On the family of sets of approximate limit numbers

by

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Let f be a bounded real function of a real variable. The number y_0 is called the approximate limit number of f at x_0 if and only if for each $\varepsilon > 0$ the set $f^{-1}\big((y_0 - \varepsilon, y_0 + \varepsilon)\big)$ has the upper exterior density at x_0 greater than zero. The set of all approximate limit numbers of f at x_0 will be denoted by $L_{ap}(f, x_0)$. Right- and left-sided approximate limit numbers are defined similarly in an obvious way and the sets of those numbers are denoted by $L^+_{ap}(f, x_0)$ and $L^-_{ap}(f, x_0)$ respectively.

This work includes some characterization of the family of sets $\{L_{ap}(f, x)\}_{x \in [a,b]}$. A similar characterization (but of the family of sets of ordinary limit numbers) is given in [1].

We shall use the following notation: if $\mathfrak{A}$ is an arbitrary plane set and E is an arbitrary linear set, then $\mathfrak{A}^{-1}(E) = \{x : \bigvee_{y \in E} \big((x, y) \in \mathfrak{A}\big)\}$.

Definition. If $\mathfrak{A}$ is a plane set, then the number y_0 is called the *approximate limit number of* $\mathfrak{A}$ *at* x_0 if and only if for each $\varepsilon > 0$ the set $\mathfrak{A}^{-1}\big((y_0 - \varepsilon, y_0 + \varepsilon)\big)$ has the upper exterior density at x_0 greater than zero.

Let us observe that in the case where $\mathfrak{A}$ is the graph of some function f, this definition is equivalent to the definition of the approximate limit number of f.

The set of all approximate limit numbers of $\mathfrak{A}$ at x_0 will be denoted by $L_{ap}(\mathfrak{A}, x_0)$. It is not difficult to see that for each plane set $\mathfrak{A}$ and for each x_0 the set $L_{ap}(\mathfrak{A}, x_0)$ is closed.

Now let us suppose that the bounded real function f is defined on the open interval (a, b). Let us put:

$$\mathfrak{L}_{ap}(f, x) = \{x\} \times L_{ap}(f, x) \quad \text{for } x \in [a, b], \qquad \mathfrak{L}_{ap}(f) = \bigcup_{x \in [a,b]} \mathfrak{L}_{ap}(f, x)$$

$\big($of course $\mathfrak{L}_{ap}(f, a) = \mathfrak{L}^+_{ap}(f, a)$ and $\mathfrak{L}_{ap}(f, b) = \mathfrak{L}^-_{ap}(f, b)\big)$ and analogously for the plane set $\mathfrak{A}$:

$$\mathfrak{L}_{ap}(\mathfrak{A}, x) = \{x\} \times L_{ap}(\mathfrak{A}, x), \qquad \mathfrak{L}_{ap}(\mathfrak{A}) = \bigcup_{x \in R} \mathfrak{L}_{ap}(\mathfrak{A}, x)$$

(R denotes here the set of all real numbers).

THEOREM 1. *If f is a bounded real function defined on the open interval (a, b), then*

$$\mathfrak{L}_{ap}(f) = \mathfrak{L}_{ap}\big(\mathfrak{L}_{ap}(f)\big).$$

Proof. Let $(x_0, y_0) \in \mathfrak{L}_{ap}(f)$ and let ε be an arbitrary positive number. The set $f^{-1}\big((y_0-\frac{1}{2}\varepsilon, y_0+\frac{1}{2}\varepsilon)\big)$ has the upper exterior density at x_0 greater than zero. Let C_d denote the set consisting of all points of the exterior density of $f^{-1}\big((y_0-\frac{1}{2}\varepsilon, y_0+\frac{1}{2}\varepsilon)\big)$. It is well known that the set C_d has the upper exterior density at x_0 greater than zero, and that for each $x \in C_d$ there exists an $y \in [y_0-\frac{1}{2}\varepsilon, y_0+\frac{1}{2}\varepsilon] \cap L_{ap}(f, x)$. Hence $\big(\mathfrak{L}_{ap}(f)\big)^{-1}\big((y_0-\varepsilon, y_0+\varepsilon)\big) \supset C_d$, and so the set $\big(\mathfrak{L}_{ap}(f)\big)^{-1}\big((y_0-\varepsilon, y_0+\varepsilon)\big)$ has the upper exterior density at x_0 greater than zero and $(x_0, y_0) \in \mathfrak{L}_{ap}\big(\mathfrak{L}_{ap}(f)\big)$.

Let $(x_0, y_0) \in \mathfrak{L}_{ap}\big(\mathfrak{L}_{ap}(f)\big)$ and let ε be an arbitrary positive number. The set $\big(\mathfrak{L}_{ap}(f)\big)^{-1}\big((y_0-\varepsilon, y_0+\varepsilon)\big)$ has the upper exterior density at x_0 greater than zero. Let us observe that if $x_1 \in \big(\mathfrak{L}_{ap}(f)\big)^{-1}\big((y_0-\varepsilon, y_0+\varepsilon)\big)$, then the set $f^{-1}\big((y_0-\varepsilon, y_0+\varepsilon)\big)$ has the upper exterior density at x_1 greater than zero. Indeed, if $y_1 \in L_{ap}(f, x_1) \cap (y_0-\varepsilon, y_0+\varepsilon)$ and $\varepsilon_1 = \min(y_1-y_0+\varepsilon, y_0+\varepsilon-y_1)$, then $f^{-1}\big((y_0-\varepsilon, y_0+\varepsilon)\big) \supset f^{-1}\big((y_1-\varepsilon_1, y_1+\varepsilon_1)\big)$ and this last set has the upper exterior density at x_1 greater than zero. We shall prove that the set $f^{-1}\big((y_0-\varepsilon, y_0+\varepsilon)\big)$ also has the upper exterior density at x_0 greater than zero. If suffices to prove that for every interval $[a_0, b_0] \subset [a, b]$ the following inequality is true:

$$|f^{-1}\big((y_0-\varepsilon, y_0+\varepsilon)\big) \cap [a_0, b_0]|_e \geqslant |\big(\mathfrak{L}_{ap}(f)\big)^{-1}\big((y_0-\varepsilon, y_0+\varepsilon)\big) \cap [a_0, b_0]|_e$$

(where $|A|_e$ denotes the exterior measure of A).

Let us suppose that there exists an interval $[a_0, b_0] \subset [a, b]$ such that the above inequality is not fulfilled. Let F be a measurable cover of the set $f^{-1}\big((y_0-\varepsilon, y_0+\varepsilon)\big)$. Then almost every point of $[a_0, b_0]-F$ is a point of dispersion of F and also of the exterior dispersion of $f^{-1}\big((y_0-\varepsilon, y_0+\varepsilon)\big)$. Hence there exists a point $x' \in \Big[\big(\mathfrak{L}_{ap}(f)\big)^{-1}\big((y_0-\varepsilon, y_0+\varepsilon)\big)-f^{-1}\big((y_0-\varepsilon, y_0+\varepsilon)\big)\Big] \cap [a_0, b_0]$ which is a point of the exterior dispersion of the set $f^{-1}\big((y_0-\varepsilon, y_0+\varepsilon)\big)$. This is impossible in virtue of the first part of the proof. So the set $f^{-1}\big((y_0-\varepsilon, y_0+\varepsilon)\big)$ has the upper exterior density at x_0 greater than zero and $(x_0, y_0) \in \mathfrak{L}_{ap}(f)$. The equality is proved.

Let us observe that for a bounded real function defined on the open interval (a, b) for every $x \in [a, b]$ the set $L_{ap}(f, x)$ is non-empty. It follows from the fact that for such a function extreme approximate limits at x (see Saks [2], p. 219) are finite numbers and belong to this set. This remark is made to justify the additional assumption in the following converse theorem.

THEOREM 2. *If $\mathfrak{A}$ is a bounded plane set fulfilling the following conditions:*

1° *the set $\{y: (x_0, y) \in \mathfrak{A}\}$ is non-empty if and only if x_0 belongs to a closed interval $[a, b]$,*

2° $\mathfrak{L}_{ap}(\mathfrak{A}) = \mathfrak{A}$,

then there exists a bounded function f defined on the open interval (a, b) such that $\mathfrak{L}_{ap}(f) = \mathfrak{A}$.

Before proving this theorem we shall prove the following lemma (under the assumption of the continuum hypothesis):

LEMMA. *For all linear bounded sets A, B and for all families of sets $\{A_t\}_{t \in T}$ and $\{B_u\}_{u \in U}$, where $A_t \subset A$ for each $t \in T$, $B_u \subset B$ for each $u \in U$ and the sets T, U have powers not greater than the power of the continuum, there exist two sets A', B' fulfilling the following conditions:*

a. $A' \subset A$, $B' \subset B$, $A' \cap B' = \emptyset$, $A' \cup B' = A \cup B$,

b. $|A'|_e = |A|_e$, $|B'|_e = |B|_e$,

and for each $t \in T$, $u \in U$

$$|A' \cap A_t|_e = |A_t|_e, \qquad |B' \cap B_u|_e = |B_u|_e.$$

Proof of the lemma. If the set $A \cap B$ is of measure zero, then there is nothing to prove, but in the proof we shall not make this assumption. On the other hand, we shall assume that every set from these two families is of positive exterior measure (for the sets of measure zero the condition b will always be fulfilled). Let us suppose, for convenience, that $A = A_{t_0}$ for some $t_0 \in T$ and similarly $B = B_{u_0}$ for some $u_0 \in U$.

Let $\mathcal{G}_t$ be the family of all open sets G such that $|G| < |A_t|_e$ and let $\mathcal{G}_u$ be the family of all open sets G such that $|G| < |B_u|_e$. Let us write

$$\mathcal{G}'_t = \{G \times \{t\}: G \in \mathcal{G}_t\}, \qquad \mathcal{G}_A = \bigcup_{t \in T} \mathcal{G}'_t,$$

$$\mathcal{G}'_u = \{G \times \{u\}: G \in \mathcal{G}_u\}, \qquad \mathcal{G}_B = \bigcup_{u \in U} \mathcal{G}'_u.$$

The sets $\mathcal{G}_A$ and $\mathcal{G}_B$ are of the power of the continuum, because for each $t \in T$ and $u \in U$ the families $\mathcal{G}_t$ and $\mathcal{G}_u$ are of the power of the continuum and the sets T, U are of power not greater than the power of the continuum. These sets may be well-ordered and the ordinal number of each of them is equal to Ω. This fact follows from the assumption of the continuum hypothesis. We obtain sets A' and B' in the following way. If $G_{A,1} \times \{t_1\}$ is the first element of $\mathcal{G}_A$, then let us choose $x_1 \in A_{t_1}-G_{A,1}$. It is possible because $G_{A,1} \in \mathcal{G}_{t_1}$, and so $|G_{A,1}| < |A_{t_1}|_e$ and $A_{t_1}-G_{A,1}$ is a non-empty set. If $G_{B,1} \times \{u_1\}$ is the first element of $\mathcal{G}_B$, then let us choose $y_1 \in B_{u_1}-(G_{B,1} \cup \{x_1\})$. It is possible because $G_{B,1} \in \mathcal{G}_{u_1}$, and so $|G_{B,1}| < |B_{u_1}|_e$; hence $B_{u_1}-G_{B,1}$ is non-denumerable and $B_{u_1}-(G_{B,1} \cup \{x_1\})$ is non-empty.

Now let $\alpha < \Omega$. Suppose that for every $\beta < \alpha$ we have already chosen x_β, y_β. If $G_{A,\alpha} \times \{t_\alpha\}$ is the αth element of $\mathcal{G}_A$, then let us choose $x_\alpha \in A_{t_\alpha}-$

$-(G_{A,\alpha}\cup\{x_\beta\colon \beta<\alpha\}\cup\{y_\beta\colon \beta<\alpha\})$. It is possible because $\mathcal{G}_{A,\alpha}\in\mathcal{G}_{t_\alpha}$, and so $|G_{A,\alpha}|<|A_{t_\alpha}|_e$; hence $A_{t_\alpha}-G_{A,\alpha}$ is non-denumerable and the sets $\{x_\beta\colon \beta<\alpha\}$ and $\{y_\beta\colon \beta<\alpha\}$ are denumerable, and so $A_{t_\alpha}-(G_{A,\alpha}\cup\{x_\alpha\colon \beta<\alpha\}\cup\{y_\beta\colon \beta<\alpha\})$ is non-empty. If $G_{B,\alpha}\times\langle u_\alpha\rangle$ is the αth element of $\mathcal{G}_B$, then let us choose $y_\alpha\in B_{u_\alpha}-(G_{B,\alpha}\cup\{x_\beta\colon \beta\leqslant\alpha\}\cup\{y_\beta\colon \beta<\alpha\})$. It is possible because this set is non-empty for the same reasons as above. By transfinite induction we obtain x_α, y_α for every $\alpha<\Omega$. Let

$$A'=(A-B)\cup\{x_\alpha\colon \alpha<\Omega\},\qquad B'=(A\cup B)-A'.$$

From this definition it follows that $A'\cap B'=\emptyset$ and $A'\cup B'=A\cup B$. Also the elements x_α were chosen from the set A, and so $\{x_\alpha\colon \alpha<\Omega\}\subset A$. Hence $A'\subset A$. We have $A'\supset A-B$; then $B'\subset(A\cup B)-(A-B)=B$, and so the condition a is fulfilled. Now we shall prove that b is also fulfilled. Let $t\in T$ and let G_0 be an arbitrary open set such that $|G_0|<|A_t|_e$. Then $G_0\times\{t\}\in\mathcal{G}_t'\subset\mathcal{G}_A$, and so $G_0\times\{t\}=G_{A,\alpha}\times\{t_\alpha\}$ for some $\alpha<\Omega$. From the construction it follows that $x_\alpha\in A'$, $x_\alpha\in A_t$ and $x_\alpha\notin G_0$, so $x_\alpha\in(A'\cap A_t)-G_0$. Hence for every open set G for which $G\supset A'\cap A_t$ we have $|G|\geqslant|A_t|_e$. Then $|A_t|_e\geqslant|A'\cap A_t|_e=\inf\limits_{G\supset A'\cap A_t}|G|\geqslant|A_t|_e$, and so the first part of the condition b is proved. To prove the second part let us observe that $\{y_\alpha\colon \alpha<\Omega\}\subset B'$. Indeed, we have $y_\alpha\in B$ for each α and $\{x_\alpha\colon \alpha<\Omega\}\cap\cap\{y_\alpha\colon \alpha<\Omega\}=\emptyset$. The proof of the fact that $|B'\cap B_u|_e=|B_u|_e$ is similar to the proof in the case of the set A_t.

Proof of the theorem. The set $\mathfrak{A}$ is bounded. Let $[c,d)$ be such an interval that $\mathfrak{A}$ is included in the rectangle $[a,b]\times[c,d)$. Let us introduce the notation $I_{n,k}=[c+k\cdot(d-c)\cdot 2^{-n},\ c+(k+1)\cdot(d-c)\cdot 2^{-n})$, $A_{n,k}=\mathfrak{A}^{-1}(I_{n,k})$ for $n=1,2,\dots$, $k=0,1,\dots,2^n-1$. Of course we have $A_{n,k}=A_{n+1,2k}\cup\cup A_{n+1,2k+1}$.

From the lemma it follows that for the sets $A_{1,0}$ and $A_{1,1}$ and for families $\{A_{n,k}\}_{n=2,3,\dots,k=0,1,\dots,2^{n-1}-1}$ and $\{A_{n,k}\}_{n=2,3,\dots,k=2^{n-1},\dots,2^n-1}$ of their subsets there exist two sets $A'_{1,0}$ and $A'_{1,1}$ fulfilling all conditions of the lemma. Similarly, for the sets $A'_{1,0}\cap A_{2,0}$ and $A'_{1,0}\cap A_{2,1}$ and the families $\{A'_{1,0}\cap A_{n,k}\}_{n=3,\dots,k=0,\dots,2^{n-2}-1}$ and $\{A'_{1,0}\cap A_{n,k}\}_{n=3,\dots,k=2^{n-2},\dots,2^{n-1}-1}$ of their subsets there exist two sets $A'_{2,0}$ and $A'_{2,1}$ fulfilling all conditions of the lemma. The same situation is for the sets $A'_{1,1}\cap A_{2,2}$ and $A'_{1,1}\cap A_{2,3}$ with corresponding families of subsets of type $A'_{1,1}\cap A_{n,k}$. Continuing this procedure successively for every pair of sets $A'_{n,k}\cap A_{n+1,2k}$ and $A'_{n,k}\cap A_{n+1,2k+1}$, we obtain the family of sets $\{A'_{n,k}\}_{n=1,2,\dots,k=0,\dots,2^n-1}$ such that

1. for each n,k $A'_{n,k}\subset A_{n,k}$,
2. for each n $\bigcup\limits_{k=0}^{2^n-1}A'_{n,k}=[a,b]$,

3. for fixed n the sets $A'_{n,k}$ are mutually disjoint,
4. for each n,k $|A'_{n,k}|_e=|A_{n,k}|_e$.

The first condition is obviously fulfilled. The second, third and fourth conditions may easily be proved by finite induction. It suffices only to observe that they are fulfilled for $n=1$ and that for each n,k

$$A'_{n+1,2k}\cup A'_{n+1,2k+1}=(A'_{n,k}\cap A_{n+1,2k})\cup(A'_{n,k}\cap A_{n+1,2k})=A'_{n,k}\cap A_{n,k}=A'_{n,k}$$

(second condition); if $k_1\neq k_2$, then $A'_{n,k_1}\cap A'_{n,k_2}=0$ if $k_1=2k_0$, $k_2=2k_0+1$ for some k_0 and $A'_{n,k_1}\subset A'_{n-1,[k_1/2]}$, $A'_{n,k_2}\subset A'_{n-1,[k_2/2]}$, where $[k_1/2]\neq[k_2/2]$ in another case (third condition); $|A'_{n,k}|_e=|A'_{n-1,[k/2]}\cap A_{n,k}|_e$ (fourth condition).

(Here $[a]$ denotes the largest integer not exceeding a.)

Now we shall define an auxiliary sequence of functions $\{f_n\}$. Let f_n be a function associating with an element $x\in A'_{n,k}$ $(k=0,1,\dots,2^n-1)$ such a number $y\in I_{n,k}$ that $(x,y)\in\mathfrak{A}$. This sequence of functions is convergent (it is even uniformly convergent), because from 1, from the equality $A_{n,k}=A_{n+1,2k}\cup A_{n+1,2k+1}$ and from the definition of $A_{n,k}$ we have $|f_{n+1}(x)-f_n(x)|<2^{-n}\cdot(d-c)$ for every $x\in[a,b]$.

Let us put $f_0=\lim\limits_{n\to\infty}f_n$. We shall prove that $\mathfrak{L}_{ap}(f_0)=\mathfrak{A}$. For each n the graph of f_n is included in $\mathfrak{A}$. From the fact that for every $x\in[a,b]$ the set $\mathfrak{L}_{ap}(\mathfrak{A},x)$ is closed we infer that the graph of f_0 is also included in $\mathfrak{A}$. Hence $\mathfrak{L}_{ap}(f_0)\subset\mathfrak{L}_{ap}(\mathfrak{A})=\mathfrak{A}$. Now let $(x_0,y_0)\in\mathfrak{A}$ and let ε be an arbitrary positive number. Suppose first that y_0 is not of the form $c+k\cdot(d-c)\cdot 2^{-n}$. Let us find an interval $I_{n,k}$ such that $y_0\in I_{n,k}$ and $\bar I_{n,k}\subset(y_0-\varepsilon,y_0+\varepsilon)$ ($\bar I_{n,k}$ means the closure of $I_{n,k}$). From the construction of f_0 we have $f_0^{-1}(\bar I_{n,k})\supset f_n^{-1}(I_{n,k})=A'_{n,k}$. But $A'_{n,k}\subset A_{n,k}$ and $|A'_{n,k}|_e=|A_{n,k}|_e$, and so it is not difficult to prove that the upper exterior density of $A'_{n,k}$ is equal to the upper exterior density of $A_{n,k}$ at an arbitrary x. From the assumption of the theorem and from the inclusion $f_0^{-1}((y_0-\varepsilon,y_0+\varepsilon))\supset f_0^{-1}(\bar I_{n,k})$ we infer that the set $f_0^{-1}((y_0-\varepsilon,y_0+\varepsilon))$ has the upper exterior density at x_0 greater than zero and so $(x_0,y_0)\in\mathfrak{L}_{ap}(f_0)$. If y_0 is of the form $c+\!+k\cdot(d-c)\cdot 2^{-n}$, then we choose two intervals I_{n_0,k_0} and I_{n_0,k_0+1} such that y_0 is the common end-point of $\bar I_{n_0,k_0}$ and I_{n_0,k_0+1} and $\overline{I_{n_0,k_0}\cup I_{n_0,k_0+1}}\subset(y_0-\varepsilon,y_0+\varepsilon)$ and we proceed as above, observing only that from the assumption of the theorem it follows that at least one of the sets A_{n_0,k_0}, A_{n_0,k_0+1} has the upper exterior density at x_0 greater than zero. Thus in both cases $(x_0,y_0)\in\mathfrak{L}_{ap}(f)$. If we put $f=f_0|(a,b)$ (f_0 reduced to (a,b)), then it is easy to see that f fulfills all requirements.

The generalization of these theorems to real functions defined in Euclidean spaces of any number of dimensions and to unbounded functions offers no difficulty.

References

[1] J. Jędrzejewski and W. Wilczyński, *On the family of sets of limit numbers*, Bull. Acad. Pol. Sc. XVIII, 8 (1970), pp. 453–460.
[2] S. Saks, *Theory of the integral*, Warszawa-Lwów 1937.

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Cohomotopy groups and shape in the sense of Fox

by

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In [2] K. Borsuk introduced the relations of *fundamental domination* and *fundamental equivalence* in the class of compact metrizable spaces and proved that: (i) homotopy domination (equivalence) implies fundamental domination (equivalence), (ii) in the class of absolute neighbourhood retracts fundamental domination (equivalence) implies homotopy domination (equivalence). In [3] K. Borsuk introduced the notion of the *shape* of a compactum X; it is the collection of all compacta fundamentally equivalent to X. In [4] R. H. Fox extends the notion of shape to arbitrary metrizable spaces such that for compacta the extended notion coincides with Borsuk's original notion of shape and the properties (i) and (ii) are preserved.

In [5] and [6] I proved that in the class of compacta cohomotopy groups are invariances of shape and that if a compactum X fundamentally dominates a compactum Y and there exists an nth cohomotopy group $\pi^n(X)$ of the compactum X, then there exists an nth cohomotopy group $\pi^n(Y)$ of the compactum Y and $\pi^n(Y)$ is a divisor of $\pi^n(X)$.

The aim of this paper is to extend my results mentioned above to arbitrary metrizable spaces.

§ 1. Basic notions. In this section we recall the notions introduced by R. H. Fox in [4].

Consider an arbitrary category E and let $\sim$ be a compositive equivalence relation on the collection $\mathrm{Mor}E$ of morphisms of E. Two morphisms of E are *concurrent* if they have the same domain and the same range. If $u_1, u_2 \in \mathrm{Mor}E$ are concurrent and if $u \in \mathrm{Mor}E$ is a morphism such that $u_1 u \sim u_2 u$, then u is an *equalizer* of u_1 and u_2. An object $U \in \mathrm{Ob}E$ is a *predecessor* of an object $U' \in \mathrm{Ob}E$ in E if there exists a morphism $u \in \mathrm{Mor}E$ with domain U and range U', $u\colon U \to U'$.

A subcategory $\boldsymbol{U}$ of E is called an *inverse system* if

(1.1) any two objects of $\boldsymbol{U}$ have a common predecessor in $\boldsymbol{U}$

and

(1.2) any two concurrent morphisms of $\boldsymbol{U}$ have an equalizer in $\boldsymbol{U}$.