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The category of recursive functions

by

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1. Introduction. This paper was inspired by the appendix of [6] where counter-examples are given which illustrate some difficulties involved in developing a homomorphism theory for recursive groups. The major difficulty has nothing to do with groups but hinges on the existence of one-one onto maps which are not isomorphisms. The principal aim of this paper is to show that if we restrict ourselves to a suitable class of onto maps (which we shall call "proper onto") then a homomorphism theory is obtained which satisfies the homomorphism theorems of universal algebra. Moreover, the spirit of [5] suggests that the definition of proper onto is reasonable.

In a preliminary section we give a survey of some of the properties of the category of partially recursive functions. Unfortunately, the most non-trivial results are counter-examples which show pathology. Nevertheless, we feel that there is a value in having an explicit record on the nature of this category.

For concepts in category theory we use the notation in [7]. For concepts in recursive function theory and notations (e.g., the use of "ϵ" for the set of non-negative integers) we use the papers of J. C. E. Dekker [1]–[5].

2. The category. We consider the category whose objects are sets of non-negative integers and whose morphisms are restrictions of partial recursive functions.

The null set is a conull object and a unit set is a null object as in the category of sets. In addition monics are one-one and epics are onto. Fortunately, the family of partial recursive functions is at least rich enough to enable us to use essentially the same proofs as in the category of sets.

PROPOSITION 1. *Two sets are categorically equivalent iff they are recursively equivalent.*

This is essentially proved in [2] without using the language of category theory. The result shows that the concept of R. E. T. arises naturally out of a categorical point of view.

Proposition 2. *The coproduct is the Dekker sum and the product is the Dekker product.*

Proof. We indicate the mappings only. The proofs are trivial. Given α and β, define $\gamma = [J(x,0)\colon x \in \alpha] \cup [J(x,1)\colon x \in \beta]$. Then the coproduct is the pair of maps $\alpha \xrightarrow{u_1} \gamma$, $\beta \xrightarrow{u_2} \gamma$ with $u_1(x) = J(x,0)$ and $u_2(y) = J(y,1)$. Given α and β define $\gamma = [J(x,y)\colon x \in \alpha \text{ and } y \in \beta]$. Then the product is the pair of maps $\gamma \xrightarrow{p_1} \alpha$, $\gamma \xrightarrow{p_2} \beta$ where $p_1(x) = k(x)$ and $p_2(x) = l(x)$. Note that these represent the coproduct and product in the category of sets as well.

The category is not balanced, i.e., there exist mono-epics which are not isomorphisms. In fact any regressive immune set $\{t_n\}$ may be regarded as an epi-subobject of ε by means of the map $t_n \to n$. Not all epi-subobjects of ε are of this form. For example, if s_n and t_n are two separable regressive sets whose union is not regressive then the map $s_n \to 2n$, $t_n \to 2n+1$ is mono-epic.

The existence of monics which do not correspond to subsets in the usual manner suggests consideration of extremal monics. f is an extremal monic if f is a monic and for all factorizations $f = gh$ with h epic, h is necessarily an isomorphism. In the category of all topological spaces, for example, extremal monics correspond to actual subspaces. We now show that this happens also in our category.

Proposition 3. *Extremal monics correspond to subsets.*

Proof. Suppose $\alpha \xrightarrow{f} \beta$ is extremal monic. Let $\gamma = \operatorname{im} f$ and factor $\alpha \xrightarrow{f} \beta$ as $\alpha \xrightarrow{g} \gamma \xrightarrow{i} \beta$ where i is the inclusion map. Since g is epic, g is an isomorphism. Hence f is equivalent to i.

Suppose $\alpha \xrightarrow{i} \beta$ is an inclusion map. Let $\alpha \xrightarrow{i} \beta = \alpha \xrightarrow{p} \gamma \xrightarrow{f} \beta$ where p is epic. Then p is one-one onto. Hence $\gamma \xrightarrow{f} \beta$ factors as $\gamma \xrightarrow{g} \alpha \xrightarrow{i} \beta$. Since p and g are inverses of each other, p is an isomorphism.

Equalizers are obtained by the same construction as in the category of sets. The pullback of

$$\begin{array}{ccc} & & \alpha \\ & & \downarrow f \\ \beta & \xrightarrow{g} & \gamma \end{array} \quad \text{is} \quad \begin{array}{ccc} \delta & \xrightarrow{k} & \alpha \\ l\downarrow & & \downarrow f \\ \beta & \xrightarrow{g} & \gamma \end{array}$$

where $\delta = [J(x,y)\colon x \in \alpha,\ y \in \beta \text{ and } f(x) = g(y)]$ and k and l have their usual meanings in recursive function theory. Thus the category is finitely complete. The following is immediate but worth noting.

Proposition 4. *For regressive epi-subobjects of ε the intersection is the Dekker min. function* [3].

The category is not finitely cocomplete. (It is, of course, intuitively plausible to expect more difficulties on the right than on the left.) We shall construct a diagram without a pushout. It follows from ([7] dual of Proposition I, 17.3) that coequalizers do not always exist.

Let α be a non-recursive set. Then the diagram $\begin{array}{cc} \alpha & \xrightarrow{i} \varepsilon \\ i\downarrow & \\ \varepsilon & \end{array}$ where i is the inclusion mapping has no pushout (incidentally if α is recursive the obvious construction does lead to a pushout). Suppose $\begin{array}{ccc} \alpha & \xrightarrow{i} & \varepsilon \\ i\downarrow & & \downarrow f \\ \varepsilon & \xrightarrow{g} & \beta \end{array}$ were a pushout. β may be chosen so that $\beta \neq \varepsilon$. Then f and g are recursive. $x \in \alpha \to f(x) = g(x)$. Now $[x\colon f(x) = g(x)]$ is recursive. Hence $\exists(x)\,[x \notin \alpha$ and $f(x) = g(x)]$. Let $a \notin \alpha$ such that $f(a) = g(a)$ and let $b \notin \beta$. We define a diagram $\begin{array}{ccc} \alpha & \xrightarrow{i} & \varepsilon \\ i\downarrow & & \downarrow f' \\ \varepsilon & \xrightarrow{g'} & \gamma \end{array}$ as follows. $\gamma = \beta \cup \{b\}$, $g'(x) = g(x)$, $f'(x) = f(x)$ if $x \neq a$ and $f'(a) = b$. Then $x \in \alpha \to f'(x) = g'(x)$ i.e. the diagram commutes. By definition of pushout $\exists(\beta \xrightarrow{h} \gamma)$ such that $hf = f'$ and $hg = g'$. Then $f'(a) = hf(a) = hg(a) = g'(a)$. This contradicts the construction of f' and g'.

This result might suggest that if a colimit exists then it is the same as the colimit in the category of sets. This is not true either. Let α be non-recursive and suppose $a \in \alpha$ and $b \in \alpha'$. Consider the diagram $\alpha \underset{i_1}{\overset{e_1}{\rightrightarrows}} \varepsilon \underset{i_2}{\overset{e_2}{\leftleftarrows}} \alpha'$ where e_i is the restriction of the identity map, $(\forall x)\,[i_1(x) = a]$ and $(\forall x)\, i_2(x) = b$. Then the colimit in the category of sets is $\varepsilon \xrightarrow{f} \{a, b\}$ where $x \in \alpha \Rightarrow f(x) = a$ and $x \in \alpha' \Rightarrow f(x) = b$. Of course, f is not a morphism in our category. We claim that the constant map $\varepsilon \xrightarrow{c} \{0\}$ is a colimit. Suppose we have a commutative diagram

$$\begin{array}{ccccc} \alpha & \underset{i_1}{\overset{e_1}{\rightrightarrows}} & \varepsilon & \underset{i_2}{\overset{e_2}{\leftleftarrows}} & \alpha' \\ & & \downarrow f & & \\ & & \beta & & \end{array} \qquad \text{i.e. } fe_1 = fi_1 \text{ and } fe_2 = fi_2.$$

Then f is constant on α and on α'. Since a set of constancy must be recursive f is necessarily constant. Thus we obtain an obvious unique map $\{0\} \xrightarrow{g} \beta$ which satisfies $gc = f$.

The categorical approach permit alternative proofs for the elementary properties of addition, multiplication, and the min. function [1], [3].

Although the usual proofs are trivial this may have expository value. We cannot expect this general approach to enable us to prove deeper theorems in the theory of R.E.T.'s. For example, more structure is needed to handle something as basic as inductive definitions categorically.

3. Homomorphism theory. We discuss sets first and consider universal algebras later. First, a reasonable definition of quotient object is suggested in [5]. A quotient object of a is a good choice (abbreviated as g.c.) set of a g.c. decomposition of a. The theory in [5] thrives on the fact that all g.c. sets of a given decomposition are recursively equivalent. We now state an elementary result.

PROPOSITION 5. *For any* R.E.T. B, A *is a quotient object of* B *iff* $A \leqslant B$ *and* $A \neq 0$.

Proof. First suppose that $A \leqslant B$. Then $(\exists a \in A)(\exists \beta \in B)(\exists \gamma)$ $(a \cup \gamma = \beta$ and a is separable from $\gamma)$. Let $a \in a$. Define $f(x) = x$ if $x \in a$ and $f(x) = a$ if $x \in \gamma$. f has a partial recursive extension and induces a g.c. decomposition with a as a g.c. set.

Now suppose $a \in A$ and let f induce a g.c. decomposition of a. Let $\beta = f(a)$. Then β is separable from $a - \beta$ since $\beta = [x \in a\colon f(x) = x]$ and $a - \beta = [x \in a\colon f(x) \neq x]$. If B is the R.E.T. of β then $B \leqslant A$.

It is clear from the existence of immune epi-subobjects of ε that not all onto maps lead to quotient objects. We will study certain classes of onto maps.

DEFINITION 1. $a \xrightarrow{f} \beta$ is *proper onto* iff $\exists(\beta \xrightarrow{g} a)$ such that $fg = 1$ (i.e. f is a retraction).

DEFINITION 2. $a \xrightarrow{f} \beta$ is *strongly proper onto* if f has a partial recursive extension f' with domain δ such that $f'(\delta - a) \cap \beta = \emptyset$.

Although the definition of proper onto might appear to be too restrictive note that all g.c. maps are proper onto. Note also that the class of proper onto maps and of strongly proper onto maps are both closed with respect to composition.

LEMMA 1. *All strongly proper onto maps are proper onto but not conversely. In fact there exists a g.c. map which is not strongly proper onto.*

Proof. Suppose f is strongly proper onto and choose f' as in the definition. By elementary recursive function theory there exists a partial recursive function g defined at least on β and range contained in δ such that $f'g = 1$. Since $f'(\delta - a) \cap \beta = \emptyset$, the range of g is necessarily included in a. This shows that f is proper onto.

Let a be non r.e. and consider the decomposition consisting of a only. Let $a \in a$. The constant map into a is clearly a g.c. map. However, since the inverse image of an r.e. set is r.e. the map is not strongly proper onto. Q.E.D.

LEMMA 2. *For one-one maps the following are equivalent*

(1) f *is proper onto,*

(2) f *is strongly proper onto,*

(3) f *has a partial recursive one-one extension.*

Proof. (2) ⇒ (1) by Lemma 1. (1) ⇒ (3) by [2]. It is obvious that (3) ⇒ (2).

Thus a proper onto map is a generalization of an isomorphic map in the case where the map is not necessarily one-one. Lemma 1 suggests that it is more reasonable to consider proper maps than strongly proper maps.

We now come to the main theorems of this paper.

FIRST ISOMORPHISM THEOREM. *If* f *is a proper map from* a *onto* β, *then the partition induced in* a *by* f *is a g.c. decomposition and* f *induces a recursive equivalence between a g.c. set of the decomposition and* β.

Proof. Choose $\beta \xrightarrow{g} a$ so that $fg = 1$. Then gf is a g.c. map for the decomposition induced by f and the image of gf is a g.c. set satisfying the required property. Q.E.D.

This result together with the fact that g.c. maps are proper onto justifies the study of proper onto maps. The class of strongly proper onto maps is too small for our present purpose but has potentialities in future studies.

SECOND ISOMORPHISM THEOREM. *Let* H *be an equivalence relation on* a *and* γ *a subset of* a *with* $\gamma H = a$. *Suppose that* H *induces a g.c. decomposition of* a *with at least one g.c. set* δ *included in* γ. *Then* $\frac{\gamma}{H/\gamma}$ *is recursively isomorphic to* $\frac{a}{H}$.

Proof. This is trivial since δ is a g.c. subset of both $\frac{\gamma}{H/\gamma}$ and $\frac{a}{H}$. Of course the restriction of the g.c. function on a to γ serves as the g.c. function on γ.

THIRD ISOMORPHISM THEOREM. *Let* H *be an equivalence relation on* a *and let* β *be a g.c. set with g.c. function* f *of the corresponding decomposition of* a. *Then there is a one-one correspondence between g.c. decompositions of* β *and g.c. decompositions of* a *containing* H *such that the corresponding g.c. sets are recursively isomorphic.*

Proof. Identify β with $\frac{a}{H}$ and consider the usual one-one correspondence in the category of sets between equivalence relations on β and equivalence relations on α containing H. We first show that the g.c. property is preserved both ways. First, let g be a g.c. function on β with g.c. set γ. Then gf is a g.c. function with the same g.c. set which induces the required partition. Conversely, if G is an equivalence relation containing H and g is the corresponding g.c. function, then fg is a g.c. function which induces the required partition on β. The last statement of the theorem is clear since the g.c. sets may even by chosen so as to be identical (as in the first part of the proof).

We now turn to universal algebras. No recursiveness properties are required on the operations. It seems natural to relegate recursiveness to the morphisms and have the objects as general as possible. First, if a congruence relation induces a g.c. decomposition then the quotient algebra induces an algebra on any g.c. set in an obvious manner and the natural recursive isomorphism of the underlying g.c. sets is an algebra isomorphism. Incidentally, if any operation is recursive, then the induced operation on the g.c. set is recursive. It is easy to see that the isomorphism theorems remain valid for universal algebras.

4. Extensions to non-onto maps. If we could restrict the class of epimorphisms to proper onto maps the category would be balanced. We had hoped for a suitable restriction on the class of all maps to give us a well-behaved category, e.g., an abelian category for the case of abelian groups. Unfortunately no concept has been found for general maps which works as well as proper onto maps for onto maps. We now indicate some pathology involved in the attempts at generalization.

DEFINITION 1. Suppose we define $\alpha \xrightarrow{f} \beta$ as *proper* iff $\alpha \xrightarrow{f} \operatorname{im} f$ is proper onto. This seems like a natural definition; however the class of proper maps under this definition is not closed with respect to composition!

EXAMPLE. Let $\alpha = \{\tau_n\}$ be regressive immune with regressing function k and let $\beta = \gamma = \varepsilon$. Let $\alpha \xrightarrow{f} \beta$ be the inclusion map and let $\beta \xrightarrow{g} \gamma$ be defined as $g(n) = \mu m[k^{(m+1)}(n) = k^{(m)}(n)]$. Then f and g are proper. As an inverse for g it suffices to take $J(0, \overbrace{1, 1, \ldots}^{n \text{ times}})$. On the other hand gf maps τ_n into n hence the only possible inverse would map n onto τ_n. Since $\{\tau_n\}$ is immune such a map is not recursive.

DEFINITION 2. We define $\alpha \xrightarrow{f} \beta$ as *proper* iff $(\exists \operatorname{map} g)\ g(\beta) \subset \alpha$ and $fg(x) = x$ if $x \in \operatorname{im} f$. For onto maps this reduces to the definition of proper onto just as in the previous case; however this definition gives some importance to $\beta - f(\alpha)$. For example if α is not r.e. then the inclusion map $\alpha \to \varepsilon$ is not proper by Definition 2 although it is by Definition 1. The class of proper maps under this definition is not closed under composition.

EXAMPLE. Let $\alpha = \{2\tau_n\}$ where $\{\tau_n\}$ is regressive immune, let $\beta = \alpha \cup \{2n+1\}$, and let $\gamma = \varepsilon$. Let $\alpha \xrightarrow{f} \beta$ be the inclusion map, $\beta \xrightarrow{g} \gamma$ be defined as $g(2\tau_n) = g(2n+1) = n$. Since α is separable from $\{2n+1\}$, f has an inverse function. $n \to 2n+1$ is an inverse function for g. The only possible inverse function for gf would be $n \to 2\tau_n$ which is not recursive since $n \to \tau_n$ is not recursive.

We will try to extend the concept of strongly proper onto maps.

DEFINITION 3. $\alpha \xrightarrow{f} \beta$ is *proper* iff $\alpha \xrightarrow{f} \operatorname{im} f$ is strongly proper onto.

EXAMPLE. Let α be non r.e., let $\alpha \subset \beta$ with β recursive and let $\gamma = \{x\}$ where $x \in \beta$. Define $\alpha \xrightarrow{f} \beta$ as the inclusion map and $\beta \xrightarrow{g} \gamma$ as the constant map. It is easy to see that f and g are proper but gf is not proper.

Finally we give some importance to $\beta - f(\alpha)$.

DEFINITION 4. $\alpha \xrightarrow{f} \beta$ is *proper* if f has a partial recursive extension f' with domain δ such that $f'(\delta - \alpha) \cap \beta = \emptyset$.

In contrast to the other cases the class of such maps is closed under composition. Let $\alpha \xrightarrow{f} \beta$ and $\beta \xrightarrow{g} \gamma$ be proper and choose $\delta_1 \supset \alpha$ and $\delta_2 \supset \beta$ and extensions f' and g' such that $f'(\delta_1 - \alpha) \cap \beta = \emptyset$ and $g'(\delta_2 - \beta) \cap \gamma = \emptyset$. Let $\delta_3 = \delta_1 \cap f'^{-1}(\delta_2)$. Then δ_3 is an r.e. set satisfying $\alpha \subset \delta_3 \subset \delta_1$. Let f'' be the restriction of f' to δ_3. Then $g'f''$ is a partial recursive extension of gf and $g'f''(\delta_3 - \alpha) \cap \gamma = \emptyset$. Since an identity map is proper under this definition the class of proper maps forms a subcategory.

In spite of this initial success for definition 4 serious weaknesses remain. For example projection maps are not necessarily in this subcategory, i.e. if α is non r.e., $\beta = \{x\}$ and $\gamma = [J(a, x)\colon a \in \alpha]$ then the restriction of the map l [with the usual meaning: $lJ(x, y) = y$] to γ is not in this subcategory.

We show finally that α and β do not have a product in this subcategory. All we have shown so far is that the category product with its projections is not in the subcategory. Let γ be a product of α and β with maps $\gamma \xrightarrow{f} \alpha$ and $\gamma \xrightarrow{g} \beta$. It is easy to see that for every pair (m, n) such that $m \in \alpha$ and $n \in \beta$ there exists a unique $s \in \gamma$ such that $f(s) = m$ and $g(s) = n$. Note that the usual proof in the category of sets works since the maps used are proper in the sense of Definition 4. We define $h(m, n)$ to be this unique s. Then $fh(m, n) = m$ and $gh(m, n) = n$. f and g have partial recursive extensions by definition of morphism. On the other hand we have no a priori recursiveness properties for h.

So far we have not made use of the given nature of α and β. Since f is onto a non r.e. set, γ is non r.e. Thus g maps a non r.e. set into the single number x and is therefore not proper. This contradiction completes the proof.

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Non-alternating mappings

by

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1. Introduction. In [5] Whyburn introduced the concept of non-aternating mapping which generalizes the concept of monotone mapping. Kuratowski in [1] and Kuratowski and Lacher in [2] have given sufficient conditions for spaces of monotone mappings to be topologically complete. McAuley in [3] has proved a similar theorem for non-alternating mappings, where the range space is the unit interval. In this paper we continue this line of investigation for spaces of non-alternating mappings. We also consider the problem of when non-alternating mappings are monotone, and under what condition there exist non-alternating mappings between two spaces; these problems were originally investigated by Whyburn in [6], [7], and [8].

2. Definition. A mapping f from a space X into a space Y is said to be *non-alternating* if for each pair of points a, b in $Y, f^{-1}(a)$ does not separate $f^{-1}(b)$ in X.

A compact metric continuum D is a *dendrite* if it is locally connected and contains no simple closed curve. A dendrite D has the following properties, see [8]:

(i) every two points of D are separated by a third point;

(ii) every point of D is either a cut point or an end point;

(iii) one and only one arc exists between any two points of D.

For $a, b \in D$, the unique arc between a and b is denoted by $[a, b]$.

If X and Y are topological spaces, the space of all mappings (continuous functions) from X into Y is denoted by $C(X, Y)$ and is topologized by the compact-open topology. The subspace of $C(X, Y)$ consisting of all non-alternating mappings from X onto Y is denoted by $N(X, Y)$.

If X is a metric space, the metric for X will be denoted by $d(-,-)$; if $x \in X$ and $\varepsilon > 0$, then the set $\{y \in X: d(x, y) < \varepsilon\}$ will be denoted by $B(x, \varepsilon)$. Euclidean n-space is denoted by R^n. Fr denotes the frontier of a set.