are isomorphic. Two fc-*spaces* X *and* Y *are homeomorphic iff the lattices* $L_R(X)$ *and* $L_R(Y)$ *are isomorphic.*

Remark. The pc- and fc-spaces play roles in the theory of T_0-spaces analogous to the roles of compact and realcompact spaces in the theory of Tychonoff spaces. It is interesting to note that it is possible to define a concept which is analogous to pseudocompactness. A T_0-space X is said to be a pseudo-pc-space if each element of (X, R) is bounded above on every irreducible closed subset of X.

It is clear from Lemma 4.1 that a T_0-space X is a pseudo-pc-space iff every irreducible closed subset of X has the FCI-property and also that a T_0-space is a pc-space iff it is both an fc- and a pseudo-pc-space.

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On the position of the set of monotone mappings in function spaces

by

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K. Kuratowski and R. C. Lacher have shown in [5] that if X and Y are compact topological spaces and Y is locally connected, then the set of all monotone mappings of X onto Y is closed in Y^X (endowed with the compact-open topology). In an earlier paper [4] K. Kuratowski showed that if the space X is compact and metric and Y an arbitrary metric space, then the monotone mappings of X into Y form a G_δ-set in Y^X.

In this connection the question arises whether the above theorems can be generalized by dropping the assumption of the compactness of X and restricting the considerations to perfect mappings. More generally, in the space Y^X can consider subset $\Phi \subset \Psi \subset Y^X$ (we shall be interested in closed or perfect monotone mappings), and, under certain assumptions on X and Y, one can prove that Φ is closed (or that is a G_δ-set) in Ψ. Below we shall prove a few facts of this type and give examples illustrating role of the assumptions which have been made.

We adopt the terminology and notation of [2] and [3]. All the spaces considered below are Hausdorff spaces. The space Y^X of mappings of X into Y will be considered with the compact-open topology. The symbol $\mathrm{M}(A, B)$, where $A \subset X$, $B \subset Y$, will denote the set $\{f \in Y^X |\ f(A) \subset B\}$.

LEMMA 1. *Let X be an arbitrary space, Y a locally connected space and Φ the set $\{f\colon X \to Y |\ f^{-1}(S) \subset X$ is connected for all open and connected $S \subset Y\}$.*

If the mapping $f\colon X \xrightarrow[\text{onto}]{} Y$ satisfies the conditions

(i) *the boundary $\mathrm{Fr} f^{-1}(y)$ is compact for every $y \in Y$,*

(ii) *if $y \in Y$ and U is a neighbourhood of the set $f^{-1}(y)$, then there exists an open set $V \subset X$ such that $f^{-1}(y) \subset V \subset U$ and the boundary $\mathrm{Fr} V$ is compact,*

(iii) $f \in \overline{\Phi}$,

then f is a monotone, closed mapping.

Proof. We shall first prove that f is a closed mapping.

Let A be an arbitrary closed subset of X and assume that $y \in \overline{f(A)} \setminus f(A)$. Hence $f^{-1}(y) \subset X \setminus A$. By (ii) there exists an open set $V \subset X$ such that $f^{-1}(y) \subset V \subset X \setminus A$ and the boundary $\mathrm{Fr}V$ is compact. Since the compact set $f(\mathrm{Fr}V)$ does not contain the point y and Y is a locally connected Hausdorff space, there exists a connected and open neighbourhood S of y such that $\bar{S} \cap f(\mathrm{Fr}V) = \emptyset$. Since $y \in \overline{f(A)}$ and f is onto, there exist a point $a \in A$ such that $f(a) \in S$ and a point $b \in f^{-1}(y)$. Since $f \in \bar{\Phi}$, there exists a $g \in \mathrm{M}(\mathrm{Fr}V, Y \setminus \bar{S}) \cap \mathrm{M}(\{a, b\}, S) \cap \Phi$. But $b \in g^{-1}(S) \cap V$, $a \in g^{-1}(S) \cap (X \setminus V)$ and $g^{-1}(S) \cap \mathrm{Fr}V = \emptyset$; hence $g^{-1}(S)$ is not connected, contrary to the assumption that $g \in \Phi$.

Now we shall prove that f is a monotone mapping.

Let us assume that for a certain $y \in Y$ we have $f^{-1}(y) = A_1 \cup A_2$, where $\bar{A}_i = A_i$, $A_i \neq \emptyset$ for $i = 1, 2$, and $A_1 \cap A_2 = \emptyset$. As A_1 and A_2 are separated, we have $\mathrm{Fr}(A_1 \cup A_2) = \mathrm{Fr}A_1 \cup \mathrm{Fr}A_2$. From (i) it follows that the sets $\mathrm{Fr}A_i$ are compact, and X being a Hausdorff space, there exist open disjoint sets G_1, G_2 such that $A_i \subset G_i$. Since $G_1 \cup G_2 \supset A_1 \cup A_2 = f^{-1}(y)$, by (ii) there exists an open set $V \subset X$ such that $A_1 \cup A_2 \subset V \subset G_1 \cup G_2$ and the boundary $\mathrm{Fr}V$ is compact. The sets $W_1 = G_1 \cap V$ and $W_2 = G_2 \cap V$ are separated and $\mathrm{Fr}(W_1 \cup W_2) = \mathrm{Fr}W_1 \cup \mathrm{Fr}W_2$. Therefore the set $\mathrm{Fr}W_1$ is compact, $W_1 \supset A$ and $\bar{W}_1 \cap A_2 = \emptyset$. Since the compact set $f(\mathrm{Fr}W_1)$ does not contain the point y and Y is a Hausdorff and locally connected space, there exists a connected and open neighbourhood S of y such that $\bar{S} \cap f(\mathrm{Fr}W_1) = \emptyset$. Take $a_i \in A_i$ for $i = 1, 2$. The set $M = \mathrm{M}(\mathrm{Fr}W_1, Y \setminus \bar{S}) \cap \mathrm{M}(\{a_1, a_2\}, S)$ is a neighbourhood of f in Y^X. If $g \in M$, then $g^{-1}(S) \cap \mathrm{Fr}W_1 = \emptyset$, $a_1 \in g^{-1}(S) \cap W_1$, $a_2 \in g^{-1}(S) \cap (X \setminus W_1)$; hence the set $g^{-1}(S)$ is not connected and $g \notin \Phi$. It follows that $M \cap \Phi = \emptyset$, contrary to (iii). Therefore f is monotone.

LEMMA 2. *If $f: X \xrightarrow[\text{onto}]{} Y$ is a closed mapping such that for all $y \in Y$ the boundary $\mathrm{Fr}f^{-1}(y)$ is compact, then for every compact $Z \subset Y$ the set $\bigcup_{z \in Z} \mathrm{Fr}f^{-1}(z)$ is compact.*

Proof. Let $X_1 = \bigcup_{y \in Y} \mathrm{Fr}f^{-1}(y)$, $Y_1 = f(X_1)$ and $f_1 = f|X_1$. The set X_1 is closed in X, hence f_1 is a perfect mapping of X_1 onto the closed subset Y_1 of Y (see [2], Problem 3.X) and the inverse image $f_1^{-1}(Z \cap Y_1)$ is compact. The lemma now follows from the equality

$$\bigcup_{z \in Z} \mathrm{Fr}f^{-1}(z) = f_1^{-1}(Z \cap Y_1).$$

(1) We say that X is a *rim-compact* space if for every $x \in X$ and every neighbourhood U of x there exists a neighbourhood V of x such that $V \subset U$ and $\mathrm{Fr}V$ is compact.

LEMMA 3. *If the mapping $f: X \xrightarrow[\text{onto}]{} Y$ is closed, the boundary $\mathrm{Fr}f^{-1}(y)$ is compact for every $y \in Y$, and Y is a rim-compact (1) space, then for every $y \in Y$ the set $f^{-1}(y)$ has a neighbourhood system consisting of open sets with compact boundaries.*

Proof. For an arbitrary $A \subset Y$ we have

$$\mathrm{Fr}f^{-1}(A) \subset \bigcup_{y \in \mathrm{Fr}A} \mathrm{Fr}f^{-1}(y). \tag{1}$$

Indeed, $\bigcup_{y \in Y} \mathrm{Int}f^{-1}(y) \subset \mathrm{Int}f^{-1}(A) \cup \mathrm{Int}(X \setminus f^{-1}(A)) = X \setminus \mathrm{Fr}f^{-1}(A)$, and thus $\mathrm{Fr}f^{-1}(A) \subset \bigcup_{y \in Y} \mathrm{Fr}f^{-1}(y)$; from the continuity of f we have $\mathrm{Fr}f^{-1}(A) \subset f^{-1}(\mathrm{Fr}A)$; hence $\mathrm{Fr}f^{-1}(A) \subset \big(\bigcup_{y \in Y} \mathrm{Fr}f^{-1}(y)\big) \cap f^{-1}(\mathrm{Fr}A) = \bigcup_{y \in \mathrm{Fr}A} \mathrm{Fr}f^{-1}(y)$.

Now take an arbitrary point $y \in Y$ and let G be an open neighbourhood of $f^{-1}(y)$. Since the mapping f is closed and Y is rim-compact, we can find an open set H such that $y \in H \subset Y \setminus f(X \setminus G)$ and the boundary $\mathrm{Fr}H$ is compact. For $V = f^{-1}(H)$ we have $f^{-1}(y) \subset V \subset G$ and by virtue of (1), $\mathrm{Fr}V \subset \bigcup_{z \in \mathrm{Fr}H} \mathrm{Fr}f^{-1}(z)$. By Lemma 2 we infer that the set $\mathrm{Fr}V$ is compact.

THEOREM 1. *If X is a paracompact space, and Y is locally connected, rim-compact and satisfies the first axiom of countability, then the set of all monotone closed mappings of X onto Y is closed in the set of all closed mappings of X onto Y.*

Proof. By Michael's generalization of Vaĭnšteĭn's theorem ([6]), it follows that if $f: X \xrightarrow[\text{onto}]{} Y$ is a closed mapping, then $\mathrm{Fr}f^{-1}(y)$ is compact for every $y \in Y$. Hence — by Lemma 3 — f has properties (i) and (ii) of Lemma 1. Our theorem now follows from Lemma 1 and the theorem ([3], § 46, I, Theorem 9), which states that if $f: X \xrightarrow[\text{onto}]{} Y$ is a monotone closed mapping then the inverse image $f^{-1}(S)$ is connected for every connected $S \subset Y$.

K. Morita ([7], simplified proof in [1]) proved that if $f: X \xrightarrow[\text{onto}]{} Y$ is a closed monotone mapping such that $\mathrm{Fr}f^{-1}(y)$ is compact for every $y \in Y$ and X is rim-compact, then Y is also rim-compact. Hence, from the quoted above result of Michael and our Theorem 1, we obtain

THEOREM 2. *If X is a paracompact and rim-compact space and Y is locally connected and satisfies the first axiom of countability, then the set of all monotone closed mappings of X onto Y is closed in the set of all closed mappings of X onto Y.*

THEOREM 3. *If X is a paracompact space and Y is a locally connected, locally compact space, then the set of all monotone closed mappings of X onto Y is closed in the set of all closed mappings of X onto Y.*

Proof. This follows from the argument given in the proof of Theorem 1, because the Vaĭnšteĭn–Michael theorem is valid also under the assumption of local compactness of Y, and every locally compact space is obviously rim-compact.

THEOREM 4. *If X is a paracompact space and Y is a locally connected, locally compact space, then the set of all monotone, perfect mappings of X onto Y is closed in the set of all closed mappings of X onto Y.*

Proof. Let $\Lambda = \{f\colon X \xrightarrow[\text{onto}]{} Y \mid f$ is a monotone, perfect mapping$\}$, and let $f\colon X \xrightarrow[\text{onto}]{} Y$ be an arbitrary closed mapping such that $f \in \overline{\Lambda}$. By the Vaĭnšteĭn–Michael theorem for every $y \in Y$ the boundary $\mathrm{Fr} f^{-1}(y)$ is compact and from Theorem 3 we infer that

(2) $$f \text{ is a monotone mapping}.$$

Suppose that there exists a $y_0 \in Y$ such that $f^{-1}(y_0)$ is not compact. Let us observe first that

(3) $$\mathrm{Fr} f^{-1}(y_0) \neq \emptyset .$$

Indeed, suppose that $\mathrm{Fr} f^{-1}(y_0) = \emptyset$. Then, as f is closed, y_0 is an isolated point in Y. Let us take an arbitrary point $a \in f^{-1}(y_0)$. As $f \in \overline{\Lambda}$, there exists a $g \in \mathrm{M}(\{a\}, \{y_0\}) \cap \Lambda$. Since $g(f^{-1}(y_0)) \ni y_0$ and $f^{-1}(y_0)$ is connected by (2), we have $g(f^{-1}(y_0)) = y_0$. Thus $f^{-1}(y_0) \subset g^{-1}(y_0)$, contrary to the assumption that $f^{-1}(y_0)$ is not compact.

Now take two open sets $V_1, V_2 \subset Y$ such that

(4) $$y_0 \in V_1 \subset \overline{V}_1 \subset V_2 \quad \text{and} \quad \overline{V}_2 \text{ is compact.}$$

Let $\{P_s\}_{s\in S}$ be the family of all components of $Y \setminus \overline{V}_1$. Since $Y \setminus \overline{V}_1$ is locally connected, $\{P_s\}_{s\in S}$ is an open covering of $Y \setminus \overline{V}_1$. The compact set $\mathrm{Fr} V_2$ is contained in $Y \setminus \overline{V}_1$, hence there exist $s_1, \ldots, s_k \in S$ such that

(5) $$\mathrm{Fr} V_2 \subset P_{s_1} \cup \ldots \cup P_{s_k}.$$

Take points $x_i \in X$ such that $f(x_i) \in P_{s_i}$ for $i = 1, \ldots, k$.

Since $f \in \overline{\Lambda}$, there exists a mapping

(6) $$g \in \mathrm{M}(\mathrm{Fr} f^{-1}(y_0), V_1) \cap \bigcap_{i=1}^{k} \mathrm{M}(\{x_i\}, P_{s_i}) \cap \Lambda .$$

Let $A = f^{-1}(y_0) \setminus g^{-1}(\overline{V}_1)$. We shall show that

(7) $$A \text{ is open} \quad \text{and} \quad g^{-1}(g(A)) = A .$$

By (6) $g^{-1}(V_1) \supset \mathrm{Fr} f^{-1}(y_0)$ and $A = \mathrm{Int} f^{-1}(y_0) \setminus g^{-1}(\overline{V}_1)$ is an open set. Since $g^{-1}(g(A)) \cap \mathrm{Fr} f^{-1}(y_0) = \emptyset$ and g is a monotone mapping, we have $g^{-1}(g(a)) \subset f^{-1}(y_0)$ for all $a \in A$. It follows that $g^{-1}(g(A)) = \bigcup_{a\in A} g^{-1}(g(a)) \subset f^{-1}(y_0)$ and $g^{-1}(g(A)) = A$.

We shall now prove that

(8) $$g(A) \text{ is open-and-closed in } Y \setminus \overline{V}_1 .$$

By (7), $g(A)$ is open even in Y. Since $\overline{g(A)} = g(\overline{A}) \subset g(A \cup g^{-1}(\overline{V}_1)) = g(A) \cup \overline{V}_1$, $g(A)$ is closed in $Y \setminus \overline{V}_1$.

As $f^{-1}(y_0) \subset g^{-1}(g(f^{-1}(y_0)))$, the set $f^{-1}(y_0)$ is not compact and g is a perfect mapping, we infer that $g(f^{-1}(y_0))$ is not compact; thus $g(f^{-1}(y_0)) \cap (Y \setminus \overline{V}_2) \neq \emptyset$. By virtue of (3) and (6) $g(f^{-1}(y_0)) \cap V_1 \neq \emptyset$. From the connectedness of $g(f^{-1}(y_0))$ and from (4) we have $g(f^{-1}(y_0)) \cap \mathrm{Fr} V_2 \neq \emptyset$, and thus $g(A) \cap \mathrm{Fr} V_2 \neq \emptyset$. Take $z \in g(A) \cap \mathrm{Fr} V_2$; by virtue of (5), $z \in P_{s_{i_0}}$ and by (8) $P_{s_{i_0}} \subset g(A)$. From (6) it follows that $g(x_{i_0}) \in P_{s_{i_0}}$, then $g(x_{i_0}) \in g(A)$ and, by (7) $x_{i_0} \in A \subset f^{-1}(y_0)$. So we have $f(x_{i_0}) = y_0$, contrary to the choice of x_{i_0}. The contradiction shows that $f^{-1}(y_0)$ must be compact, which completes the proof.

Remark 1. Theorems 2 and 4 can be slightly generalized as follows:

THEOREM 2′. *Let X be a rim-compact space, Y a locally connected space, $\Xi = \{f\colon X \xrightarrow[\text{onto}]{} Y \mid f$ is closed and monotone$\}$ and $\Psi = \{f\colon X \xrightarrow[\text{onto}]{} Y \mid \mathrm{Fr} f^{-1}(y)$ is compact for every $y \in Y\}$.*

Then the intersection $\Xi \cap \Psi$ is closed in the set Ψ.

Indeed, if X is rim-compact and $f \in \Psi$, then f satisfies conditions (i), (ii) of Lemma 1; furthermore if $f \in \Xi$, then $f^{-1}(S)$ is connected for every connected $S \subset Y$. Theorem 2′ follows now from Lemma 1.

THEOREM 4′. *If X is an arbitrary space and Y is a locally connected, locally compact space, then the set Λ of all perfect monotone mappings X onto Y is closed in the set $\Psi = \{f\colon X \xrightarrow[\text{onto}]{} Y \mid \mathrm{Fr} f^{-1}(y)$ is compact for every $y \in Y\}$.*

We can assume that $\Lambda \neq \emptyset$. Then X is locally compact (see [2], Problem 3.Y) and *a fortiori* rim-compact. By Theorem 2′ we have $\overline{\Lambda} \cap \Psi \subset \Xi \cap \Psi$ and, as in the proof of Theorem 4, we can show that $\overline{\Lambda} \cap \Psi = \Lambda$.

Remark 2. Under the assumptions of Theorem 2 the set of all monotone closed mappings of X onto Y is identical (as observed by Morita in [7]) with the set of all monotone, quotient [2] mappings of X onto Y for which the boundaries of inverse images of points are compact. Indeed, in this case such a quotient mapping f satisfies conditions (i) and (ii) of Lemma 1; furthermore, if S is an open and connected subset of Y, then $f^{-1}(S)$ is connected, thus the condition (iii) is also satisfied.

Now we shall show that the assumptions in our theorems are essential. We shall use the fact that the compact-open topology in Y^X is identical

[2] A mapping $f\colon X \xrightarrow[\text{onto}]{} Y$ is *quotient* if $f^{-1}(A)$ is closed in X implies that A is closed in Y.

with the topology of uniform convergence on compacta induced in Y^X by an arbitrary uniformity compactible with the topology of Y (see [2], Theorem 8.2.3). Let E^2 denote the Euclidean plane, and for $a, b \in E^2$ let $[a, b]$ denote the closed segment with end-points a and b.

EXAMPLE 1. The assumptions "*Y is rim-compact*" in Theorem 1, "*X is rim-compact*" in Theorem 2 and "*Y is locally compact*" in Theorem 3 cannot be omitted.

Take in E^2 the points $x_0 = (0, 0)$, $y_0 = (1, 0)$ and $x_n = (0, 1/n)$, $y_n = (1, 1/n)$ for $n = 1, 2, \ldots$ and $X = \bigcup_{n=1}^{\infty} [x_n, y_n] \cup \{y_0, y_1] \cup [x_0, x_1]$. Denote by R the decomposition of X into the sets $\{x\}$ for $x \notin [x_0, x_1] \cup [y_0, y_1]$ and $\{[x_0, x_1] \cup [y_0, y_1]\}$. The quotient space $Y = X/R$ is metrizable, because the natural quotient mapping $p\colon X \to X/R$ is perfect (see [2], Problem 4.S); let d be a metric in Y. For every $\varepsilon > 0$ there exist open, disjoint squares K_1, K_2 with sides parallel to the axes and centres at x_0 and y_0, such that $X \cap (K_1 \cup K_2) \subset p^{-1}(B(p(x_0), \varepsilon/2))$, where $B(y, r)$ is a ball of radius r with centre at y. Let $Z \subset X$ be an arbitrary compact set. Take the first n_0 such that $Z \cap [x_n, y_n] \subset K_1 \cup K_2$ for $n > n_0$ and $f_{\varepsilon,Z}\colon X \xrightarrow[\text{onto}]{} Y$ defined by

$$f_{\varepsilon,Z}(x) = \begin{cases} p(x) & \text{for} \quad x \in \bigcup_{n=1}^{n_0} [x_n, y_n], \\ p(x_0) & \text{for} \quad x \in [x_{n_0+1}, y_{n_0+1}] \cup [x_0, x_1] \cup [y_0, y_1], \\ p(x + (x_{n-1} - x_n)) & \text{for} \quad x \in [x_n, y_n] \quad \text{and} \quad n > n_0 + 1. \end{cases}$$

It is easy to see that $\sup_{z \in Z} d(f_{\varepsilon,Z}(x), p(x)) < \varepsilon$, hence

$$p \in \overline{\{f_{\varepsilon,Z} \mid \varepsilon > 0, Z \subset X \text{ is compact}\}}.$$

The mappings $f_{\varepsilon,Z}$ are perfect and monotone, and p is perfect, but not monotone, as $p^{-1}(p(x_0)) = [x_0, x_1] \cup [y_0, y_1]$ is not connected.

EXAMPLE 2. The assumption "*Y satisfies the first axiom of countability*" in Theorem 2 is essential.

Let X and R be as in Example 1. Take $X' = X \setminus \{x_0, y_0\}$, $R' = R \cap (X' \times X')$ and $p'\colon X' \to X'/R' = Y'$. Let $Z \subset X'$ be a compact set. Take the first n_0 such that $Z \cap [x_n, y_n] = \emptyset$ for $n > n_0$. Let

$$f_Z(x) = \begin{cases} p'(x) & \text{for} \quad x \in \bigcup_{n=1}^{n_0} [x_n, y_n], \\ p'(x_1) & \text{for} \quad x \in [x_{n_0+1}, y_{n_0+1}] \cup (x_0, x_1] \cup (y_0, y_1], \\ p'(x + (x_{n-1} - x_n)) & \text{for} \quad x \in [x_n, y_n] \text{ and } n > n_0 + 1. \end{cases}$$

The mapping $f_Z\colon X' \xrightarrow[\text{onto}]{} Y'$ is closed and monotone; furthermore

$f_Z|Z = p'|Z$; thus $p' \in \overline{\{f_Z \mid Z \subset X' \text{ is compact}\}}$, but p' is closed and not monotone.

EXAMPLE 3. The assumption "*Y is locally compact*" in Theorem 4 cannot be replaced by rim-compactness.

Let $z_n = \left(\cos\frac{\pi}{4n}, \sin\frac{\pi}{4n}\right)$, $X_n = \{\lambda z_n \mid 0 \leqslant \lambda < 1/n\}$, $n = 1, 2, \ldots$, and $X = \bigcup_{n=1}^{\infty} X_n$. Define $f\colon X \xrightarrow[\text{onto}]{} X$ by

$$f(\lambda z_n) = \begin{cases} (0, 0) & \text{for} \quad n = 1 \text{ and } 0 \leqslant \lambda < 1, \\ \dfrac{n}{(n-1)} \lambda z_{n-1} & \text{for} \quad n > 1 \text{ and } 0 \leqslant \lambda < \dfrac{1}{n}, \end{cases}$$

and for $m = 2, 3, \ldots$ $f_m\colon X \to X$ by

$$f_m(\lambda z_n) = \begin{cases} \dfrac{\lambda}{m} z_m & \text{for} \quad n = 1 \text{ and } 0 \leqslant \lambda < 1, \\ \dfrac{n}{n-1} \lambda z_{n-1} & \text{for} \quad 1 < n \leqslant m \text{ and } 0 \leqslant \lambda < \dfrac{1}{n}, \\ \lambda z_n & \text{for} \quad n > m \text{ and } 0 \leqslant \lambda < \dfrac{1}{n}. \end{cases}$$

For $z \in X$ we have $|f(z) - f_m(z)| \leqslant 1/m$ and the homeomorphisms f_m converge uniformly to the closed, monotone, but not perfect mapping f.

It has been shown in [5] that the assumption of local connectedness of Y cannot be omitted in all the above theorems. In this context we will show a little more:

EXAMPLE 4. The assumption "*Y is locally connected*" in Theorem 4 cannot be omitted, even if we restrict ourselves to the set of monotone and closed mappings.

Let $z_n = \left(\cos\frac{\pi}{4n}, \sin\frac{\pi}{4n}\right)$ for $n = 1, 2, \ldots$, $z_0 = (1, 0)$ and $X_n = \{\lambda z_n \mid 0 \leqslant \lambda < 1\}$ for $n = 0, 1, \ldots$ Put $X = \bigcup_{n=0}^{\infty} X_n$ and define the mapping $f\colon X \xrightarrow[\text{onto}]{} X$ by

$$f(\lambda z_n) = \begin{cases} (0, 0) & \text{for} \quad n = 1 \text{ and } 0 \leqslant \lambda < 1, \\ \lambda z_{n-1} & \text{for} \quad n > 1 \text{ and } 0 \leqslant \lambda < 1, \\ \lambda z_0 & \text{for} \quad n = 0 \text{ and } 0 \leqslant \lambda < 1. \end{cases}$$

Let $\varphi_m\colon [0,1]\to[0,1]$ for $m=2,3,\ldots$ be defined by

$$\varphi_m(\lambda)=\begin{cases}\dfrac{\lambda}{m-1} & \text{for} \quad 0\leqslant\lambda<1-\dfrac{1}{m},\\ (m-1)\left(\lambda-\dfrac{m-1}{m}\right)+\dfrac{1}{m} & \text{for} \quad 1-\dfrac{1}{m}\leqslant\lambda\leqslant 1,\end{cases}$$

and define $f_m\colon X\to X$ by

$$f_m(\lambda z_n)=\begin{cases}\varphi_m(\lambda)z_m & \text{for} \quad n=1 \text{ and } 0\leqslant\lambda<1,\\ \lambda z_{n-1} & \text{for} \quad 1<n\leqslant m \text{ and } 0\leqslant\lambda<1,\\ \lambda z_n & \text{for} \quad n>m,\ n=0 \text{ and } 0\leqslant\lambda<1.\end{cases}$$

For $z\in X_0\cup\{\lambda z_1|\ 0\leqslant\lambda\leqslant 1-1/m\}\cup X_2\cup\ldots$ we have $|f_m(z)-f(z)|\leqslant 1/m$ and homeomorphisms f_m converge in the compact-open topology to the closed, monotone, but not perfect mapping f.

LEMMA 4 (cf. [5]). *If X, Y, Z are arbitrary spaces such that $Y\subset Z$ and F is a compact subset of X, then the set $M_1=\{(f,z)\in Y^X\times Z|\ z\in f(F)\}=\{(f,y)\in Y^X\times Y|\ y\in f(F)\}$ is closed in $Y^X\times Z$; furthermore if Y is metrizable, then M_1 is a G_δ-set in $Y^X\times Y$.*

Proof. Note that

(9) $$M_2=\{(f,z)\in Y^F\times Z|\ z\in f(F)\} \text{ is closed in } Y^F\times Z.$$

Indeed, if $(f,z)\notin M_2$ we have $z\notin f(F)$ and there exists an open set $U\subset Z$ such that $f(F)\subset U$ and $z\notin\overline{U}$; now the set $\mathrm{M}(F,U)\times(Z\setminus\overline{U})$ is a neighbourhood of (f,z) in $Y^F\times Z$ disjoint with M_2.

Consider the restriction $\varphi\colon Y^X\times Z\to Y^F\times Z$, where $\varphi(f,z)=(f|F,z)$, for $f\in Y^X$ and $z\in Z$. We have

(10) $$\varphi^{-1}(M_2)=\{(f,y)\in Y^X\times Y|\ y\in(f|F)(F)\}=\{(f,y)\in Y^X\times Y|\ y\in f(F)\}=M_1.$$

The mapping φ is continuous; hence by (9) and (10) $M_1=\varphi^{-1}(M_2)$ is closed in $Y^X\times Z$. If Y is metrizable, the space Y^F is also metrizable, and M_2 is a G_δ-subset of $Y^F\times Y$ as a closed subset of a metrizable space. This implies that $\varphi^{-1}(M_2)=M_1$ is a G_δ-set in $Y^X\times Y$, which completes the proof.

THEOREM 5. *Let X be a rim-compact and σ-compact (3), metrizable space and Y a metrizable space. Then the set $\Gamma=\{f\in Y^X|\ f^{-1}(y)$ is compact and connected for every $y\in Y\}$ is a G_δ-set in the set $\Omega=\{f\in Y^X|\ f^{-1}(y)$ is compact for every $y\in Y\}$.*

(3) We say that X is a *σ-compact* space if $X=\bigcup_{i=1}^{\infty}X_i$, where X_i are compact subspaces of X. Obviously, every σ-compact metric space is separable.

Proof. Let αY be an arbitrary compactification of Y and let $P=\Omega\times Y$, $Q=\Omega\times\alpha Y$. For an open set $G\subset X$ such that $\mathrm{Fr}\,G$ is compact put $[G]=\{(f,u)\in Q|\ f^{-1}(u)\cap G\neq\emptyset,\ f^{-1}(u)\cap(X\setminus\overline{G})\neq\emptyset$ and $f^{-1}(u)\cap\mathrm{Fr}\,G=\emptyset\}$.

We shall show first that

(11) $$[G] \text{ is an } F_\sigma\text{-set in } Q.$$

Let us note that $[G]=A\cap B_1\cap B_2$, where

$$A=\{(f,u)\in Q|\ f^{-1}(u)\cap\mathrm{Fr}\,G=\emptyset\},$$
$$B_1=\{(f,u)\in Q|\ f^{-1}(u)\cap G\neq\emptyset\} \quad\text{and}$$
$$B_2=\{(f,u)\in Q|\ f^{-1}(u)\cap(X\setminus\overline{G})\neq\emptyset\}.$$

We have $A\cap P=\{(f,y)\in\Omega\times Y|\ f^{-1}(y)\cap\mathrm{Fr}\,G=\emptyset\}$; hence by Lemma 4 the set $A\cap P$ is an F_σ-set in P. Thus $A\cap P=\bigcup_{n=1}^{\infty}F_n$, where $F_n=\overline{F}_n^P=\overline{F}_n^Q\cap P$ and we obtain

$$A=(A\cap P)\cup\big(A\cap(Q\setminus P)\big)=\Big[\Big(\bigcup_{n=1}^{\infty}\overline{F}_n^Q\Big)\cap P\Big]\cup[A\cap(Q\setminus P)].$$

We shall prove now that both sets B_i are F_σ-sets in Q and that $B_i\subset P$. By the symmetry of assumptions we may limit ourselves to B_1. Since X is metrizable, G is an F_σ-set in X and $G=\bigcup_{n=1}^{\infty}A_n$, where $\overline{A}_n^X=A_n$. Since X is σ-compact, $X=\bigcup_{n=1}^{\infty}X_n$, where X_n are compact. Hence $G=\bigcup_{n,m=1}^{\infty}A_n\cap X_m$ and $B_1=\bigcup_{n,m=1}^{\infty}\{(f,u)\in Q|\ f^{-1}(u)\cap(A_n\cap X_m)\neq\emptyset\}$. Since the sets $A_n\cap X_m$ are compact, it follows from Lemma 4 that B_1 is an F_σ-set in Q. If $u\in\alpha Y\setminus Y$ and $f\in\Omega$, then $f^{-1}(u)\cap G\subset f^{-1}(u)=\emptyset$; hence $B_1\subset P$. Finally we have

$$A\cap B_1\cap B_2=\Big\{\Big[\Big(\bigcup_{n=1}^{\infty}\overline{F}_n^Q\Big)\cap P\Big]\cup[A\cap(Q\setminus P)]\Big\}\cap B_1\cap B_2=\bigcup_{n=1}^{\infty}\overline{F}_n^Q\cap B_1\cap B_2.$$

The set on the right side is an F_σ-set in Q, which proves (11).

Since X is a rim-compact, separable and metrizable space, there exists (see [2], Theorem 1.1.7) a countable base whose elements have compact boundaries. Let $G_1, G_2, \ldots$ be all the finite unions of elements of such a base. Let us denote by p the projection $p\colon\Omega\times\alpha Y\to\Omega$ and let $\Psi=p\big(\bigcup_{n=1}^{\infty}[G_n]\big)$. Since p is closed (see [2], Theorem 3.2.8) Ψ is an F_σ-set in Ω. Now it suffices to show that $\Psi=\Omega\setminus\Gamma$. If $f\in\Omega\setminus\Gamma$, then there exists a point $y\in Y$ such that $f^{-1}(y)=A_1\cup A_2$, $\overline{A}_i=A_i$, $A_i\neq\emptyset$, $i=1,2$ and

$A_1 \cap A_2 = \emptyset$. We can find a k satisfying $G_k \supset A_1$, $\bar{G}_k \cap A_2 = \emptyset$. This implies that $(f, y) \in [G_k]$ and that $f \in \Psi$. Conversely, if $f \in \Psi$ there exist a k and a point $y \in Y$ such that $(f, y) \in [G_k]$. By the definition of $[G_k]$ it follows that $f^{-1}(y)$ is not connected, hence $f \in \Omega \backslash \Gamma$.

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Errata to the paper "On shape"

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It has been pointed out to me by Julian Eisner that my proof of Lemma 5.5 is incorrect. A correct proof under the slightly stronger hypothesis that P is metrizable can be found on p. 240 of the paper of J. Dugundji, *Absolute Neighborhood Retracts and Local Connectedness in Arbitrary Metric Spaces*, Compositio Mathematicae 13 (1958), pp. 229–246.