Borel-complete topological spaces [1]

by

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A topological space will be called Borel-complete if each ultrafilter of Borel sets has non-void total intersection when each of its countable subfamilies does; equivalently, the space is complete in the uniformity generated by Borel-measurable functions to the real line. Our study of spaces with this property is in two parts. The first (§ 2) concerns the behaviour of Borel-completeness under topological operations: Borel-completeness is (completely) hereditary, additive for a countable family of Borel sets, countably productive (and no nontrivial uncountable product is Borel-complete), and preserved under one-to-one Borel measurable pre-image. The second (§ 3) concerns the relation with realcompactness. We prove: a Borel-complete space has nonmeasurable power, and is realcompact if countable paracompact and normal; a space is Borel-complete if hereditarily realcompact, or if realcompact and each open set can be derived from the closed sets by the Souslin operation.

There are a few related results in the literature. In 1948, Marczewski and Sikorski proved that for metric spaces, Borel completeness is equivalent to nonmeasurability [18]; in 1950, Hewitt proved that *Baire*-completeness is equivalent to realcompactness [9]; recently, Dykes has studied spaces complete relative to the *closed* sets [3]. The Hewitt and Dykes Theorems are discussed in § 1, and will be used a few times in this paper. Each of our results relating Borel-completeness to nonmeasurability and realcompactness can be viewed as an improvement of one half or the other of the Marczewski-Sikorski Theorem; and our argument to establish 3.8 below is similar to their proof. (Observe that the Marczewski-Sikorski Theorem and the Hewitt Theorem yield the corollary: for metric spaces, realcompactness is equivalent to nonmeasurability. This nontrivial result is usually obtained as a corollary of the later Theorems of Katětov and Shirota. See [20] for a direct proof.)

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Many questions remain:

Among compact spaces, Borel-complete spaces include metric spaces and one-point (or $\aleph_0$-point) compactifications of nonmeasurable discrete sets (or locally compact Borel-complete spaces); some non-Borel-complete spaces are the ordinal space ω_1+1, the spaces $2^{\aleph_1}$ and βN. But the structure of compact Borel-complete spaces is largely a mystery.

We conjecture that no nontrivial βX is Borel-complete, but the issue of Borel-complete extensions (of a necessarily Borel-complete space) is largely unresolved. E. G., when is the Hewitt extension vX Borel-complete? What spaces have compact Borel-complete extensions? Etc.

1. Background. Let $\mathcal{P}(X)$ be the power set of X and let $\mathcal{A} \subset \mathcal{P}(X)$. An $\mathcal{A}$-filter is a family $\mathcal{F} \subset \mathcal{A}$ such that $F_1, \ldots, F_n \in \mathcal{F}$ implies $\bigcap F_i \neq \emptyset$, and $\mathcal{A} \ni A \supset F \in \mathcal{F}$ implies $A \in \mathcal{F}$. The $\mathcal{A}$-filter $\mathcal{F}$: has the *countable intersection property*, or cip, if $F_1, F_2, \ldots \in \mathcal{F}$ implies $\bigcap F_i \neq \emptyset$; is *fixed* if $\bigcap \mathcal{F} \neq \emptyset$, and *free* otherwise.

In general, the $\mathcal{A}$-filter $\mathcal{F}$ is an ultrafilter (i.e. maximal) if $A \in \mathcal{A}$ and $A \cap F \neq \emptyset$ for $F \in \mathcal{F}$ implies $A \in \mathcal{F}$. In case $\mathcal{A}$ is closed under complementation (e.g. a σ-algebra), $\mathcal{F}$ is an ultrafilter iff for $A \in \mathcal{A}$, either $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$. Note that $\mathcal{A}$-ultrafilters with cip are closed under countable intersection if $\mathcal{A}$ is. All families are assumed to have this property.

If $p \in X$, $\mathcal{F}_p$ denotes the fixed $\mathcal{A}$-ultrafilter $\{A \in \mathcal{A}: p \in A\}$; clearly, any fixed ultrafilter is of this form (and p is unique if $\mathcal{A}$ separates points of X).

If each $\mathcal{A}$-ultrafilter with cip is fixed we shall say that X is $\mathcal{A}$-complete. We shall consider completeness relative to $\mathcal{P}(X)$ itself, and to families which arise as follows.

Let X be a topological space, always assumed completely regular Hausdorff. $\mathcal{Z}(X)$ is the family of zero-sets, the sets of the form $\{x: f(x) = 0\}$ for some continuous real-valued function f. $\mathcal{B}a(X)$ is the family of Baire sets of X, the σ-algebra generated by $\mathcal{Z}(X)$. $C(X)$ is the family of closed sets, and $\mathcal{B}o(X)$, the Borel sets, is the generated σ-algebra.

Given $\mathcal{A} \subset \mathcal{P}(X)$, let $\sigma(\mathcal{A})$ denote the least σ-algebra in $\mathcal{P}(X)$ which contains $\mathcal{A}$, $\varrho(\mathcal{A})$ the least family closed under countable intersection and union which contains $\mathcal{A}$, Souslin $(\mathcal{A})$ the family of sets derivable from $\mathcal{A}$ by the Souslin operation. Then $\mathcal{B}a(X) = \sigma(\mathcal{Z}(X)) = \varrho(\mathcal{Z}(X)) \subset \text{Souslin}(\mathcal{Z}(X))$ (see [5]), while in general $\mathcal{B}o(X) = \sigma(C(X))$ properly contains $\varrho(C(X))$, and is incomparable with Souslin$(C(X))$; $\varrho(C(X)) \subset \text{Souslin}(C(X))$ holds.

As is well-known, $\mathcal{Z}(X)$-completeness is equivalent to Hewitt's realcompactness of X, and $\mathcal{P}(X)$-completeness is equivalent to ($\mathcal{Z}(X)$-completeness of discrete X, and to) Ulam non-measurability of the cardinal number $|X|$; see [6]. Less well-known is Hewitt's nontrivial theorem that $\mathcal{Z}(X)$-completeness and $\mathcal{B}a(X)$-completeness are equivalent [9]. (Actually, Hewitt proves that X is realcompact iff each nonzero Baire measure on X is a point mass. This condition on measures is seen easily to be equivalent to $\mathcal{B}a(X)$-completeness.) This result has been reproved and explicated by Hayes [8] and Frolík [5]; in these treatments, the result derives from general theorems implying equivalence of $\mathcal{A}$-completeness and $\sigma(\mathcal{A})$-completeness whenever $\sigma(\mathcal{A}) = \varrho(\mathcal{A})$ (Hayes), or $\sigma(\mathcal{A}) \subset \text{Souslin}(\mathcal{A})$ (Frolík).

For $C(X)$ and $\mathcal{B}o(X)$, we have only this:

1.1. THEOREM. *$\mathcal{B}o(X)$-completeness implies $C(X)$-completeness; the converse holds if each open set is in Souslin$(C(X))$ (e.g., when open sets are F_σ).*

Proof. The first assertion follows from [8], the second from [5].

$C(X)$-completeness itself has been (defined dually, called a-realcompactness and) studied by Dykes [3]. She shows that $\mathcal{Z}(X)$-completeness implies $C(X)$-completeness, and the converse holds if X is a cb-space (e.g., countably paracompact and normal, or countably compact [13]).

2. Borel completeness under topological operations. The following lemma will be used in the proofs of our three main theorems.

2.1 LEMMA. *If $f: X \to Y$ is Borel-measurable (e.g., continuous), and $\mathcal{F}$ is a $\mathcal{B}o(X)$-ultrafilter with cip, then $f(\mathcal{F}) \equiv \{B \in \mathcal{B}o(Y): f^{-1}(B) \in \mathcal{F}\}$ is a $\mathcal{B}o(Y)$-ultrafilter with cip.*

Proof. Evidently, $f(\mathcal{F})$ is a $\mathcal{B}o(Y)$-filter with cip. If $B \notin f(\mathcal{F})$, then $f^{-1}(B) \notin \mathcal{F}$, and $X - f^{-1}(B) \in \mathcal{F}$, because $\mathcal{F}$ is maximal. But $X - f^{-1}(B) = f^{-1}(Y - B)$, so $Y - B \in f(\mathcal{F})$, and $f(\mathcal{F})$ is maximal.

We first shall consider subspaces.

2.2 LEMMA. *For $E \subset X$, $\mathcal{B}o(E) = \mathcal{B}o(X) \cap E$.*

Proof. By $\mathcal{B}o(X) \cap E$ we mean $\{B \cap E: B \in \mathcal{B}o(X)\}$, and likewise for any $\mathcal{A} \subset \mathcal{P}(X)$. [7], Theorem E, p. 25, asserts that $\sigma(\mathcal{A} \cap E) = \sigma(\mathcal{A}) \cap E$. Using $\mathcal{A} = C(X)$, and the fact that $C(E) = C(X) \cap E$, the result follows.

2.3 THEOREM. *Let $E \subset X$. If X is Borel-complete then E is Borel-complete.*

Proof. Let $\mathcal{F}$ be a $\mathcal{B}o(E)$-ultrafilter with cip, and let f be the inclusion map of E into X. By 2.2, f is Borel-measurable, so by 2.1 and $\mathcal{B}o(X)$-completeness, there is $p \in \bigcap f(\mathcal{F})$. Since $f(\mathcal{F}) = \{B \in \mathcal{B}o(X): B \cap E \in \mathcal{F}\}$, it follows that $p \in \bigcap \mathcal{F}$ and E is $\mathcal{B}o(E)$-complete.

This result is trickier than it looks. More generally, let $\mathcal{A} \subset \mathcal{P}(X)$ be a σ-algebra and call $E \subset X$ $\mathcal{A}$-closed if $E = \bigcap \mathcal{E}$ for some $\mathcal{E} \subset \mathcal{A}$. It can be shown that X $\mathcal{A}$-complete and E $\mathcal{A}$-closed imply that E is $\mathcal{A}\cap E$-complete (and this is a special case of a theorem from the theory of uniform spaces). 2.3 follows from this after noting 2.2 and that fact that all subsets of X are $\mathcal{B}o(X)$-closed (because any $\{p\}\in\mathcal{B}o(X)$). Taking $\mathcal{A}=\mathcal{B}a(X)$, and proving further that $\mathcal{B}a(X)\cap E$-completeness implies $\mathcal{B}a(E)$-completeness — another special case of a theorem from the theory of uniform spaces — we find that if E is Baire-closed in X and X is $\mathcal{B}a(X)$-complete (i.e., realcompact, from § 1), then E is $\mathcal{B}a(E)$-complete, i.e., realcompact. Compare [8]. This implies Negrepontis' theorem that a Baire-set in a realcompact space is realcompact [19], and Mrówka's theorem [16] that a Q-closed subset of a realcompact space is realcompact (because "Q-closed" is equivalent to "Baire-closed", as can be shown).

The same method can be used to prove that a Q-closed subset of an N-compact space is N-compact; see [17].

2.4 Theorem. *If $X=\bigcup X_n$, with each $X_n\in\mathcal{B}o(X)$ and Borel-complete, then X is Borel-complete.*

Proof. Let $\mathcal{F}$ be a $\mathcal{B}o(X)$-ultrafilter with cip. By cip, there is m with $X_m\in\mathcal{F}$. Then $\mathcal{F}\cap X_m$ is a $\mathcal{B}o(X_m)$-ultrafilter with cip; hence some $\{p\}\in\bigcap\mathcal{F}\cap X_m)$. Evidently, $\{p\}\in\bigcap\mathcal{F}$, so X is Borel-complete.

2.5 Corollary. *If X is Borel-complete and locally compact then the one-point compactification is Borel-complete.*

The analogue of 2.4 for Baire-completeness holds, though the proof requires some attention to detail along the lines of the comments following 2.3. That is: *if X is the union of a sequence of realcompact Baire-sets then X is realcompact.* This has escaped prior notice, and should be compared with Mrówka's theorem [15] that X is realcompact if X is normal and the union of a sequence of closed realcompact subspaces.

2.6 Theorem. *If $f\colon X\to Y$ is one-one and Borel-measurable, then X is Borel-complete if Y is.*

Proof. By 2.3, we may suppose that f is onto. If $\mathcal{F}$ is a $\mathcal{B}o(X)$-ultrafilter with cip, and Y is $\mathcal{B}o(Y)$-complete, then by 2.1 there is $p\in\bigcap f(\mathcal{F})$. Since f is one-one, $f^{-1}(p)\in\bigcap\mathcal{F}$, and X is $\mathcal{B}o(X)$-complete.

The analogue of 2.6 for Baire-completeness (i.e., realcompactness) and Baire-measurable, or even continuous, maps fails: map a measurable discrete space one-one onto a compact space. However, the one-one continuous (or Baire-measurable) pre-image of a *hereditarily* realcompact space is hereditarily realcompact [6]. This property resembles Borel-completeness somewhat; see 3.6 and 3.7 below.

2.7 Corollary. *If X is Borel-complete, then $|X|$ is nonmeasurable.*

Proof. Let D be discrete X, and f the identity $D\to X$. 2.6 applies, so D is Borel-complete. Since $\mathcal{B}o(D)=\mathcal{P}(X)$, $|X|$ is nonmeasurable.

There is an analogue of 2.7 for realcompactness, due to Juhasz [11]: if X is realcompact, and each point of X is the intersection of nonmeasurably many open sets, then $|X|$ is nonmeasurable. While Juhasz' proof is quite indirect, there is a direct method which proves a simultaneous generalization of this and 2.7 [10]. A version of this argument has been noticed also by Comfort and Negrepontis, and will appear in [2].

We turn to products.

2.8 Theorem. *Let $X=\pi\{X_\lambda\colon \lambda\in\Lambda\}$. Then, X is Borel-complete iff each X_λ is Borel-complete and $|X_\lambda|=1$ for all but countably many indices λ.*

Proof. Suppose Λ is the positive integers, and each X_n is Borel-complete. Let $\mathcal{F}$ be a $\mathcal{B}o(X)$-ultrafilter with cip. For each n, $\pi_n(\mathcal{F})$ is a $\mathcal{B}o(X_n)$-ultrafilter with cip, by 2.1, and there is $p_n\in X_n$ with $\{p_n\}\in\pi_n(\mathcal{F})$. Thus, $\pi_n^{-1}\{p_n\}\in\mathcal{F}$, and hence $\{(p_n)\}=\bigcap\pi_n^{-1}\{p_n\}\in\mathcal{F}$. So X is Borel-complete.

Now suppose $X=\pi\{X_\lambda\colon \lambda\in\Lambda\}$ is Borel-complete. For each λ, X_λ is homeomorphic to a subspace of X, and hence is Borel-complete by 2.3. Suppose $|X_\lambda|>1$ for uncountably many λ. Then X contains a homeomorph of the space $2^{\aleph_1}$, and by 2.3, $2^{\aleph_1}$ must be Borel-complete. But it is not, by 2.10 below.

2.9 Example. The space W of countable ordinals is not Borel-complete (or even $C(W)$-complete).

First proof. If W were Borel-complete, it would be $C(W)$-complete by 1.1. Now W is countably compact, and if it were $C(W)$-complete, it would be realcompact (by Dykes theorem stated after 1.1). But W is not realcompact [6].

Second proof. The family $\mathcal{F}$ of Borel sets which contain an unbounded closed set in W is a free $\mathcal{B}o(W)$-ultrafilter with cip [7], p. 231.

2.10 Corollary. *$2^{\aleph_1}$ is (compact, realcompact, closed-complete but) not Borel-complete.*

Proof. W has a basis of clopen-sets, and a basis of power $\aleph_1$. By a theorem of Alexandroff [1], W embeds homeomorphically into $2^{\aleph_1}$. By 2.9 and 2.3, $2^{\aleph_1}$ is not Borel-complete.

It is known that $\pi\{X_\lambda\colon \lambda\in\Lambda\}$ is realcompact iff each X_λ is [6]. One can prove this (using Baire-completeness) much as 2.8 is proved, but the argument is more technical (*per* the comments following 2.3) and is strongly reminiscent of this proof: Recall that X is realcompact iff X is complete in the uniformity generated by the real-valued continuous functions [6], ch. 15. Now the uniform product of complete uniform spaces is complete, and the pre-image under a uniformly continuous homeomorphism of a complete space is complete. The result now follows using the identity map on πX_1. (We have not seen this proof in the literature.) This method can also be used to prove the corresponding part of 2.8.

3. Borel-completeness and realcompactness. The first result is trivial.

3.1 THEOREM. *In case $\mathcal{B}o(X) = \mathcal{B}a(X)$ (e.g., if X is perfectly normal), then X is Borel-complete iff X is realcompact.*

Proof. By Hewitt's theorem [9] (see §1), X is realcompact iff X is $\mathcal{B}a(X)$-complete.

We conjecture that X *must* be realcompact if $\mathcal{B}o(X) = \mathcal{B}a(X)$ and if X has no closed discrete set of measurable power. It does not seem to be known if (a) $\mathcal{B}o(X) = \mathcal{B}a(X)$ implies X is perfectly normal, or if (b) a perfectly normal space without closed discrete sets of measurable power must be realcompact. Question (a) was raised by Katětov [12], and (b) by R. L. Blair and R. M. Stephenson, Jr., in separate letters to one of us.

The following is implicit in the argument of 2.8.

3.2 THEOREM. *Let X be a cb-space (e.g., countably paracompact and normal, or countably compact). If X is Borel-complete, then X is realcompact.*

Proof. If X is $\mathcal{B}o(X)$-complete, it is $C(X)$-complete, by 1.1. By Dykes' theorem [3] (mentioned after 1.1), X is realcompact if *cb*.

The hypothesis that X be *cb* in 3.2 cannot be dropped. The space ψ of [6], 5I has power $\leqslant c$ and each subset is a G_δ; so $\mathcal{B}o(\psi) = \mathcal{P}(\psi)$ and ψ is Borel-complete. But ψ is not realcompact (so isn't *cb*).

Of course the converse of 3.2 fails badly, e.g., $2^{\aleph_1}$ (by 2.10). Another example comes from 3.2.

3.3 COROLLARY. *βN is not Borel-complete.*

Proof. If βN is Borel-complete, then for $p \in \beta N - N$, $\beta N - \{p\}$ is Borel complete by 2.3. But $\beta N - \{p\}$ is countably compact and not compact [4], p. 148, hence not realcompact. By 3.2, $\beta N - \{p\}$ cannot be Borel-complete.

Whenever X is not pseudocompact, $\beta N \subset \beta X$ [6], 9.10 and by 3.3 and 2.3, βX is not Borel-complete. If X is not compact, but is pseudocompact and normal, then X is countably compact [4], p. 149, and not realcompact [4], p. 153, and by 3.2 is not Borel-complete; by 2.3 βX is not Borel-complete. A reasonable conjecture would seem to be: *if X is not compact, then βX is not Borel-complete.*

In the other direction, we have two theorems, the first improving part of 3.1.

3.4 THEOREM. *If X is realcompact, and each open set is in Souslin $(C(X))$, then X is Borel-complete.*

Proof. By 1.1, it is enough that X be $C(X)$-complete, and this follows from realcompactness by Dykes' theorem [3] (mentioned after 1.1).

3.4 does not focus very sharply, largely because the property "each open set $\in$ Souslin$(C(X))$" is rather unfamiliar.

We don't know if 3.4 is subsumed by 3.6 below, that is, if a realcompact space in which open sets $\in$ Souslin$(C(X))$ is hereditarily realcompact. (From [8] and [21], X is hereditarily realcompact if X is realcompact and open sets $\in \varrho(C(X))$.)

3.5 LEMMA. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are closed under countable intersection, that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{P}(X)$, and that $\mathcal{A}$ has the property: if $A \in \mathcal{A}$, then there are $A_1, A_2, \ldots \in \mathcal{A}$ with $X - A = \bigcup A_n$. Then, whenever $\mathcal{F}$ is a $\mathcal{B}$-ultrafilter with cip, then $\mathcal{F} \cap \mathcal{A}$ is an $\mathcal{A}$-ultrafilter with cip.*

Proof. Clearly, $\mathcal{F} \cap \mathcal{A}$ is an $\mathcal{A}$-filter with cip. For maximality, suppose that $A \in \mathcal{A}$ and $A \cap F \neq \emptyset$ if $F \in \mathcal{F} \cap \mathcal{A}$. Write $X - A = \bigcup A_n$. Since each $A \cap A_n = \emptyset$, $A_n \notin \mathcal{F}$ and there is $F_n \in \mathcal{F}$ with $A_n \cap F_n = \emptyset$. Then, $A \supset \bigcap F_n \in \mathcal{F}$, and $A \in \mathcal{F} \cap \mathcal{A}$.

3.5 applies to $\mathcal{A} = \mathcal{Q}(X)$, because $\{x: f(x) \neq 0\} = \bigcup \{x: |f(x)| \geqslant 1/n\}$.

3.6 THEOREM. *If X is hereditarily realcompact, then X is Borel-complete.*

Proof. Let $\mathcal{F}$ be a free $\mathcal{B}o(X)$-ultrafilter with cip. By 3.5, $\mathcal{F} \cap \mathcal{Q}(X)$ is a $\mathcal{Q}(X)$-ultrafilter with cip. Since X is realcompact, there is $p \in X$ with $\{p\} = \bigcap (\mathcal{F} \cap \mathcal{Q}(X))$. Let $X' = X - \{p\}$, and $\mathcal{F}' = \{F \cap X' : F \in \mathcal{F}\}$. Since $\{p\} \notin \mathcal{F}$, $\mathcal{F}'$ is closed under countable intersection. By 2.2, $\mathcal{F}' \subset \mathcal{B}o(X')$, and is a filter; using 2.2 again, one shows that $\mathcal{F}'$ is maximal. Again by 3.5, $\mathcal{F}' \cap \mathcal{Q}(X')$ is a $\mathcal{Q}(X')$-ultrafilter with cip. But $\bigcap (\mathcal{F}' \cap \mathcal{Q}(X')) \subset X' \cap \bigcap (\mathcal{F} \cap \mathcal{Q}(X)) = X' \cap \{p\} = \emptyset$, and thus X' cannot be realcompact.

By 3.2 and 2.3, the converse of 3.6 holds when X is hereditarily *cb*.

Some further comment is in order concerning Borel-completeness *versus* hereditary realcompactness. Both properties are hereditary (2.3), and preserved under one-one continuous pre-image (2.6 and comments thereafter). In contrast to the $\aleph_0$-productive property of Borel-completeness (2.8), however:

3.7 EXAMPLE (Moran [14]). Let $X = N \cup \{\omega_0\}$, the one-point compactification of the countable discrete space N. Let $Y = D \cup \{\infty\}$, D discrete of power $\aleph_1$, and neighborhoods of ∞ having countable complement (one-point Lindelöfification of D). X and Y are hereditarily realcompact, but $X \times Y - \{(\omega_0, \infty)\}$ is not realcompact (and its Hewitt realcompactification is $X \times Y$).

This space is Borel-complete (3.6 and 2.8), realcompact (as the product of two realcompact spaces [6]) and not hereditarily realcompact.

Finally, we consider paracompact spaces. A famous theorem of Katětov asserts that a paracompact space without closed discrete subsets of measurable power is realcompact [12] (and more generally, a topologically complete space with this property is realcompact [21], [6]).

Observe that in the proof of 3.6, it is sufficient that X and each subspace $X-\{p\}$ be realcompact; this implies hereditary realcompactness by a theorem of Shirota [21], [6].

From Katětov's theorem, and 3.6, we obtain:

3.8 COROLLARY. *Suppose that X has no closed discrete subspace of measurable power. If X and each subspace $X-\{p\}$ are paracompact, then X is Borel-complete.*

Now Katětov's theorem is quite deep. Thus it seems worthwhile to sketch the following relatively simple proof of 3.8.

Proof. Let $\mathcal{F}$ be a free $\mathcal{B}o(X)$-ultrafilter with cip. Let $\mathcal{G}=\{G\subset X:$ G is open and $G\notin\mathcal{F}\}$. By 3.5, $\mathcal{F}\cap\mathcal{Q}(X)$ is a $\mathcal{Q}(X)$-ultrafilter with cip. Let $\mathcal{G}'=\{X-Z:\ Z\in\mathcal{F}\cap\mathcal{Q}(X)\}$; clearly, $\bigcup\mathcal{G}'\subset\bigcup\mathcal{G}$. If $\mathcal{F}\cap\mathcal{Q}(X)$ is free, $\bigcup\mathcal{G}'=X$, and if fixed, $\bigcup\mathcal{G}'=X-\{p\}$ for some $p\in X$. In either case, $\bigcup\mathcal{G}$ is paracompact by hypothesis, and $\mathcal{G}$ is an open cover. By [4], p. 212, there is a σ-discrete open refinement $\mathcal{U}$ of $\mathcal{G}$. Write $\mathcal{U}=\bigcup\mathcal{U}_n$, with each $\mathcal{U}_n$ discrete. Let $G_n=\bigcup\{U:\ U\in\mathcal{U}_n\}$. Since $X=\bigcup G_n$, there is m with $G_m\in\mathcal{F}$. Define $\mathcal{F}^*$, a free $\mathcal{P}(\mathcal{U}_m)$-ultrafilter with cip as follows: if $\mathcal{E}\subset\mathcal{U}_m$, $\mathcal{E}\in\mathcal{F}^*$ iff $\bigcup\{E:\ E\in\mathcal{E}\}\in\mathcal{F}$. This shows that $|\mathcal{U}_m|$ is measurable. A closed discrete set of measurable power is obtained by picking one point from each member of $\mathcal{U}_m$.

From Katětov's theorem and the theorem of Dykes mentioned after 1.1 follows: *X is $\mathcal{C}(X)$-complete if (merely) paracompact without closed discrete sets of measurable power.* A modification of the above proof of 3.8 yields a direct proof, and with Dykes' theorem that X is realcompact if $\mathcal{C}(X)$-complete and *cb*, we obtain Katětov's theorem. Compare [20].

We might note that Moran has improved Katětov's theorem: *if X is weakly paracompact and normal, and has no closed discrete set of measurable power, then X is realcompact* [14] (*hence $\mathcal{C}(X)$-complete*). With example 3.7, he shows that normality cannot be dropped. This example is still $\mathcal{C}(X)$-complete, though, and we don't know if there is a non-$\mathcal{C}(X)$-complete example.

Added in proof (4.7.1972): (1) Concerning 2.5, it can be shown that the topological sum of non-measurably many Borel-complete spaces is Borel-complete. (2) Concerning the remarks following 3.8, the second two authors have shown that X is $\mathcal{C}(X)$-complete if just *weakly*-paracompact without closed discrete sets of measurable power.

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