This process may be described also as "changing a given topology into a coarser one".

In this paper, in the preceding section, a class of selfcomplete topologies T on Z was constructed. It is not difficult to prove, that for each of these the set of continuous characters contains a Cantor set $C \subset \hat{Z} = R/Z$.

The method used for the construction of selfcomplete topologies can be very roughly described as "constructing so fine a topology on Z, that all possible Cauchy nets are convergent to an element of Z".

So it seems difficult to reconcile the two aims in the following

6.1. Problem. Does there exist a minimally almost periodic and selfcomplete topology on Z?

References

[1] L. Fuchs, *Abelian groups*, Budapest 1958.
[2] M. I. Graev, *Free topological groups*, Izv. Akad. Nauk SSSR, Ser. Mat. 12 (1948), pp. 278–324 (Russian); A.M.S. Translations Series One, 35 (1951).
[3] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis, I*, New York 1963.
[4] L. A. Hinrichs, *Integer topologies*, Proc. Amer. Math. Soc. 15 (1964), pp. 991–995.
[5] A. Kertész and T. Szele, *On the existence of non-discrete topologies in infinite abelian groups*, Publ. Math. Debrecen 3 (1953), pp. 187–189.
[6] J. W. Nienhuys, *Not locally compact monothetic groups* I, II, Indag. Math. 32 = Proc. Kon. Nederl. Akad. v. Wetensch. Ser. A, 73 (1970), pp. 295–326.
[7] — *Corrections to "Not locally compact monothetic groups"*, Indag. Math. 33 = Proc. Kon. Nederl. Akad. v. Wetensch., Ser. A, 74 (1971), p. 59.
[8] — *A Solenoidal and Monothetic Minimally Almost Periodic Group*, Fund. Math. 73 (1971), pp. 167–169.
[9] S. Rolewicz, *Some remarks on monothetic groups*, Coll. Math. 13 (1964), pp. 27–28.

Reçu par la Rédaction le 24. 9. 1970

Fixed point sets of homeomorphisms on dendrites (1)

by

Helga Schirmer (Ottawa)

1. Introduction. It has been known for some time that not every non-empty closed subset of a dendrite D can be the fixed point set of a homeomorphism. G. E. Schweigert [5] proved that such a fixed point set cannot consist of one end point only, and several further restrictions can be found in [4]. These restrictions are mainly concerned with the behaviour of the fixed point set on the end points and branch points of D.

Here we show that the fixed point set of a homeomorphism of D is in fact to a large extent determined by the end points and branch points which it contains. More precisely: if the fixed point set F of a homeomorphism f of D contains points of the closure V of the set of all end points and branch points of D, then we can construct an isotopy relative to V which transforms f into a homeomorphism which is fixed point free on $D \setminus V$ (Theorem 1). If, on the other hand, F contains no end points and branch points, then F consists of a single point of order two (Theorem 2).

Many, but not all, of the known restrictions on the fixed point set of a homeomorphism of D also hold for monotone surjective self-maps [4], [6]. It is shown in § 4 that Theorem 1 cannot be extended to monotone maps. I do not know whether Theorem 2 (suitably modified) is still true in the monotone case.

2. Dendrites. The purpose of this paragraph is to collect the properties of dendrites needed in this paper. They can be found in [2], [4], [6], [7] and [8].

A dendrite D is a metric continuum (i.e. compact connected Hausdorff space) in which every pair of distinct points is separated by a third point. It has a partial order structure which was developed by L. E. Ward,

(1) This research was partially supported by the National Research Council of Canada (Grant A 7579).

Jr. [6], [7]. Select any point $r \in D$ as a root, and define $x \leqslant y$ if $x = r$, x separates r and y, or $x = y$. Then $r \leqslant x$ for every $x \in D$. If

$$L(p) = \{x \in D|\ x \leqslant p\},$$
$$M(p) = \{x \in D|\ p \leqslant x\},$$

then the sets $L(p)$ and $M(p)$ form a subbasis for the closed sets of D [6], p. 148. The set $[p, q] = M(p) \cap L(q)$ is a non-empty closed chain (i.e. it is linearly ordered) if $p < q$. We write (p, q) for $[p, q] \setminus (\{p\} \cup \{q\})$.

A point $m \in A$ is called a *maximum* (*minimum*) *of the subset* A *of* D, denoted by $\max A (\min A)$, if $m \nless x$ ($x \nless m$) for every $x \in A$. It is shown in [6], Theorem 1, that every non-empty closed subset of D contains a maximum and a minimum.

The order $o(p)$ of a point $p \in D$ is defined in [8], p. 48. If either $o(p)$ or the number of components of $D \setminus \{p\}$ is finite, then these two numbers are equal [8], p. 88. The point p is called an *end point* if $o(p) = 1$, and a *branch point* if $o(p) \geqslant 3$.

D is not only connected, but also arcwise connected, and the arc between any two of its points p and q, written arc pq, is unique [8], p. 89. We have arc $pq = [m, p] \cup [m, q]$, where $m = \max(L(p) \cap L(q))$ [4], Lemma 2.3. The point p is a branch point of D if and only if there are at least three arcs in D with p as a common end point which are pairwise disjoint except for p [2], p. 44.

The proofs of our results lean heavily on the fact that a homeomorphism of D preserves its order structure. We use the following lemma.

LEMMA. *A homeomorphism $f\colon D \twoheadrightarrow D$ is strictly isotone (i.e. $x < y$ implies $f(x) < f(y)$).— A strictly isotone bijective transformation $f\colon D \twoheadrightarrow D$ is a homeomorphism.*

Proof. The first part is proved in [6], Lemma 13 and p. 156. The second part is a consequence of the facts that for a strictly isotone bijection $f(L(x)) = L(g(x))$ and $f(M(x)) = M(f(x))$ for all $x \in D$, and that $L(x)$ and $M(x)$ form a subbasis for the closed sets of D.

3. Fixed point sets of homeomorphisms on dendrites. Let E be the set of all end points and B be the set of all branch points of the dendrite D, and define $V = \mathrm{Cl}(E \cup B)$, where Cl denotes the closure. V reduces to the set of all vertices of D if D is a finite graph. We first consider the case where the fixed point set F of a homeomorphism contains points of both V and $D \setminus V$, and show that then the fixed points in $D \setminus V$ can be eliminated by an isotopy.

THEOREM 1. *Let $f\colon D \twoheadrightarrow D$ be a homeomorphism of a dendrite D with fixed point set F. If $F \cap V \neq \emptyset$, then there exists an isotopy f_t $(0 \leqslant t \leqslant 1)$ relative to V such that $f_0 = f$, and $f_1 = g$ has a fixed point set which is a subset of V.*

Proof. As $F \cap V \neq \emptyset$, we can select a root $r \in F \cap V$. For every $x \in D \setminus V$, consider the sets $L(x) \cap V$ and $M(x) \cap V$. They are closed, the first is non-empty as $r \in L(x) \cap V$, and the second as $M(x)$ contains a maximum which is clearly a maximum of D and hence [4], Lemma 2.2 an end point. Therefore both sets contain a maximum and a minimum. As $L(x)$ is a chain, the maximum of $L(x) \cap V$ is unique. But the minimum of $M(x) \cap V$ is unique also: if there exist, for some $x \in D \setminus V$, two points n_1 and n_2 such that both are a minimum of $M(x) \cap V$, then it follows from the definition of a minimum that n_1 and n_2 are not comparable. As $n_i \in M(x)$ for $i = 1, 2$, we have $x < n_i$, and hence $x \leqslant k$ if $k = \max[L(n_1) \cap L(n_2)]$, so that $k \in M(x)$. As arc $n_1 n_2 = [k, n_1] \cup [k, n_2]$, the point k is a common end point of the otherwise disjoint arcs $[k, n_1]$, $[k, n_2]$ and $[r, k]$. Therefore k is a branch point and hence $k \in V$. So $k \in M(x) \cap V$, and by definition $k \leqslant n_i$. As n_1 and n_2 are not comparable, this implies $k < n_1$ and $k < n_2$, which is impossible if n_i are minima. So we see that the minimum of $M(x) \cap V$ must be unique. Put

$$\left.\begin{aligned} m(x) &= \max[L(x) \cap V] \\ n(x) &= \min[M(x) \cap V] \end{aligned}\right\} \quad \text{for every } x \in D \setminus V.$$

Let $Q = \bigcup \big(m(x), n(x)\big)$, where the union is taken over all $x \in D \setminus V$ for which $\big(m(x), n(x)\big) \cap F \neq \emptyset$. Note that then $V \subset D \setminus Q$. In order to define the desired isotopy we first show that $m(x) \in F$ and $n(x) \in F$ if $x \in Q$. As every point of $\big(m(x), n(x)\big)$ is of order two, we see that for any point $p \in \big(m(x), n(x)\big) \cap F$

$$m(x) = \max[L(x) \cap V] = \max[L(p) \cap V],$$
$$n(x) = \min[M(x) \cap V] = \min[M(p) \cap V].$$

As $p \in F$, the lemma implies $f\big(L(p)\big) = L(p)$ and $f\big(M(p)\big) = M(p)$. For a homeomorphism clearly $f(V) = V$, hence $f\big(L(p) \cap V\big) = L(p) \cap V$ and $f\big(M(p) \cap V\big) = M(p) \cap V$. But f is strictly isotone, so it maps a maximum (minimum) of a set onto a maximum (minimum) of the image set. Therefore $f\big(m(x)\big) = m(x)$ and $f\big(n(x)\big) = n(x)$.

We now define a homeomorphism $g\colon D \twoheadrightarrow D$. If $x \in D \setminus Q$, put $g(x) = f(x)$. Now assume $x \in Q$. Then $m(x) \in F$ and $n(x) \in F$, hence the lemma implies $f(x) \in \big(m(x), n(x)\big)$. Give D a convex metric [1], [3] and express x in the form

$$x = \lambda m(x) + (1-\lambda) n(x), \quad \text{where } \lambda = \lambda(x) \text{ is such that } 0 < \lambda < 1.$$

Define $g(x)$ by

$$g(x) = \lambda^2 m(x) + (1-\lambda^2) n(x).$$

Then $0<\lambda^2<1$, so that $g(x)\in(m(x),n(x))$. Both f and g induce, for all $x\in Q$, a bijection of $[m(x),n(x)]$ onto itself, and $g=f$ on $D\backslash Q$, therefore we see that g is a bijection of D onto itself. The fact that f is strictly isotone implies that g is strictly isotone, and hence the lemma shows that g is a homeomorphism. By construction it is fixed point free on $D\backslash V\subset Q$.

If we define f_t by

$$f_t(x)=tg(x)+(1-t)f(x)\qquad (x\in D;\ 0\leqslant t\leqslant 1)\,,$$

then $f_0=f$ and $f_1=g$, so that f_t is a homotopy from f to g. It is relative to V, as $g(x)=f(x)$ for every $x\in V$. As f_t maps, for all $0\leqslant t\leqslant 1$, each set $(m(x),n(x))\subset Q$ onto itself, and $f_t=f$ on $D\backslash Q$, each f_t is a surjection. Both f and g are strictly isotone, and therefore f_t is strictly isotone for all t, so that it is also an injection. Hence f_t is the desired isotopy, and Theorem 1 is proved.

If $F\cap V=\emptyset$, then an inspection of the proof shows that it still works if V is replaced by $V\cup\{r\}$, where the root $r\in F\cap(D\backslash V)$. In this case the isotopy f_t transforms the homeomorphism f into a homeomorphism g with fixed point set G such that $G\subset V\cup\{r\}$ and $G=F$ on V, hence $G=\{r\}$ consists of a single point of order two. We can strengthen this statement in two ways: the isotopy is in fact the identity, and the assumption $F\cap V=\emptyset$ can be replaced by $F\cap(E\cup B)=\emptyset$. This can be seen from the next theorem.

THEOREM 2. *Let $f\colon D\twoheadrightarrow D$ be a homeomorphism of a dendrite D with fixed point set F. If $F\cap(E\cup B)=\emptyset$, then F consists of a single point p of order two, and the two components of $D\backslash\{p\}$ are homeomorphic and interchanged by f.*

Proof. $F\neq\emptyset$ as D has the fixed point property, and $o(x)=2$ for every $x\in F$ as $F\cap(E\cup B)=\emptyset$. Select an arbitrary point $p\in F$. Then $D\backslash\{p\}$ has two components, say K_1 and K_2. As they are open in $D\backslash\{p\}$, they are open in D, and hence $K_j\cup\{p\}=D\backslash K_i$ $(i,j=1,2;\ i\neq j)$ is closed in D, and therefore [8], p. 88 a subdendrite. The point p is an end point of $K_i\cup\{p\}$, as K_i is connected in D and hence in $K_i\cup\{p\}$.

As f is a homeomorphism we either have $f(K_1)=K_2$ and $f(K_2)=K_1$, or $f(K_1)=K_1$ and $f(K_2)=K_2$. In the first case Theorem 2 holds. In the second case $f|(K_1\cup\{p\})$ is a homeomorphism of a dendrite which leaves the end point p fixed. Hence it also has a fixed point on $K_1\cup\{p\}$ which is of order $\neq 2$ [4], Theorem 4.5, and therefore contained in $E\cup B$. But we have assumed that $F\cap(E\cup B)=\emptyset$, so this case is impossible.

4. Remarks.

(i) It is not possible to replace V in Theorem 1 by $E\cup B$. This is shown by the following example.

In the xy-plane, select the points

$$e_+=(0,2),\qquad e_-=(0,-2),$$
$$a_+=(0,1),\qquad a_-=(0,-1),$$
$$c_0=(0,0),\qquad d_0=(0,1),$$
$$c_n=\left(0,\frac{n}{n+1}\right),\qquad d_n=\left(\frac{1}{n+1},\frac{n}{n+1}\right),$$
$$c_{-n}=\left(0,-\frac{n}{n+1}\right),\qquad d_{-n}=\left(\frac{1}{n+1},-\frac{n}{n+1}\right),\qquad n=1,2,3,\ldots$$

Join them by the segments $[e_-,e_+]$ and $[c_i,d_i]$ for $i=0,\pm1,\pm2,\ldots$ to obtain a dendrite D with $E=\{e_+,e_-,d_i|\ i=0,\pm1,\pm2,\ldots\}$, $B=\{c_i|\ i=0,\pm1,\pm2,\ldots\}$ and $V=E\cup B\cup\{a_+,a_-\}$. Define f on V by

$$f(e_+)=e_+,\qquad f(e_-)=e_-,$$
$$f(a_+)=a_+,\qquad f(a_-)=a_-,$$
$$f(c_i)=c_{i+1},\quad f(d_i)=d_{i+1},\qquad i=0,\pm1,\pm2,\ldots,$$

and extend it linearly over all $[c_i,d_i]$ and $[c_i,c_{i+1}]$ and as a fixed point free homeomorphism of (e_+,a_+) respectively (e_-,a_-) onto itself. Then f is a homeomorphism with fixed points $e_+,e_-\in E$, and $a_+,a_-\in V$ but $\notin E\cup B$. It is not possible to construct an isotopy relative to $E\cup B$ so that it frees f from the fixed points on $D\backslash(E\cup B)$.

(ii) If D and hence $V=E\cup B$ is finite, then it follows from [4], § 6, that Theorem 1 can be generalized to monotone surjective self-maps, with the isotopy replaced by a homotopy $f_t\operatorname{rel}V$ so that f_t is a monotone surjection for all $0\leqslant t\leqslant 1$. But for arbitrary dendrites Theorem 1 cannot be extended to monotone surjective maps. In the example at the end of § 4 in [4] a monotone surjective map is constructed which has one fixed point contained in E and one in $D\backslash V$, and for which no homotopy $f_t\operatorname{rel}V$ exists such that f_t is a monotone surjection for all $0\leqslant t\leqslant 1$ and f_1 has fixed points different from $f_0=f$. I do not know whether Theorem 2 (with obvious modifications) holds for monotone surjective self-maps.

References

[1] R. H. Bing, *Partitioning continuous curves*, Bull. Amer. Math. Soc. 58 (1952), pp. 536–556.

[2] C. Eberhart, *Metrizability of trees*, Fund. Math. 65 (1969), pp. 43–50.

[3] R. L. Plunkett, *A fixed point theorem for continuous multivalued transformations*, Bull. Amer. Math. Soc. 7 (1956), pp. 160–163.

[4] H. Schirmer, *Properties of fixed point sets on dendrites*, Pacific J. Math. 36 (1971), pp. 795–810.
[5] G. E. Schweigert, *Fixed elements and periodic types for homeomorphisms on s. l. c. continua*, Amer. J. Math. 66 (1944), pp. 229–244.
[6] L. E. Ward, Jr., *Partially ordered topological spaces*, Proc. Amer. Math. Soc. 5 (1954), pp. 144–161.
[7] — *A note on dendrites and trees*, Proc. Amer. Math. Soc. 5 (1954), pp. 992–994.
[8] G. T. Whyburn, *Analytic topology*, Providence, R. I., 1942.

CARLETON UNIVERSITY,
Ottawa

Reçu par la Rédaction le 17. 11. 1970

Continua which are a one-to-one continuous image of $[0, \infty)$

by

Sam B. Nadler, Jr. (New Orleans)

1. Introduction. In [5] it was shown that if a locally connected and locally compact metric space is a one-to-one continuous image of the real line, then it is one of the following five objects: an open interval, a figure eight, a dumbbell, a letter theta, or a noose. Noting that each of these objects is embeddable in the plane, the author asked the following question: If a continuum (in this paper the term *continuum* will mean a nonempty compact connected metric space, not necessarily locally connected) is a one-to-one continuous image of the real line, is it embeddable in the plane? As the Example below shows, the answer to this question is no.

However, if a continuum is a one-to-one continuous image of the half-line $[0, \infty)$, then it *is* embeddable in the plane. The primary purpose of this paper is to give a proof of this statement. In doing this we obtain a characterization of the continua which are a one-to-one continuous image of $[0, \infty)$ (see the Structure Theorem below). This yields a characterization (see the Corollary at the end of section 3) of the arcwise connected inverse limits of circles with onto bonding maps in terms of one-to-one continuous images of $[0, \infty)$ (cf. Theorem 6 of [7]).

Throughout this paper the term *circle* means a space homeomorphic to $\{z$ in the plane: $|z| = 1\}$ and the term *half-ray* means a space homeomorphic to $[0, \infty)$. The symbol $\bar{S}$ means the closure of S.

Now we present the Example mentioned above of a continuum which is a one-to-one continuous image of the real line but which is not embeddable in the plane. The author wishes to thank G. S. Young for his help with this example.

EXAMPLE. Let T be the triod in the plane in 3-space formed by the union of the line segment from $(0, 0, 0)$ to $(0, 1, 0)$ and the line segment from $(-1, 0, 0)$ to $(1, 0, 0)$. Let β be the quarter of the unit circle in the plane in 3-space from $(-1, 0, 0)$ to $(0, 1, 0)$, i.e., $\beta = \{(x, y, 0)$ in 3-space: $x^2+y^2 = 1, -1 \leqslant x \leqslant 0$, and $0 \leqslant y \leqslant 1\}$. Let γ be the semi-