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Reçu par la Rédaction le 27. 10. 1969

Non-alternating mappings

by

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1. Introduction. In [5] Whyburn introduced the concept of non-aternating mapping which generalizes the concept of monotone mapping. Kuratowski in [1] and Kuratowski and Lacher in [2] have given sufficient conditions for spaces of monotone mappings to be topologically complete. McAuley in [3] has proved a similar theorem for non-alternating mappings, where the range space is the unit interval. In this paper we continue this line of investigation for spaces of non-alternating mappings. We also consider the problem of when non-alternating mappings are monotone, and under what condition there exist non-alternating mappings between two spaces; these problems were originally investigated by Whyburn in [6], [7], and [8].

2. Definition. A mapping f from a space X into a space Y is said to be *non-alternating* if for each pair of points a, b in Y, $f^{-1}(a)$ does not separate $f^{-1}(b)$ in X.

A compact metric continuum D is a *dendrite* if it is locally connected and contains no simple closed curve. A dendrite D has the following properties, see [8]:

(i) every two points of D are separated by a third point;
(ii) every point of D is either a cut point or an end point;
(iii) one and only one arc exists between any two points of D.

For $a, b \in D$, the unique arc between a and b is denoted by $[a, b]$.

If X and Y are topological spaces, the space of all mappings (continuous functions) from X into Y is denoted by $C(X, Y)$ and is topologized by the compact-open topology. The subspace of $C(X, Y)$ consisting of all non-alternating mappings from X onto Y is denoted by $N(X, Y)$.

If X is a metric space, the metric for X will be denoted by $d(-, -)$; if $x \in X$ and $\varepsilon > 0$, then the set $\{y \in X: d(x, y) < \varepsilon\}$ will be denoted by $B(x, \varepsilon)$. Euclidean n-space is denoted by R^n. Fr denotes the frontier of a set.

If A and B are subsets of a space, A meets B means $A \cap B \neq \emptyset$. The closure of A is denoted by $\mathrm{Cl}(A)$.

3. Spaces of non-alternating mappings. In [2] the following is proved.

THEOREM A. *If X and Y are compact Hausdorff spaces and Y is locally connected, then the space of all monotone mappings from X onto Y is closed in $C(X, Y)$.*

A similar theorem for the non-alternating case was proved in [3]:

THEOREM B. *If X is a compact metric continuum and I is the unit interval, then $N(X, I)$ is closed in $C(X, I)$.*

Theorem 3.2 below is a direct generalization of Theorem B. The example following Theorem 3.2 shows that Theorem A does not generalize to the non-alternating case, however, Theorem 3.3 gives a sufficient condition for $N(X, Y)$ to be topologically complete.

LEMMA 3.1. *Let X be a compact Hausdorff space, Y a 1st countable, locally connected Hausdorff space, and $f\colon X \to Y$ a non-alternating mapping of X onto Y. Let $a \in Y$ and suppose C is a component of $Y - \{a\}$, then $f^{-1}(C)$ is connected.*

Proof. Suppose $f^{-1}(C)$ is separated, so that $f^{-1}(C) = A \cup B$, where A and B are non-empty, disjoint open sets in $X - f^{-1}(a)$. We may assume that there exists $p \in f(A)$ such that $p \in \mathrm{Cl}(f(B))$. Thus there exists a sequence, $\{p_i\}_{i=1}^{\infty}$, in $f(B)$ which converges to p.

Since C and $Y - (\{a\} \cup C)$ are open in Y, A and $X - (A \cup f^{-1}(a))$ are open in X. Thus $f^{-1}(p) \subset A$ and $f^{-1}(p_i) \subset B$. Since X is compact, there are points in $f^{-1}(p)$ which are limit points of $\bigcup\{f^{-1}(p_i)\colon i = 1, 2, \ldots\}$, but this contradicts A and B being disjoint open sets in $X - f^{-1}(a)$.

THEOREM 3.2. *If X is a compact Hausdorff space and D is a dendrite, then $N(X, D)$ is closed in $C(X, D)$.*

Proof. Let f be in the closure of $N(X, D)$. It is clear that f is onto. There exists a sequence $\{f_i\}_{i=1}^{\infty}$ in $N(X, D)$ such that $\{f_i\}_{i=1}^{\infty}$ converges to f.

Let $a, b \in D$, we wish to show that $f^{-1}(b)$ does not separate $f^{-1}(a)$ in X.

Let c separate b from a in D. Let the component of $D - \{b\}$ containing a be denoted by C_b. Let the component of $D - \{c\}$ containing a be denoted by C_c. Note that $c \in C_b$. By Lemma 3.1, $f_i^{-1}(C_b)$ and $f_i^{-1}(C_c)$ are connected. We have for any i:

$$\mathrm{Cl}(f_i^{-1}(C_c)) \subset f_i^{-1}(C_c) \cup f_i^{-1}(c) \subset f_i^{-1}(C_b),$$

where $\mathrm{Cl}(f_i^{-1}(C_c))$ is a continuum.

Let $K = \{x \in X\colon$ every neighborhood of x meets an infinite number of $\mathrm{Cl}(f_i^{-1}(C_c))\}$. K is a continuum.

We shall now show that $f^{-1}(C_c) \subset K$. If $f(x) \in C_c$, let $\varepsilon = d(f(x), D - C_c)$.

There exists an integer N such that $d(f_i(x), f(x)) < \varepsilon$ for $i > N$. Thus $f_i(x) \in C_c$ and hence $x \in f_i^{-1}(C_c)$ for $i > N$. Therefore $x \in K$.

We shall now show that $f^{-1}(b) \cap K = \emptyset$. Let $f(x) = b$ and $\varepsilon = d(b, \mathrm{Cl}(C_c))$. Let N be an integer, where $d(f_i, f) < \varepsilon/3$ for $i > N$. Then $f^{-1}(B(b, \varepsilon/3)) \cap f_i^{-1}(\mathrm{Cl}(C_c)) = \emptyset$, since $y \in f^{-1}(B(b, \varepsilon/3))$ implies that

$$d(b, f_i(y)) < d(b, f(y)) + d(f(y); f_i(y)) < \varepsilon .$$

Since $\mathrm{Cl}(f_i^{-1}(C_c)) \subset f_i^{-1}(\mathrm{Cl}(C_c))$ it follows that $f^{-1}(b) \cap K = \emptyset$.

Since $f^{-1}(a) \subset K$ and K is connected, we have that f is non-alternating.

EXAMPLE. Let B^3 be a 3-cell and B^2 be a 2-cell. We shall show that $N(B^3, B^2)$ is not closed. Let $B^2 = \{x \in R^2\colon \|x\| \leqslant 1\}$ and $B^3 = B^2 \times [-1, 1]$. We denote the origin in R^n by 0, and the boundary of B^2, the unit circle, by S. We shall use the vector space structure of R^2 in describing points in the interior of B^2; thus if $x \in S$, the radius from 0 to x is equal to $\{tx\colon 0 \leqslant t \leqslant 1\}$. Hence $B^3 = \{(tx, s)\colon x \in S, 0 \leqslant t \leqslant 1, -1 \leqslant s \leqslant 1\}$.

We shall define a sequence of mappings, $\{f_i\colon B^3 \to B^2\}_{i=1}^{\infty}$, where each f_i is a non-alternating onto mapping, $\{f_i\}_{i=1}^{\infty} \to f$, and f is not non-alternating.

We now define f_i for $i = 1, 2, 3, \ldots$

(i) $f_i(tx, 1) = f_i(tx, -1) = tx$.

(ii) For $0 \leqslant t \leqslant 1$, $f_i(tx, 0) = tx/i$.

(iii) For $0 < |s| < 1$, $f_i(x, s) = |s|x + (1 - |s|)x/i$.

(iv) For $0 \leqslant t < 1$, $f_i(tx, s) = tf_i(x, s)$.

Note that f_1 is the projection map. To picture f_i for $i \geqslant 2$, let D be a diameter of B^2, choose α and β such that $0 < \alpha < 1/i < \beta < 1$. Let the

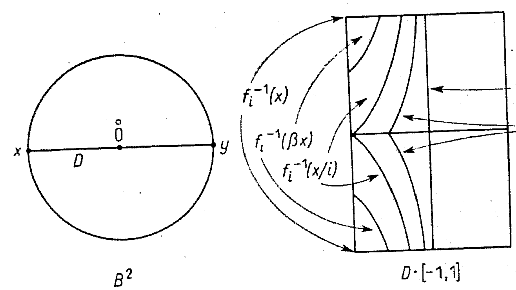

Fig. 1.

endpoints of D be x and y. In Figure 1 we have D and $D \times [-1, 1]$, a verticle cross-section of B^3, with $f_i^{-1}(\alpha x)$, $f_i^{-1}(x/i)$, $f_i^{-1}(\beta x)$, $f_i^{-1}(x)$, and $f_i^{-1}(0)$ indicated; the first three of these are hyperbolas.

It is easy to see that f_i is non-alternating. To show that f is not non-alternating, note that $B^2\times\{0\}\subset f^{-1}(0)$, while for $x\in S^1$, $f^{-1}(x)=\{x\}\times\times\{1,-1\}$. Thus $f^{-1}(0)$ separates $f^{-1}(x)$ in B^3.

THEOREM 3.3. *If X and Y are Peano continua, then $N(X,Y)$ is a G_δ-set in $C(X,Y)$, and is thus topologically complete.*

Proof. Let $G_n=\{f\in C(X,Y)$: there exist $x,y\in Y$, with $d(x,y)\geqslant 1/n$, such that $f^{-1}(x)$ separates $f^{-1}(y)$ in $X\}$. Note that $N(X,Y)=C(X,Y)-\bigcup\{G_n\colon n=1,2,\ldots\}$.

We shall show that G_n is closed for each n, thus proving the theorem. Suppose $\{f_i\}_{i=1}^{\infty}\subset G_n$ and $\{f_i\}_{i=1}^{\infty}$ converges to $f\in C(X,Y)$. For each i there exists $x_i,y_i\in Y$ such that $d(x_i,y_i)\geqslant 1/n$ and $f_i^{-1}(x_i)$ separates $f_i^{-1}(y_i)$ in X. Without loss of generality we can assume $\{x_i\}_{i=1}^{\infty}$ converges to x and $\{y_i\}_{i=1}^{\infty}$ converges to y in Y.

We want to show $f^{-1}(x)$ separates $f^{-1}(y)$ in X. Suppose the contrary, then, since $X-f^{-1}(x)$ is locally connected, there is a component, C, of $X-f^{-1}(x)$ containing $f^{-1}(y)$. Thus there is a continuum K in C, containing $f^{-1}(y)$. There is a connected neighborhood, $N(K)$, of K, such that $\mathrm{Cl}(N(K))\subset C$. Finally, there is a neighborhood, M, of $f^{-1}(x)$, such that $\mathrm{Cl}(N(K))\cap\mathrm{Cl}(M)=\emptyset$.

Now we show that there exists an I, such that for $i>I$, $f_i^{-1}(y_i)\subset N(K)$. Suppose the contrary, then without loss of generality we can assume for each i there exists $z_i\in X$ such that $f_i(z_i)=y_i$ and $z_i\notin N(K)$. Suppose $\{z_i\}_{i=1}^{\infty}$ converges to z. Thus $z\notin f^{-1}(y)$, but

$$d\big(f(z),y\big)\leqslant d\big(f(z),f(z_i)\big)+d\big(f(z_i),f_i(z_i)\big)+d\big(f_i(z_i),y\big)\,,$$

and the sum on the right can be made arbitrarily small, which implies $f(z)=y$. This contradiction implies the existence of the desired I. Similarly there exists I' such that for $i>I'$, $f_i^{-1}(y_i)\subset N(K)$ and $f_i^{-1}(x_i)\subset M$, so that $f_i^{-1}(x_i)$ cannot separate $f_i^{-1}(y_i)$ in X, another contradiction. Hence G_n is closed.

4. Non-alternating mappings on simply connected Peano continua. We first prove a theorem about simply connected Peano continua which follows from a similar theorem about R^2.

THEOREM 4.1 (See [4], Theorem 9.2). *If F_1 and F_2 are two closed disjoint sets in R^2, then two points which are connected in both R^2-F_1 and R^2-F_2 are connected in $R^2-(F_1\cup F_2)$.*

We prove the following:

COROLLARY 4.2. *Theorem 3.1 holds if R^2 is replaced by a simply connected Peano continuum P.*

Proof. Let x and y be connected in $P-F_1$ and in $P-F_2$. Let the circle S be expressed as the union of two arcs a_1 and a_2, where $a_1\cap a_2=\{x',y'\}$. Then there exists a mapping $g\colon S\to P$, such that $g(x')=x$, $g(y')=y$, $g(a_1)\subset P-F_1$, and $g(a_2)\subset P-F_2$. Now g can be extended to all of B^2.

There exist x_1 and y_1 in $B^2-\big(S\cup g^{-1}(F_1\cup F_2)\big)$, such that there are arcs from x' to x_1, and y' to y_1 in $B^2-g^{-1}(F_1\cup F_2)$. Note that B^2-S is topologically R^2, and that x_1 is not separated from y_1 by either $g^{-1}(F_1)$ or $g^{-1}(F_2)$ since $g^{-1}(F_i)\cap a_i=\emptyset$, for $i=1,2$. Thus by Theorem 4.1, x_1 and y_1 are connected in $B^2-\big(S\cup g^{-1}(F_1\cup F_2)\big)$ and the corollary follows.

Using Corollary 4.2, we can duplicate the proof of Theorem 14.3 of [4], to obtain:

THEOREM 4.3. *If x and y are separated by the closed set F in a simply connected Peano continuum P, then they are separated by a component of F.*

We shall apply Theorem 4.3 in the proof of the main theorem of this section:

THEOREM 4.4. *Let X be a simply connected Peano continuum and D be a dendrite. If $f\colon X\to D$ is a non-alternating mapping of X onto D, then f is monotone.*

Before we prove this theorem we list the following lemmas, some of which are stated without proof.

LEMMA 4.5. *For any $x\in D$, $D-\{x\}$ has at most a countable number of components.*

LEMMA 4.6. *If C is a connected space, O is an open subset of C, and the frontier of O is connected, then $C-O$ is connected.*

LEMMA 4.7. *Let X, D, and f be as in Theorem 4.4. Let $x\in D$ and suppose C is a component of $X-f^{-1}(x)$. Then $\mathrm{Fr}\,C\subset f^{-1}(x)$ and $\mathrm{Fr}\,C$ is connected.*

Proof. Let $x_1\in f(C)$. There exists a sequence $\{x_i\}_{i=2}^{\infty}$ in $[x_1,x]$ such that $[x_i,x]\subset[x_{i-1},x]$ for $i=2,3,\ldots$, and $\{x_i\}_{i=1}^{\infty}\to x$. If $y\in f(C)$, then $[y,x]\cap[x_1,x]=[a,x]$, where $a\neq x$ and $[y,a]\cup[a,x_1]=[y,x_1]$. Thus for some i, x_i separates x from y in D.

Now $f^{-1}(x_2)$ separates $f^{-1}(x)$ from $f^{-1}(x_1)$ in X. Let $u\in f^{-1}(x)$ and $v\in f^{-1}(x_1)$; then there exists Q_2, a component of $f^{-1}(x_2)$, such that Q_2 separates u from v in X. But $f^{-1}(x)$ is in the same component of $X-Q_2$ as is u, similarly for $f^{-1}(x_1)$ and v. Thus Q_2 separates $f^{-1}(x)$ from $f^{-1}(x_1)$ in X.

By induction we obtain Q_i, a component of $f^{-1}(x_i)$ such that Q_i separates $f^{-1}(x)$ from $f^{-1}(x_{i-1})$ in X.

Let $Q=\{p\in X$: every neighborhood of p meets an infinite number of the $Q_i\}$. Note $Q\subset f^{-1}(x)$. We now show that $Q=\mathrm{Fr}\,C$. We have that $Q\subset\mathrm{Fr}\,C$. If $p\in\mathrm{Fr}\,C$, let U be an open connected set containing p, then there exists $q\in U\cap C$. Thus there exists I such that $i>I$ implies x_i

separates x from $f(q)$ in D, and hence Q_{i+1} separates p from q in X and also in U. Therefore $Q_{i+1} \cap U \neq \emptyset$ for $i > I$, so $p \in Q$. Thus $Q = \operatorname{Fr} C$ and is connected.

Proof of 4.3. For any $x \in D$, we wish to show that $f^{-1}(x)$ is connected. Since x is either an end point or a cut point of D, we have two cases.

Case I. x is an end point. There exist $\{U_i\}_{i=1}^{\infty}$, a sequence of neighborhoods of x where $U_{i+1} \subset U_i$, $\bigcap_{i=1}^{\infty} U_i = \{x\}$, diameter $U_i < 1/i$, and each U_i has one boundary point x_i. Thus $\{x_i\}_{i=1}^{\infty} \to x$. Let C_i be the component of $X - f^{-1}(x_i)$ such that $f^{-1}(x) \subset C_i$. Then $f(C_i) \subset U_i$ and $\operatorname{Cl}(C_i)$ is connected. It is easy to see that $f^{-1}(x) = \bigcap_{i=1}^{\infty} \operatorname{Cl}(C_i)$ and thus is connected.

Case II. x is a cut point. Thus $f^{-1}(x)$ separates X. By Lemmas 3.1 and 4.5 we have that $X - f^{-1}(x)$ has at most a countable number of components, $\{C_i\}_{i=1}^{\infty}$ (all but a finite number of the C_i may be empty). Let $X_1 = X - C_1$. By induction define $X_i = X_{i-1} - C_i$ for $i = 2, 3, \ldots$ By Lemmas 4.6 and 4.7, X_i is connected for each i, thus $f^{-1}(x)$ is connected since $f^{-1}(x) = \bigcap_{i=1}^{\infty} X_i$.

Our final result is an application of Theorem 4.3.

THEOREM 4.8. *There is no non-alternating mapping from a simply connected Peano continuum X onto the circle S.*

Proof. Suppose there were such a mapping f. Let $x, y \in S$, by Lemma 3.1 both $X - f^{-1}(x)$ and $X - f^{-1}(y)$ are connected. But $X - \left(f^{-1}(x) \cup \cup f^{-1}(y)\right)$ must be separated. Thus there exists a component C of $f^{-1}(x) \cup \cup f^{-1}(y)$ which separates X. Since $f(C) = x$ or y, we have a contradiction.

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Reçu par la Rédaction le 27. 1. 1970

Construction of group topologies on abelian groups

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Introduction. In this article we discuss a few methods for constructing grouptopologies on abelian groups and the relations between these methods.

In section 1 and 2 the method of Hinrichs (intended for the ring of integers, cf. [4]) is investigated together with its relation to the construction occurring in [6]. In section 3 it is explained how the topology of a given abelian group can be refined by making a character continuous.

Next the problem is studied of finding topologies T on an abelian group G such that G becomes a complete topological group with respect to T. Generalizations of the results of section 8 of [6] and section 8 of [2] are obtained in section 4 and 5 respectively. The methods used resemble those introduced in section 1 and 2.

Finally, in section 6 it is observed that this paper basically deals with refinements. The problem is posed of reaching the aims of section 4 and 5 of this article and the result of [8], which is obtained by coarsifying, at the same time.

Notations and terminology. All groups in this article will be commutative and additively written. Let G be a group and U and V subsets of it. $U+V$ is defined by $U+V = \{a+b\colon\ a \in U,\ b \in V\}$; $1U \underset{D}{=} U$ and $nU \underset{D}{=} (n-1)U + U$, for $n > 1$. Instead of $n\{x\}$ we will write nx and instead of $n\{-x, 0, x\}$ we will write $n \cdot x$.

We will denote a topological group frequently by (G, T) in which G is a group and T a topology defined on it such that the operation $(x, y) \to x - y$ is continuous in both variables together. G_d stands for (G, D), in which D is the discrete topology on G. G alone stands for the group G without a topology. We may discuss topologies defined on it. Sometimes we will also use the notation G for a topological group, if there is no danger for confusion about the topology that is meant.

$\boldsymbol{Z}$ will denote the group of integers, $\boldsymbol{R}$ the group of reals and by $\boldsymbol{N}$ we will mean the positive integers including 0.