K at ∞. Let H be a closed subset of K, let $y \in H$, and let V be any open neighborhood of y in Y. It suffices to show that $V \cap X \cap f^{-1}(H) \neq \emptyset$. Note that $f^{-1}(\{y\}) - V$ is closed in Y, hence it is compact. Also, $f^{-1}(\{y\}) - V \subset X$ and so, because X is locally compact, there exists a compact set $W \subset X$ such that $f^{-1}(\{y\}) - V \subset \mathrm{int}_X(W)$. Let $L = \mathrm{Cl}_X(X - W)$. Then L is a co-compact subset of X and hence $f[L] = K$. Let $z \in L$ such that $f(z) = y$. Then $z \in V \cap X \cap f^{-1}(H)$.

It is trivial that $2 \to 3$ and $3 \to 4$.

Assume that statement 4 is true. Let L be any co-compact subset of X. Let $y \in K$. We must show $y \in f[L]$. By Theorem 1 which appeared in section I, $K \subset \mathrm{int}_Y(\mathrm{Cl}_Y(L))$, and by the hypothesis that 4 is true, $y \in \mathrm{Cl}_Y(X \cap f^{-1}(\{y\}))$. Thus, $X \cap f^{-1}(\{y\}) \cap \mathrm{int}_Y(\mathrm{Cl}_Y(L)) \neq \emptyset$. Let $z \in X \cap f^{-1}(\{y\}) \cap \mathrm{int}_Y(\mathrm{Cl}_Y(L))$ and note that $z \in X \cap \mathrm{Cl}_Y(L) = \mathrm{Cl}_X(L) = L$, which completes the proof of the lemma.

We use the following result repeatedly in the proof of the following theorem: If the hypotheses of the preceding lemma are satisfied, then whenever $H \subset K$, $K \cap f^{-1}(H) = H$. This is true because f is the identity on K.

One of our major results is the following theorem.

THEOREM 9. *Let Y be a compactification of X and let $K = Y - X$. Let $f\colon Y \to K$ be a continuous map such that f maps onto K at ∞ and such that f is the identity on K. Let $\mathcal{K}$ be the family of all subsets of X of the form $X \cap f^{-1}(H)$ where H is an arbitrary closed subset of K. Then $Y \cong w(\mathrm{CM}(\mathcal{K}))$.*

Proof. Clearly $\emptyset, X \in \mathcal{K}$ and $\mathcal{K}$ is a ring of closed subsets of X. Let $\mathcal{F} = \mathrm{CM}(\mathcal{K})$ and observe that by Corollary 1 of Theorem 4, in order to show that $\mathcal{F}$ is a normal base, it suffices to show that $\mathcal{K}$ is normal at ∞.

To show that $\mathcal{K}$ is normal at ∞, in the definition of normality at ∞, take $J = X$ and let L be any co-compact subset of X. Let H_1, H_2 be closed in K and suppose that $[X \cap f^{-1}(H_1) \cap L] \cap [X \cap f^{-1}(H_2) \cap L] = \emptyset$. Since $f[L] = K$, we get that $H_1 \cap H_2 = \emptyset$. Since K is compact, K is normal and so H_1 and H_2 are contained in disjoint open subsets of K. Let V_1 and V_2 respectively be the complements of these open sets and then we have $H_1 \cap V_1 = \emptyset$, $H_2 \cap V_2 = \emptyset$, $V_1 \cup V_2 = K$. Therefore, since

$$[X \cap f^{-1}(H_1) \cap L] \cap [X \cap f^{-1}(V_1) \cap L] = \emptyset,$$

$$[X \cap f^{-1}(H_2) \cap L] \cap [X \cap f^{-1}(V_2) \cap L] = \emptyset,$$

and

$$[X \cap f^{-1}(V_1) \cap L] \cup [X \cap f^{-1}(V_2) \cap L] = L,$$

it follows that $\mathcal{K}$ is normal at ∞.

Since $\mathcal{F}$ is a normal base, we can complete the proof by showing that $\overline{\mathcal{F}} = \{\mathrm{Cl}_Y(F) \mid F \in \mathcal{F}\}$ is separating on Y and has the trace property with respect to X.

To show $\overline{\mathcal{F}}$ is separating, let $y \in Y$ and let H be closed in Y with $y \notin H$. We consider two cases.

Case 1. $y \in X$. In this case, there exists a compact set $B \subset X$ such that $y \in \mathrm{int}_X(B)$ and $B \cap H = \emptyset$. Let $L = \mathrm{Cl}_X(X - B)$. Since $\{y\}$ is compact and L is co-compact, we have $y \in \{y\} \in \overline{\mathcal{F}}$, $H \subset \mathrm{Cl}_Y(L) \in \overline{\mathcal{F}}$, and $\{y\} \cap \mathrm{Cl}_Y(L) = \emptyset$.

Case 2. $y \in K$. In this case, by the normality of Y, there exist sets U, V open in Y such that $y \in U$, $H \subset V$, and $\mathrm{Cl}_Y(U) \cap \mathrm{Cl}_Y(V) = \emptyset$. Define sets A_1, A_2, B_1, B_2, and B_3 as follows:

$$A_1 = H \cap f^{-1}[K \cap \mathrm{Cl}_Y(U)], \quad B_1 = \mathrm{Cl}_Y(A_1) \cap \mathrm{Cl}_Y(V),$$

$$A_2 = X \cap f^{-1}[K \cap \mathrm{Cl}_Y(V)], \quad B_2 = \mathrm{Cl}_Y(A_2) \cap \mathrm{Cl}_Y(U),$$

and

$$B_3 = B_1 \cup B_2 \cup [H \cap f^{-1}(K - V)].$$

We now show $B_3 \subset X$. Using statement 2 of the Lemma together with the fact that $K \cap \mathrm{Cl}_Y(U)$ is closed in K, we have

$$K \cap \mathrm{Cl}_Y(U) = K \cap \mathrm{Cl}_Y\big(X \cap f^{-1}(K \cap \mathrm{Cl}_Y(U))\big) = K \cap \mathrm{Cl}_Y(A_1)$$

Therefore, since f is the identity on K,

$$\begin{aligned}\mathrm{Cl}_Y(A_1) &= [K \cap \mathrm{Cl}_Y(A_1)] \cup [X \cap \mathrm{Cl}_Y(A_1)] = [K \cap \mathrm{Cl}_Y(U)] \cup A_1 \\ &= [K \cap f^{-1}(K \cap \mathrm{Cl}_Y(U)] \cup \big[X \cap f^{-1}(K \cap \mathrm{Cl}_Y(U))\big] \\ &= f^{-1}\big(K \cap \mathrm{Cl}_Y(U)\big).\end{aligned}$$

Similarly, $\mathrm{Cl}_Y(A_2) = f^{-1}\big(K \cap \mathrm{Cl}_Y(V)\big)$.

Now since f is the identity on K,

$$\begin{aligned}K \cap B_1 &= K \cap \mathrm{Cl}_Y(A_1) \cap \mathrm{Cl}_Y(V) = K \cap f^{-1}\big(K \cap \mathrm{Cl}_Y(U)\big) \cap \mathrm{Cl}_Y(V) \\ &= K \cap \mathrm{Cl}_Y(U) \cap \mathrm{Cl}_Y(V) = \emptyset.\end{aligned}$$

Therefore, $B_1 \subset X$. Similarly, $B_2 \subset X$. Finally, $H \cap f^{-1}(K - V) \cap K = H \cap (K - V) = \emptyset$ and so $H \cap f^{-1}(K - V) \subset X$ and $B_3 \subset X$. Clearly, B_3 is compact. So, there exists a compact set $B_4 \subset X$ such that $B_3 \subset \mathrm{int}_X(B_4)$. Now, define sets L, L_4, M_1, and M_2 as follows:

$$L = \mathrm{Cl}_X(X - B_3), \quad L_4 = \mathrm{Cl}_X(X - B_4),$$

$$M_1 = A_1 \cap L_4 \quad \text{and} \quad M_2 = (A_2 \cap L) \cup B_3.$$

Note that by the very definition of $\mathcal{K}$, we have $A_1, A_2 \in \mathcal{K}$. Therefore, since $\mathcal{F} = \mathrm{CM}(\mathcal{K})$, $M_1, M_2 \in \mathcal{F}$, and so $\mathrm{Cl}_Y(M_1)$, $\mathrm{Cl}_Y(M_2) \in \overline{\mathcal{F}}$. In order to complete the proof that $\overline{\mathcal{F}}$ is separating, we must establish three relations; namely, $y \in \mathrm{Cl}_Y(M_1)$, $H \subset \mathrm{Cl}_Y(M_2)$, and $\mathrm{Cl}_Y(M_1) \cap \mathrm{Cl}_Y(M_2) = \emptyset$.

Now, since L_4 is co-compact, we have by Theorem 1 that

$$K \cap \mathrm{Cl}_Y(A_1) = K \cap \mathrm{Cl}_Y(A_1 \cap L_4) \subset \mathrm{Cl}_Y(M_1).$$

Therefore, since f is the identity on K,

$$y \in K \cap \mathrm{Cl}_Y(U) = K \cap f^{-1}\big(K \cap \mathrm{Cl}_Y(U)\big) = K \cap \mathrm{Cl}_Y(A_1) \subset \mathrm{Cl}_Y(M_1).$$

We next show $H \subset \mathrm{Cl}_Y(M_2)$. Since $Y = f^{-1}(K) = f^{-1}(K \cap V) \cup f^{-1}(K - V)$, we have

$$\begin{aligned}H \cap (X - B_3) &\subset H \cap L \subset [H \cap L \cap f^{-1}(K \cap V)] \cup [H \cap L \cap f^{-1}(K - V)] \\ &\subset \big[L \cap X \cap f^{-1}(K \cap \mathrm{Cl}_Y(V))\big] \cup [H \cap f^{-1}(K - V)] \\ &\subset (L \cap A_2) \cup B_3 = M_2.\end{aligned}$$

Therefore,

$$H \cap X = (H \cap B_3) \cup \big(H \cap (X - B_3)\big) \subset B_3 \cup M_2 = M_2 \subset \mathrm{Cl}_Y(M_2).$$

Next,

$$\begin{aligned}H \cap K \subset K \cap V \subset K \cap \mathrm{Cl}_Y(V) &= K \cap f^{-1}\big(K \cap \mathrm{Cl}_Y(V)\big) = K \cap \mathrm{Cl}_Y(A_2) \\ &= K \cap \mathrm{Cl}_Y(A_2 \cap L) \subset \mathrm{Cl}_Y(M_2).\end{aligned}$$

Therefore, $H = (H \cap X) \cup (H \cap K) \subset \mathrm{Cl}_Y(M_2)$. Finally, we show $\mathrm{Cl}_Y(M_1) \cap \cap \mathrm{Cl}_Y(M_2) = \emptyset$. First, since $K \cap B_3 = \emptyset$,

$$\begin{aligned}&K \cap \mathrm{Cl}_Y(M_1) \cap \mathrm{Cl}_Y(M_2) \\ &= K \cap \mathrm{Cl}_Y(A_1 \cap L_4) \cap \mathrm{Cl}_Y(A_2 \cap L) = \cap K \cap \mathrm{Cl}_Y(A_1) \cap \mathrm{Cl}_Y(A_2) \\ &= K \cap f^{-1}\big(K \cap \mathrm{Cl}_Y(U)\big) \cap f^{-1}\big(K \cap \mathrm{Cl}_Y(V)\big) = K \cap \mathrm{Cl}_Y(U) \cap \mathrm{Cl}_Y(V) = \emptyset.\end{aligned}$$

Second, since $B_3 \subset \mathrm{int}_X(B_4)$, $L_4 \cap B_3 = \emptyset$. Therefore, since M_1 and M_2 are closed in X,

$$X \cap \mathrm{Cl}_Y(M_1) \cap \mathrm{Cl}_Y(M_2) = M_1 \cap M_2 \subset (A_1 \cap A_2 \cap L \cap L_4) \cup (L_4 \cap B_3) \subset (A_1 \cap A_2) \cup \emptyset \subset f^{-1}(\mathrm{Cl}_Y(U) \cap \mathrm{Cl}_Y(V)) = f^{-1}(\emptyset) = \emptyset .$$

Putting these two inclusions together, we have

$$\mathrm{Cl}_Y(M_1) \cap \mathrm{Cl}_Y(M_2) = [K \cap \mathrm{Cl}_Y(M_1) \cap \mathrm{Cl}_Y(M_2)] \cup [X \cap \mathrm{Cl}_Y(M_1) \cap \mathrm{Cl}_Y(M_2)] = \emptyset \cup \emptyset = \emptyset .$$

This completes the proof that $\mathcal{F}$ is separating.

To show that $\mathcal{F}$ has the trace property with respect to X, let $H_1, \ldots, H_n$ be closed in K, $L_1, \ldots, L_n$ be co-compact in X, and $B_1, \ldots, B_n$ be compact in X with $\bigcap_{i=1}^{n} \mathrm{Cl}_Y[(X \cap f^{-1}(H_i) \cap L_i) \cup B_i] \neq \emptyset$ and suppose, by way of contradiction that

$$\bigcap_{i=1}^{n} [(X \cap f^{-1}(H_i) \cap L_i) \cup B_i] = \emptyset .$$

Let $y \in \bigcap_{i=1}^{n} \mathrm{Cl}_Y[(X \cap f^{-1}(H_i) \cap L_i) \cup B_i]$ and note that $y \in K$. For each i,

$$\mathrm{Cl}_Y(X \cap f^{-1}(H_i) \cap L_i) \subset \mathrm{Cl}_Y(f^{-1}(H_i)) = f^{-1}(H_i) .$$

Also, $\bigcup_{i=1}^{n} B_i \subset X$. Therefore,

$$y \in \bigcap_{i=1}^{n} \mathrm{Cl}_Y(X \cap f^{-1}(H_i) \cap L_i) \subset \bigcap_{i=1}^{n} f^{-1}(H_i) ,$$

and since $f(y) = y$, $y \in \bigcap_{i=1}^{n} H_i$. Since $\bigcap_{i=1}^{n} L_i$ is co-compact in X, $f[\bigcap_{i=1}^{n} L_i] = K$; therefore, there exists a point $x \in \bigcap_{i=1}^{n} L_i$ such that $f(x) = y$. But then $x \in \bigcap_{i=1}^{n} (X \cap f^{-1}(H_i) \cap L_i) = \emptyset$, which is impossible, and the proof of the theorem is complete.

That Theorem 9 provides us with a rather broad spectrum of compactifications can be seen from the following theorem.

THEOREM 10. *Let Y_1 and Y_2 be compactifications of X such that $Y_1 \geqslant Y_2$. Let $K_1 = Y_1 - X$ and let $f_1\colon Y_1 \to K_1$ be a continuous map such that f_1 is the identity on K_1 and f_1 maps onto K_1 at ∞. Then Y_2 is a Wallman compactification of X.*

Proof. Let $g\colon Y_1 \to Y_2$ be a continuous map which is the identity on X. Let $K_2 = Y_2 - X$ and observe that $g[K_1] = K_2$. Define the function $f_2\colon Y_2 \to K_2$ as follows: $f_2(x) = g(f_1(x))$ for each $x \in X$ and $f_2(y) = y$ for each $y \in K_2$. To show that f_2 is continuous, let H be closed in K_2. Then H is closed in Y_2. $g^{-1}(H)$ is closed in Y_1. $K_1 \cap g^{-1}(H)$ is closed in K_1. But $g^{-1}(H) \subset K_1$ so $g^{-1}(H)$ is closed in K_1. Next, $f_1^{-1}(g^{-1}(H))$ is closed in Y_1 and, finally, $g[f_1^{-1}(g^{-1}(H))]$ is closed in Y_2. Since $f_2^{-1}(H) = g[f_1^{-1}(g^{-1}(H))]$, f_2 is continuous. Clearly, f_2 is the identity on K_2 and f_2 maps onto K_2 at ∞. Therefore, Theorem 9 implies that Y_2 is a Wallman compactification of X.

As an example, let $X = \left\{\left(x, \sin\frac{1}{x}\right) \middle|\ 0 < x \leqslant 1\right\}$, let $K_1 = \{0\} \times \times [-1, 1]$, and let $Y_1 = K_1 \cup X$. Then Y_1 is a compactification of X and if we define $f_1\colon Y_1 \to K$ by $f_1(x, y) = (0, y)$ for each $(x, y) \in Y_1$, f_1 satisfies the hypotheses of Theorem 10. Thus, any upper semi-continuous decomposition of Y_1 in which the single points of X are elements is a Wallman compactification of X; moreover, we have given an explicit construction for a normal base inducing such a compactification.

IV. Some questions. We have been able to prove that certain compactifications of locally compact spaces which were not previously known to be Wallman compactifications are indeed so. Nevertheless, we have not been able to prove that every compactification of a locally compact space is Wallman. A consequence of our results is that if a compactification is Wallman, then it is determined by a normal base which contains all compact sets and all co-compact sets. Thus, if there exists a compactification Y which is not determined by any normal base which contains all compact and all co-compact sets, then Y is not Wallman.

In [8], A. K. Steiner and E. F. Steiner present a theorem concerning a map $f\colon X \to K$ where K is the remainder of some compactification of X and where f maps onto a dense subset of K at ∞. The proof of our Theorem 9 is not valid if we so generalize f. Nevertheless, the theorem may have such a generalization which is valid. We do not know the answer to this question.

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Connectivity retracts of unicoherent Peano continua in R^n

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If X is a topological space with the connectivity function fixed point property, then each connectivity retract of X has the continuous function fixed point property. (See [7] for the definitions and background.) Since I^2 has the connectivity function fixed point property (see [6]), it was hoped that any non-separating planar continuum might be a connectivity retract of I^2 and hence have the fixed point property. Cornette, however, demonstrated in [3] that the class of unicoherent Peano continua was closed under connectivity retraction. In a subsequent joint paper, Cornette and Girolo (see [4]) raise the question: "Is there a k-coherent Peano continuum that has a connectivity retract that is not a continuous retract?".

That this question has a negative answer for sets in R^2 was proven in [2]. The purpose of this paper is to describe an example in R^n, $n \geqslant 3$, which provides an affirmative answer to the major question of [4].

In the subsequent discussion, we will use these results:

(1) If a Peano continuum X fails to be unicoherent, there is a simple closed curve which is a retract of X, and thus $H_1(X, Z) \neq 0$ (Čech homology, integral coefficients). (See [5].)

(2) If each of X and Y are unicoherent Peano continua and $X \cap Y$ is connected, then $X \cup Y$ is unicoherent. (See [8], Chapter 9.)

(3) If X is a unicoherent Peano continuum with no cut points, and $f\colon X \to X$ is a peripherally continuous function, then f is a connectivity function. (A function $f\colon X \to Y$ is peripherally continuous if for each point $p \in X$ and each pair of open sets U, V about p and $f(p)$ respectively, there is an open set W, $p \in W \subset U$ and $f(\mathrm{Bd}\,W) \subset V$. See [9].)

Let $\theta = (0, \ldots, 0)$ in R^{n-1}, and the set

$$B_k = \{x \in R^{n-1}\colon d[x, (3/2^{k+1}, 0, \ldots, 0)] \leqslant 1/2^{k+1}\}, \quad \text{and} \quad B = \{\theta\} + \sum_{k=1}^{\infty} B_k.$$