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Continua which are a one-to-one continuous image of $[0, \infty)$

by

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1. Introduction. In [5] it was shown that if a locally connected and locally compact metric space is a one-to-one continuous image of the real line, then it is one of the following five objects: an open interval, a figure eight, a dumbbell, a letter theta, or a noose. Noting that each of these objects is embeddable in the plane, the author asked the following question: If a continuum (in this paper the term *continuum* will mean a nonempty compact connected metric space, not necessarily locally connected) is a one-to-one continuous image of the real line, is it embeddable in the plane? As the Example below shows, the answer to this question is no.

However, if a continuum is a one-to-one continuous image of the half-line $[0, \infty)$, then it *is* embeddable in the plane. The primary purpose of this paper is to give a proof of this statement. In doing this we obtain a characterization of the continua which are a one-to-one continuous image of $[0, \infty)$ (see the Structure Theorem below). This yields a characterization (see the Corollary at the end of section 3) of the arcwise connected inverse limits of circles with onto bonding maps in terms of one-to-one continuous images of $[0, \infty)$ (cf. Theorem 6 of [7]).

Throughout this paper the term *circle* means a space homeomorphic to $\{z$ in the plane: $|z| = 1\}$ and the term *half-ray* means a space homeomorphic to $[0, \infty)$. The symbol $\bar{S}$ means the closure of S.

Now we present the Example mentioned above of a continuum which is a one-to-one continuous image of the real line but which is not embeddable in the plane. The author wishes to thank G. S. Young for his help with this example.

EXAMPLE. Let T be the triod in the plane in 3-space formed by the union of the line segment from $(0,0,0)$ to $(0,1,0)$ and the line segment from $(-1,0,0)$ to $(1,0,0)$. Let β be the quarter of the unit circle in the plane in 3-space from $(-1,0,0)$ to $(0,1,0)$, i.e., $\beta = \{(x, y, 0)$ in 3-space: $x^2+y^2 = 1, -1 \leqslant x \leqslant 0$, and $0 \leqslant y \leqslant 1\}$. Let γ be the semicircle in 3-space with center $(1/4, 1/4, 0)$ and radius $\sqrt{10}/4$ which contains the three points $(1, 0, 0)$, $(1/4, 1/4, \sqrt{10}/4)$, and $(-1/2, 1/2, 0)$. Finally, let H be a half-ray in 3-space which closes down precisely on T without

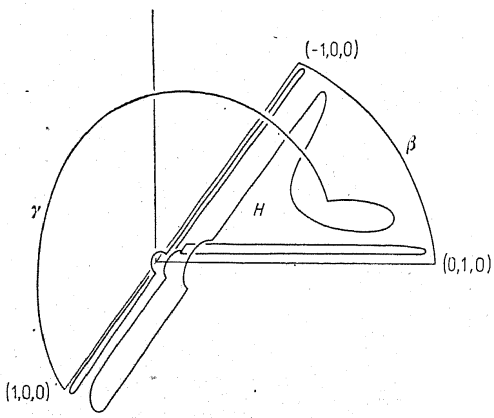

Fig. 1

touching $T \cup \beta \cup \gamma$. (See Figure 1 above.) The continuum $T \cup \beta \cup \gamma \cup H$ is clearly a one-to-one continuous image of the real line but a simple application of the Jordan Curve Theorem shows that it is not embeddable in the plane.

2. Preliminary results. Throughout this section we let $f\colon [0, \infty) \to X$ be a one-to-one continuous function onto a metric continuum (X, ϱ) and we let $K = \{x \in X\colon$ there exists a sequence $\{t_n\}_{n=1}^{\infty}$ in $[0, \infty)$ such that $t_n \to \infty$ as $n \to \infty$ and $\{f(t_n)\}_{n=1}^{\infty}$ converges to $x\}$. In this section we show that the set K is an arc or a point of the form $f([s_0, t_0])$ (see Lemma 4) and give a description of the continuum $\overline{f([t_0+1, \infty))}$ (see Lemma 8). Results in this section will be used in the next section to obtain the Structure Theorem.

LEMMA 1. *The set K is nonempty and compact.*

Proof. The lemma follows easily from the fact that X is a continuum and $K = \bigcap_{n=1}^{\infty} \overline{\{f(t)\colon t \geqslant n\}}$.

LEMMA 2. *The set K can not contain a set of the form $f([r_0, \infty))$, $0 \leqslant r_0 < \infty$.*

Proof. Suppose that there is a point $r_0 \in [0, \infty)$ such that $f([r_0, \infty)) \subset K$. Then, since f is one-to-one, $f([n-1, n]) \cap K$ is nowhere dense in K

for each $n = 1, 2, \ldots$ Thus, since $K = \bigcup_{n=1}^{\infty} (f([n-1, n]) \cap K)$, we have that K is of the first category in itself, a contradiction [3], p. 89.

LEMMA 3. *If $x = f(a)$ and $y = f(b)$, with $a \leqslant b$, are each in K, then $f([a, b])$ is contained in K.*

Proof. Suppose, on the contrary, that there exists $z = f(c)$, $a < c < b$, such that $z \notin K$. Let $s_n \to \infty$ and $t_n \to \infty$ be in $[0, \infty)$ such that $f(s_n) \to x$ and $f(t_n) \to y$. We may assume $s_n < t_n$ for each $n = 1, 2, \ldots$ Let $A_n = f([s_n, t_n])$ for each $n = 1, 2, \ldots$ Since X is compact, there is a convergent subsequence of $\{A_n\}_{n=1}^{\infty}$, whose terms we again denote by A_n, which converges to a continuum A. Also, by the above construction which defines A, $A \subset K$ and (thus) $z \notin A$. If $f^{-1}(A) \subset [0, r]$ for some $r \in [0, \infty)$, then $A \cup f([a, b])$ would be a subcontinuum of the arc $f([0, \max\{b, r\}])$. Hence, $A \cup f([a, b])$ would be an arc. However, this is impossible because $x, y \in A \cap f([a, b])$ and $z \notin A$ implies $f([a, b]) \cap A$ is not connected. We have now shown (*) $f^{-1}(A)$ is not contained in any closed and bounded subinterval of $[0, \infty)$. Since A is a subset of K, it follows from Lemma 2 that there exist points u_n $(n = 1, 2, \ldots)$ such that $u_n \to \infty$ as $n \to \infty$ and $f(u_n) \notin A$. Let $M_n = [u_{n-1}, u_n] \cap f^{-1}(A)$ for each $n = 1, 2, \ldots$, where $u_0 = 0$. From (*) infinitely many of the sets M_n are nonempty and, for what follows, we may assume without loss of generality that they are all nonempty. Since f is one-to-one, the sets $f(M_n)$, $n = 1, 2, \ldots$, are mutually disjoint. Since $A = \bigcup_{n=1}^{\infty} f(M_n)$, we now have the continuum A expressed as the union of a countable number of nonempty, mutually disjoint, compact sets. This is a contradiction [4], p. 173.

LEMMA 4. *The set K is an arc or a point of the form $f([s_0, t_0])$, $0 \leqslant s_0 \leqslant t_0 < \infty$.*

Proof. By Lemma 1 $f^{-1}(K)$ is a nonempty closed subset of $[0, \infty)$ and, therefore, has a smallest member $s_0 \in f^{-1}(K)$. By Lemma 3 we have that if $t \in f^{-1}(K)$, then $[s_0, t] \subset f^{-1}(K)$. Thus, by Lemma 2, $f^{-1}(K)$ has an upper bound and, by Lemma 1, the least upper bound of $f^{-1}(K)$ belongs to $f^{-1}(K)$. Let t_0 denote the least upper bound of $f^{-1}(K)$. Then, by Lemma 3, $[s_0, t_0] \subset f^{-1}(K)$ and, by ontoness of f, $[s_0, t_0] = f^{-1}(K)$, i.e., $f([s_0, t_0]) = K$.

From now on we let $\underline{s_0}$ and $\underline{t_0}$ $(s_0 \leqslant t_0)$ denote the points of $[0, \infty)$ such that $K = f([s_0, t_0])$.

LEMMA 5. *Let $J = [0, s_0) \cup (t_0, \infty)$. The mapping $f|J$ (i.e., f restricted to J) is a homeomorphism of J onto $f(J)$.*

Proof. Since f is assumed to be one-to-one and continuous, it suffices to show that the image under f of an open subset of J is open relative to $f(J)$. Suppose there is an open subset U of J such that $f(U)$ is not open relative to $f(J)$. Let $p \in f(U)$ such that p is the limit of a convergent sequence $\{f(t_i)\}_{i=1}^{\infty}$ such that $f(t_i) \in f(J)-f(U)$ for each $i = 1, 2, \ldots$ Since f is one-to-one and $t_i \notin U$ for each $i = 1, 2, \ldots, t_i \to \infty$ as $i \to \infty$. But then p would belong to K, a contradiction (because $f^{-1}(K) = [s_0, t_0]$ and $p = f(t)$, some $t \in J$). This completes the proof of the lemma.

An *arc component* of a continuum is defined to be a maximal arcwise connected subset of the continuum.

LEMMA 6. *If Y is a metric continuum with exactly two arc components A and H where A is an arc and H is a half-ray such that $\overline{H} \supset A$, then Y is chainable.*

Proof. The lemma is a simple consequence of a theorem due to Bing (see Theorem 11 of [1], p. 660) which states that an hereditarily decomposable metric continuum is chainable if and only if it is both a-triodic (for the definition, see [1], p. 653) and hereditarily unicoherent. Verification that Y satisfies the conditions in Bing's theorem which imply Y is chainable is straightforward and the details are not included here. However, we point out that the following fact about subcontinua of Y is useful in supplying these details. Let g be a homeomorphism of $[0, \infty)$ onto H; if C is a subcontinuum of Y which is contained neither in A nor in H (so that C is not an arc), then C is of the form $\overline{g([t, \infty))}$ for some $t \in [0, \infty)$.

The proof of the next lemma is easy and is omitted.

LEMMA 7. *For any $t \geqslant t_0+1$, $\overline{f([t, \infty))} = f([t, \infty)) \cup K$.*

LEMMA 8. *The continuum $\overline{f([t_0+1, \infty))}$ is either* (1) *an arc if K is a point or* (2) *a chainable continuum with exactly two arc components if K is an arc, the two arc components being the arc K and the half-ray $f([t_0+1, \infty))$.*

Proof. Note that, by Lemma 5, $f|[t_0+1, \infty)$ is a homeomorphism of $[t_0+1, \infty)$ onto $f([t_0+1, \infty))$ so that $f([t_0+1, \infty))$ is a half-ray. By Lemma 4, K is an arc or a point. If K were a single point, then by Lemma 7, $\overline{f([t_0+1, \infty))}$ would be an arc (a one-point compactification of a half-ray must be an arc). Next we assume K is an arc. Suppose there is an arc $a \subset \overline{f([t_0+1, \infty))}$ from a point of $f([t_0+1, \infty))$ to a point of K. Since $a \cap K \neq \emptyset \neq a \cap f([t_0+1, \infty))$ and a is connected, it follows from Lemma 7 that $a \cap f([t_0+1, \infty)) \not\subset f([t_0+1, s])$ for any $s \geqslant t_0+1$. Hence, there is a sequence $\{t_n\}_{n=1}^{\infty}$ in $[t_0+1, \infty)$ such that $f(t_n) \in a$ for each $n = 1, 2, \ldots$ and $t_n \to \infty$ as $n \to \infty$. It now follows (noting also that if $f(s), f(t) \in a$ with $t_0+1 \leqslant s < t < \infty$, then $f([s, t]) \subset a$) that $a \supset f([r, \infty))$ for some $r \geqslant t_0+1$; but then $\overline{f([r, \infty))}$ would be a subcontinuum of the arc a and, therefore, would be an arc. However, by Lemma 7 we see that $\overline{f([r, \infty))} = f([r, \infty)) \cup K$. Thus, since each point of K is a noncut point of $\overline{f([r, \infty))}$, we have uncountably many noncut points of $\overline{f([r, \infty))}$, a contradiction to $\overline{f([r, \infty))}$ being an arc. This proves that there is no arc contained in $\overline{f([t_0+1, \infty))}$ from a point of $f([t_0+1, \infty))$ to a point of K. Therefore, $f([t_0+1, \infty))$ and K are maximal arcwise connected subsets of $\overline{f([t_0+1, \infty))}$. By Lemma 7 we have now proved (assuming K is an arc) that $\overline{f([t_0+1, \infty))}$ has exactly two arc components, namely the arc K and the half-ray $f([t_0+1, \infty))$. The chainability of $\overline{f([t_0+1, \infty))}$ follows from Lemma 6 above. This completes the proof of the lemma.

3. The Structure Theorem. In this section we give necessary and sufficient conditions in order that a continuum be a one-to-one continuous image of $[0, \infty)$. These conditions are in terms of pieces of the continuum and how they intersect. In addition to using results in the previous section, we also use two results of the author stated below. We include the statements of these results not just for completeness but primarily because they are necessary for understanding the information that the Structure Theorem gives.

THEOREM 1 OF [6]. *If a chainable continuum has exactly two arc components, then one of them is an arc and the other is a half-ray.*

THEOREM 6 OF [7]. *A metric continuum Y is an arcwise connected inverse limit of circles with onto bonding maps if and only if either Y is a circle or $Y = A \cup C$ where A is an arc, C is a chainable continuum with exactly two arc components, and $A \cap C$ is exactly the two noncut points of A which are also opposite end points of C* (*in the sense of Bing* [1], p. 661).

We also use the next lemma, which follows easily from results on pages 660 and 661 of [1] (some of the proof is done in part of the proof of Theorem 6 of [7]).

LEMMA 9. *If C is a chainable continuum with exactly two components, namely an arc I and a half-ray H, then two points of C are opposite end points of C* (*in the sense of Bing* [1], p. 661) *if and only if*

(1) *one of them is the noncut point of H and*

(2) *the other is either of the noncut points of I if $\overline{H} \supset I$ or, if $\overline{H} \not\supset I$, the noncut point of I not in $\overline{H}$.*

As a consequence of the results in this section stated above, we have the following:

LEMMA 10. *If Y is an arcwise connected inverse limit of circles with onto bonding maps, then Y is a one-to-one continuous image of $[0, \infty)$. Therefore, if Y is not a circle, the image of zero under an arbitrary one-to-one continuous mapping of $[0, \infty)$ onto Y is the one and only point of Y whose complement in Y is arcwise connected.*

Proof. If Y is a circle, then Y is obviously a one-to-one continuous image of $[0, \infty)$. For the rest of this proof we assume that Y is not a circle. We first show that Y is a one-to-one continuous image of $[0, \infty)$. The continuum $Y = A \cup C$ where A and C have the properties indicated in Theorem 6 of [7]. By Theorem 1 of [6] one of the arc components of C, denoted here by I, is an arc and the other, denoted by H, is a half-ray. Let h denote the noncut point of H. Note that, by Lemma 9, $A \cap I$ is precisely one of the noncut points of I and $A \cap H = \{h\}$. We now describe a one-to-one continuous mapping f of $[0, \infty)$ onto Y as follows. Let f on $[0, 1]$ be a homeomorphism onto I such that $f(0)$ is the noncut point of I not in $A \cap C$ and $f(1)$ is the noncut point of I in $A \cap C$. Let f on $[1, 2]$ be a homeomorphism onto A with $f(2) = h$. Finally, let f on $[2, \infty)$ be a homeomorphism of $[2, \infty)$ onto H. This completes the description of f and shows that Y is a one-to-one continuous image of $[0, \infty)$. Now clearly $Y - \{f(0)\}$ is arcwise connected. We show that the complement in Y of any point other than $f(0)$ is not arcwise connected. Let $t > 0$ and suppose there is an arc γ from $f(0)$ to $f(t+1)$ such that $f(t) \notin \gamma$. Since $\gamma \cap f([0, t+1])$ is not connected, $\gamma \cup f([0, t+1])$ contains a circle. Since every proper subcontinuum of an inverse limit of circles must be chainable (use 2.8 and 2.11 of [2]), it follows that Y is a circle, a contradiction. This completes the proof of Lemma 10.

Now we state and prove one of the two main results of this paper.

STRUCTURE THEOREM. *Let M be a metric continuum. Then M is a one-to-one continuous image of $[0, \infty)$ if and only if $M = \alpha \cup \Sigma$ where α is an arc or a point, Σ is an arcwise connected inverse limit of circles with onto bonding maps, and $\alpha \cap \Sigma$ is a single point of Σ which is a noncut point of α and which, if Σ is not a circle, is the unique point of Σ whose complement in Σ is arcwise connected.*

Proof. Assume that M is a one-to-one continuous image of $[0, \infty)$ under a mapping f. Let s_0 and t_0 play the same role here as in the previous section. Let α be the arc or point $f([0, s_0])$ and let $\Sigma = f([s_0, \infty))$. Note that $\alpha \cap \Sigma$ is the one point $f(s_0)$ which is a noncut point of α. Also note that

(*) $$\Sigma = f([t_0, t_0+1]) \cup \overline{f([t_0+1, \infty))}$$

and

(**) $$f([t_0, t_0+1]) \cap \overline{f([t_0+1, \infty))} = \{f(t_0), f(t_0+1)\}.$$

By Lemma 8, $\overline{f([t_0+1, \infty))}$ is a chainable continuum which is either an arc or has exactly two arc components. If $\overline{f([t_0+1, \infty))}$ is an arc, then $f(t_0)$ and $f(t_0+1)$ are the noncut points of $\overline{f([t_0+1, \infty))}$. Hence, by (*) and (**), Σ is a circle. Now assume $\overline{f([t_0+1, \infty))}$ is not an arc. Then Σ is

the union of the arc $f([t_0, t_0+1])$ and the chainable continuum $\overline{f([t_0+1, \infty))}$ which has exactly two arc components, namely $f([s_0, t_0])$ and $f([t_0+1, \infty))$. By (**), $f([t_0, t_0+1]) \cap \overline{f([t_0+1, \infty))}$ is the two noncut points of the arc $f([t_0, t_0+1])$ which are, by Lemma 9, opposite end points of $\overline{f([t_0+1, \infty))}$. Therefore, by Theorem 6 of [7], Σ is an arcwise connected inverse limit of circles with onto bonding maps. It follows from Lemma 10 that $f(s_0)$ is the unique point of Σ whose complement in Σ is arcwise connected.

Conversely, assume $M = \alpha \cup \Sigma$ where α and Σ satisfy the conditions in the statement of the theorem. If α is a point, then $M = \Sigma$ and, therefore, Lemma 10 applies giving that M is a one-to-one continuous image of $[0, \infty)$. For the rest of the proof we assume α is not just a point; thus, there is a homeomorphism h of $[0, 1]$ onto α, such that $h(1)$ is the point in $\alpha \cap \Sigma$. By Lemma 10, Σ is a one-to-one continuous image of $[1, \infty)$ under a mapping g. If Σ is a circle, we may assume without loss of generality that g was chosen so that $g(1) = h(1)$. If Σ is not a circle, then by Lemma 10 $g(1)$ must be $h(1)$, the unique point of Σ whose complement in Σ is arcwise connected. Let $f\colon [0, \infty) \to M$ be h on $[0, 1]$ and g on $[1, \infty)$; then f is a one-to-one continuous mapping of $[0, \infty)$ onto M. This completes the proof of the theorem.

Remark. The Structure Theorem above characterizes metric continua which are one-to-one continuous images of $[0, \infty)$ in terms of a certain "decomposition property". In view of Theorem 6 of [7], this characterization can be reformulated without mentioning inverse limits as follows: A metric continuum M is a one-to-one continuous image of $[0, \infty)$ if and only if M can be written in the form $\alpha \cup C \cup A$ where α is an arc or a point, C is a chainable continuum with at most two arc components, A is an arc, $A \cap C$ is exactly the two noncut points of A which are also opposite end points of C, and $\alpha \cap (C \cup A)$ is a single point of C which is a noncut point of α and which, if C is not an arc (i.e., $C \cup A$ is not a circle), is the noncut point not in $A \cap C$ of the arc component of C which is an arc.

We mention the following characterization of the arcwise connected inverse limits of circles with onto bonding maps which is a consequence of some of the results and techniques above.

COROLLARY. *A continuum M is an arcwise connected inverse limit of circles with onto bonding maps if and only if M is a one-to-one continuous image of $[0, \infty)$ under a mapping f such that $f(0) = \lim_{n\to\infty} f(t_n)$ where $t_n \to \infty$ as $n \to \infty$.*

Proof. Assume M is an arcwise connected inverse limit of circles with onto bonding maps. Then, by Lemma 10, there exists a one-to-one continuous function f mapping $[0, \infty)$ onto M. Suppose $f(0) \neq \lim_{n\to\infty} f(t_n)$

with $t_n \to \infty$ as $n \to \infty$. Letting s_0 and t_0 play the same role here as in the previous section, we let $Z = f([0, t_0+1]) \cup f([t_0+2, \infty))$ (note that Z is a continuum). Then

$$Z - \overline{f([t_0+3, \infty))} = f([0, s_0)) \cup f((t_0, t_0+1]) \cup f([t_0+2, t_0+3))$$

so that Z is a triod [1], p. 653, a contradiction to the fact that M must be a-triodic (use 2.8 and 2.11 of [2]). This proves the first half of the Corollary. Conversely, assume the continuum M is a one-to-one continuous image of $[0, \infty)$ under a mapping f such that $f(0) = \lim_{n\to\infty} f(t_n)$ where $t_n \to \infty$ as $n \to \infty$. Then $M = \alpha \cup \Sigma$ where α and Σ satisfy the conditions in the Structure Theorem. But in the proof of the Structure Theorem (and using the same notation here) Σ was defined to be $f([s_0, \infty))$. Since the conditions assumed about $f(0)$ say $f(0) \in f([s_0, t_0])$ and since f is one-to-one, $s_0 = 0$. Hence, $f([0, \infty)) = \Sigma$, i.e., $M = \Sigma$. Therefore, M is an arcwise connected inverse limit of circles with onto bonding maps. This completes the proof of the Corollary.

4. The Embedding Theorem. In this section we prove that if a continuum M is a one-to-one continuous image of $[0, \infty)$, then M can be embedded in the plane. First we discuss briefly the main difficulty to overcome for such an embedding.

From the previous section we have $M = \alpha \cup \Sigma$ where α and Σ have the properties indicated in the Structure Theorem. By a Corollary in [7] Σ can be embedded in the plane. In 3-space, Σ may look like the object in Figure 2. When such a Σ is embedded in the plane, it may be embedded so as to look like the object in Figure 3. Hence, if M were the object in Figure 4, there would be no room with the embedding of Σ as indicated in Figure 3 to put α in the plane. If, however, Σ were embedded in the plane as in Figure 5, then α could be put in the plane so as to obtain a continuum homeomorphic to M. With this in mind we see that the apparent obstruction to embedding M in the plane is the "peculiar" embeddings of Σ in the plane, these being essentially a consequence of different ways in which a given chainable continuum with exactly two arc components can be embedded in the plane. In particular, such a chainable continuum can be embedded in the plane in such a way that some of its points are not arcwise accessible from its complement (in the plane).

The next lemma gives a specialized embedding in the plane of a chainable continuum with exactly two arc components. The lemma enables us to overcome the difficulty indicated above. The essential idea of its proof is to put the continuum in the plane, with the arc component which is an arc being a convex segment, and then to pick the half-ray up in 3-space like a spring coil and flatten it out in a plane containing the convex segment so that it lies above the convex segment.

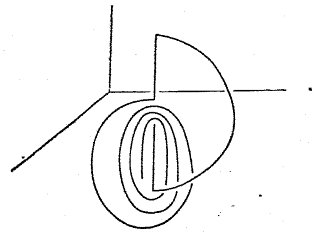

Fig. 2

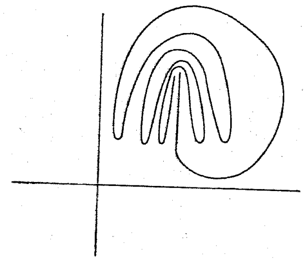

Fig. 3

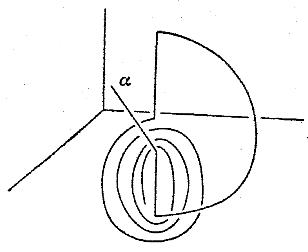

Fig. 4

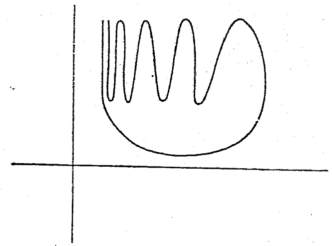

Fig. 5

LEMMA 11. *If C is a chainable continuum with exactly two arc components, then C can be embedded in the plane in such a way that the arc component of C which is an arc lies on the y-axis and each point of the arc component of C which is a half-ray has a strictly positive first coordinate.*

Proof. Let I denote the arc component of C which is an arc and let H denote the arc component of C which is a half-ray. A theorem of Bing [1] shows that we can consider C as being in the plane; thus, since for any two arcs in the plane there is a homeomorphism of the plane onto itself taking one of the arcs onto the other (see, for example, [3], p. 176), we can consider C as in the plane with I being a convex segment. We now consider $C = I \cup H$ as lying in the set of points in 3-space R^3 whose third coordinates are zero such that each point of I also has its second coordinate equal to zero. Let h be a homeomorphism of H onto $(0, 1]$ and define $g\colon C \to R^3$ by

$$g((x, y, 0)) = \begin{cases} (x, 0, 0), & \text{if } (x, y, 0) \in I, \\ (x, 0, h((x, y, 0))), & \text{if } (x, y, 0) \in H. \end{cases}$$

It is easy to verify that g is a homeomorphism onto $g(C)$. Note that each point of I remains fixed under g and that each point of $g(H)$ has a strictly positive third coordinate. This completes the proof (a simple homeomorphism can be applied to $\{(x, 0, z)\colon x$ and z are real numbers$\}$ to put everything in the plane as promised).

Now we prove the Embedding Theorem.

EMBEDDING THEOREM. *If M is a continuum which is a one-to-one continuous image of $[0, \infty)$, then M can be embedded in the plane.*

Proof. The continuum $M = \alpha \cup \Sigma$ where α and Σ have the properties indicated in the Structure Theorem. Furthermore, Σ is a circle or $\Sigma = A \cup C$ where A and C have the properties indicated in Theorem 6 of [7]. If Σ is a circle, then M is a circle if α is a point and, if α is an arc, M is a circle with a sticker; hence, M is embeddable in the plane. Now assume C has exactly two arc components. Using Lemma 11, let C' be an embedding of C in the plane by a homeomorphism $h\colon C \to C'$ such that the arc component of C' which is an arc lies on the y-axis from $(0, a)$ to $(0, b)$ and each point of the arc component H of C' which is a half-ray has a strictly positive first coordinate. Let p and q denote the opposite end points of C which make up the set $A \cap C$ (see Theorem 6 of [7]). Then, by Lemma 9, one of $h(p)$ and $h(q)$ is the noncut point of H, say $h(p)$, and the other, $h(q)$, is $(0, a)$ or $(0, b)$, say $(0, a)$. Let A' be an arc in the plane from $h(p)$ to $h(q)$ such that $A' \cap C' = \{h(p), h(q)\}$ (such a choice for A' is possible because H lies entirely to the right of the y-axis; furthermore, it may even be assumed, though it is not important here, that each point of A' different from $(0, a)$ has a strictly positive first coordinate). If α is a point, then $A' \cup C'$ is homeomorphic to M. Assume α is not a point. Since $\inf\{((x-0)^2+(y-b)^2)^{1/2}\colon (x, y) \in A'\} = \eta > 0$, there is a (nondegenerate) arc α' intersecting $C' \cup A'$ in exactly the point $(0, b)$ (such an α' is easily seen to exist by simply taking α' to be a convex segment of diameter less than η such that each point of α' different from $(0, b)$ is of the form (x, b) with $x < 0$). It is easy to see that $\alpha' \cup C' \cup A'$ is homeomorphic to M, completing the proof that M is embeddable in the plane.

Remark. In [7] the author noted that several different results in the literature (some of them algebraic) imply independently that an arcwise connected inverse limit of circles with onto bonding maps is embeddable in the plane. As demonstrated in the proof of the Embedding Theorem, Lemma 11 allows us to give a more self-contained and descriptive proof of this fact. It also allows us to choose a particularly nice embedding (compare Figures 3 and 5).

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