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Epireflections in the category of T_0-spaces*

by

L. D. Nel and **R. G. Wilson** (Ottawa, Ont.)

A T_0-space with the property that every non-empty irreducible closed set is a point closure will be called a pc-*space*. It is shown that the pc-spaces form an epireflective subcategory of the category $\mathcal{C}_0$ of all T_0-spaces, generated in $\mathcal{C}_0$ by any pc-space which contains a copy of the Alexandroff dyad.

A larger simply generated, epireflective subcategory of $\mathcal{C}_0$—the fc-spaces—is introduced and it is shown that an fc-space is an invariant of its lattice of real-valued lower semi-continuous functions. As a preliminary it is shown that equalizers in $\mathcal{C}_0$ correspond to "front-closed" subspaces.

1. Preliminaries. The set of continuous maps from X to Y will be denoted by (X, Y). The closure of A in X will be written $\mathrm{cl}_X A$ (or $\mathrm{cl}\, A$ when no confusion is possible) and for $x \in X$, $\mathrm{cl}\, x$ means $\mathrm{cl}\{x\}$. R will always denote the T_0-space obtained by endowing the real line with its lower topology (i.e. the topology having as non-trivial open sets those of the form $\{x \in R\colon x > a\}$ $a \in R$).

Let X be any T_0-space. One can define a second topology on X—the *front topology* — by specifying the front-closure operator fcl, as follows: $x \in \mathrm{fcl}\, A$ means that for any neighbourhood N of x, $N \cap \mathrm{cl}\, x \cap A \neq \emptyset$. The name is motivated by the fact that for $A \subset R$, $\mathrm{fcl}\, A$ is obtained by adjoining to A those points in $\mathrm{cl}\, A$ which lie "in front" of some points of A. It is easy to verify that fcl is a Kuratowski closure operator. This topology is the same as the b-topology of [6].

We note in passing that the front topology on X is discrete iff X is a T_D-space (see [7]) and that if X is a non-discrete T_0-space then the front topology is strictly larger than the original topology.

2. Equalizers and extremal subobjects in the category of T_0-spaces. The category of all T_0-spaces with continuous maps will be denoted by $\mathcal{C}_0$.

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If X and Y are objects of $\mathcal{T}_0$ and $F \subset (X, Y)$, the equalizer of F can be identified with the subset $K = \{x \in X\colon f(x) = g(x) \text{ for all } f, g \in F\}$ (see [3], 5.5). A characterization of equalizers will now be given which is useful for certain applications of category theory to T_0-spaces. We note that equalizers are the same as extremal subobjects in T_0 (7.1.3 and 7.2.14 in [3]).

2.1. THEOREM. *Let X, Y be T_0-spaces and $A \subset X$. A is the equalizer of some family $F \subset (X, Y)$ iff A is front-closed in X.*

Proof. Suppose A is the equalizer of $F \subset (X, Y)$. Let $x \in X - A$. Then $f(x) \neq g(x)$ for some $f, g \in F$. Then there exists $h \in (Y, R)$ such that $h(f(x)) = 0$ and $h(g(x)) = 1$ (after possible relabelling of f and g). Clearly $N = \{z \in X\colon h(g(z)) > 0\}$ is a neighbourhood of x and $\mathrm{cl}\,x \subset \{z \in X\colon h(f(z)) \leqslant 0\}$, and $f(z) \neq g(z)$ whenever $z \in N \cap \mathrm{cl}\,x$. Hence $N \cap \mathrm{cl}\,x \cap A = \emptyset$ and so A is front closed.

Conversely suppose that A is front-closed. Let $x \in X - A$. Choose an open neighbourhood N of x so that $N \cap \mathrm{cl}\,x \cap A = \emptyset$. Put $M = N \cup (X - \mathrm{cl}\,x)$ and define $u_x, v_x \in (X, R)$ as follows:

$$u_x(z) = \begin{cases} 1 & \text{if} \quad z \in M, \\ 0 & \text{if} \quad z \in X - M; \end{cases}$$

$$v_x(z) = \begin{cases} 1 & \text{if} \quad z \in X - \mathrm{cl}\,x, \\ 0 & \text{if} \quad z \in \mathrm{cl}\,x. \end{cases}$$

The mappings u_x and v_x agree precisely on $X - (M \cap \mathrm{cl}\,x)$ which contains A. Let $Y = R^{X-A}$ and define $f, g \in (X, Y)$ by

$$p_x \circ f = u_x, \quad p_x \circ g = v_x \quad (x \in X - A)$$

where p_x is the projection map onto the xth coordinate. A simple argument shows that f and g agree precisely on A. Thus A is the equalizer of $\{f, g\}$ as required.

Similar theorems are known for the category $\mathcal{T}$ of all topological spaces and the category $\mathcal{T}_2$ of all Hausdorff spaces. Thus: A full subcategory of $\mathcal{T}$ (respectively $\mathcal{T}_0$, $\mathcal{T}_2$) has equalizers iff it is closed under formation of subspaces (respectively front-closed subspaces, closed subspaces).

3. pc-spaces. Recall that a closed set is termed irreducible if it cannot be expressed as the union of two proper closed subsets. T_0-spaces in which the point-closures are the only irreducible closed sets will be called pc-*spaces*. Such spaces were studied in [1] where results were obtained which imply that the full subcategory $\mathcal{I}$ of all pc-spaces is epireflective in $\mathcal{T}_0$ (see the review of [1] by H. Herrlich, M. R. 37, No. 5851). This result may also be deduced from 2.1 and the fact that a product of pc-spaces is a pc-space.

3.1. LEMMA. *Let $A \subset X \subset Y$. A is irreducible in X iff $\mathrm{cl}_Y A$ is irreducible in Y.*

The simple proof is left to the reader.

3.2. THEOREM. (a) *A front-closed subspace of a* pc-*space is a* pc-*space.*

(b) *A* pc-*subspace of a T_0-space Y is front-closed in Y.*

Proof. (a) A direct proof of this is simple and is left to the reader. The result is also an immediate consequence of 2.1 in view of the fact that an epireflective subcategory of $\mathcal{T}_0$ has equalizers (see 9.1.2 in [3]).

(b) Let $z \in \mathrm{fcl}\,X$, where $X \subset Y$ is a pc-space, and put $A = X \cap \mathrm{cl}_Y z$. Clearly, $z = \mathrm{cl}_Y A$ and $\mathrm{cl}_Y z \subset \mathrm{cl}_Y A \subset \mathrm{cl}_Y z$. Thus $\mathrm{cl}_Y A$ is irreducible, hence so is A. Since X is a pc-space, $A = \mathrm{cl}_X x$ for some $x \in X$. But then $\mathrm{cl}_Y x = \mathrm{cl}_Y z$ which implies that $z = x \in X$ and so X is front-closed.

3.3. LEMMA. *If $\prod_{i \in I} X_i$ is a product space and $J \subset I$, then $\prod_{j \in J} X_j$ is homeomorphic to a front-closed subspace of $\prod_{i \in I} X_i$.*

The simple proof is left to the reader.

The Alexandroff dyad D is the subset $\{0, 1\}$ of R with the relative topology. It is clearly a pc-space.

3.4. THEOREM. *A T_0-space is a* pc-*space iff it is homeomorphic to a front-closed subspace of some product D^I of Alexandroff dyads. The epireflective subcategory $\mathcal{I}$ is generated in $\mathcal{T}_0$ by any* pc-*space which contains a copy of D.*

Proof. Let $\mathcal{A}$ denote the class of all spaces which are homeomorphic to a front-closed subspace of some Alexandroff cube D^I. It is well known that every T_0-space is a subspace of some cube D^I and by 3.2 any pc-space will be a front-closed subspace of such a cube. Hence $\mathcal{A}$ contains all pc-spaces. The class $\mathcal{A}$ is clearly productive and by (2.1) it has equalizers, hence it forms an epireflective subcategory of $\mathcal{T}_0$ (see 9.3.3 and 10.2.1 in [3]). On the other hand since D is a pc-space and $\mathcal{I}$ is productive and closed under the formation of front-closed subspaces, $\mathcal{I}$ contains $\mathcal{A}$. Hence $\mathcal{I}$ is the smallest epireflective subcategory of $\mathcal{T}_0$ containing D. The proof remains valid if D is replaced by any pc-space J containing a copy of D.

4. fc-spaces. We now introduce a class of T_0-spaces which contains the pc-spaces as a proper subclass. The fc-spaces resemble realcompact spaces in the same way that pc-spaces resemble compact Hausdorff spaces.

DEFINITION. A T_0-space X is called an fc-*space* if the point-closures are the only non-empty irreducible closed sets with the FCI-property; a subset $A \subset X$ has the FCI-property if an open filter $\mathcal{G}$ on X satisfies $G \cap A \neq \emptyset$ for all $G \in \mathcal{G}$ only if $\mathcal{G}$ has the countable intersection property.

It is immediate from the definition that every pc-space is an fc-space. Every T_2-space is a pc-space (see [1]) and therefore an fc-space. However, a T_1-space need not be an fc-space, for example any uncountable set with the cofinite topology. In this section we give a characterization of fc-spaces similar to 3.4 but first we need a lemma.

4.1. LEMMA. *A subset $A \subset X$ has the FCI-property iff every function $f \in (X, R)$ is bounded above on A.*

Proof. Suppose $f \in (X, R)$ is unbounded above on A. The sets $\{x: f(x) > n\}$ form a base for an open filter on X each member of which meets A, but which clearly does not have the countable intersection property.

Conversely suppose that A does not have the FCI-property. Then there exists an open filter $\mathcal{F}$ on X such that if $F \in \mathcal{F}$, $F \cap A \neq \emptyset$ but there exists a family $\{F_n\}_{n \in N} \subset \mathcal{F}$ such that $\bigcap_n F_n = \emptyset$. For each $n \in N$ define f_n as follows:

$$f_n(x) = \begin{cases} n & \text{if} \quad x \in F_1 \cap \ldots \cap F_n, \\ 0 & \text{otherwise.} \end{cases}$$

Since $A \cap \{F_1 \cap \ldots \cap F_n\} \neq \emptyset$ for each $n \in N$ it is clear that $f = \sup_n f_n$ is continuous and $\sup\{f(a): a \in A\} = \infty$.

4.2. THEOREM. *A T_0-space X is an fc-space iff X is homeomorphic to a front closed subspace of some cube R^I.*

Proof. The functions in (X, R) separate the points of X. It follows that the map $\sigma: X \to R^{(X,R)}$ defined by $p_f \circ \sigma = f$ $(f \in (X, R))$ is a homeomorphic embedding (see for instance 2.1 in [4]). We show that $\sigma(X)$ is front-closed in $Y = R^{(X,R)}$. Let $z \in \mathrm{fcl}\,\sigma(X)$ and put $A = \sigma(X) \cap \mathrm{cl}_Y z$. The closed set A is irreducible by 3.1. Let $g: \sigma(X) \to R$ be a continuous function, then $f = g \circ \sigma \in (X, R)$ and so the projection p_f is an extension of g to all of Y. But $p_f(z) \geqslant p_f(a) = g(a)$ holds for all $a \in A \subset \mathrm{cl}_Y z$. Hence g is bounded above on A and so by the previous lemma, A has the FCI-property (in X). If X is an fc-space, A is a point-closure $\mathrm{cl}_{\sigma(X)} x = \sigma(X) \cap \mathrm{cl}_Y x = \sigma(X) \cap \mathrm{cl}_Y z$. Hence $z = x \in \sigma(X)$ and so $\sigma(X)$ is front-closed in Y.

Conversely suppose that X is a front-closed subspace of $Y = R^I$ for some set I and let A be an irreducible closed set in X with the FCI-property. Fix $i \in I$. If $\{a_i: a \in A\}$ is unbounded above in R, then the sets $G_a = \{x \in R^I: x_i > a_i\}$ $(a \in A)$ form a base for an open filter $\mathcal{G}$ on X such that $G \cap A \neq \emptyset$ holds for all $G \in \mathcal{G}$, while the sequence $G_n = \{x: x_i > n\}$ in $\mathcal{G}$ satisfies $\bigcap_n G_n = \emptyset$, which is absurd. We conclude that $y_i = \sup\{a_i: a \in A\} \in R$ for each i and defines $y \in R^I$. Clearly $A \subset \mathrm{cl}_Y y$.

It is easy to see that any neighbourhood N of y in R^I meets A. Thus $y \in A \subset X$ and $A = \mathrm{cl}_X y$. It follows that X is an fc-space.

4.3. THEOREM. *The full subcategory $\mathcal{F}$ of $\mathcal{C}_0$ consisting of all fc-spaces is epireflective in $\mathcal{C}_0$. $\mathcal{F}$ is generated in $\mathcal{C}_0$ by the space R or any fc-space which contains R as a front-closed subspace.*

Proof. It follows from 4.2 and 3.3 that the subcategory $\mathcal{F}$ is productive and closed under the formation of front-closed subspaces. That is to say, $\mathcal{F}$ has products and equalizers 2.1 and so as before, it follows that $\mathcal{F}$ is epireflective in $\mathcal{C}_0$. The second statement may be proved using an argument similar to that given in the proof of 3.4.

The epireflectiveness of $\mathcal{F}$ means, in particular, that if X is any T_0-space, then there exists an essentially unique fc-space, φX and a map $\varphi: X \to \varphi X$ such that for any map $f \in (X, Y)$ with Y an fc-space, there is precisely one $f^\varphi \in (\varphi X, Y)$ for which $f^\varphi \circ \varphi = f$.

We now show that φX can be identified with a subspace of the pc-space πX (π being the epireflection for the subcategory $\mathcal{T}$) in a way analogous to that in which the realcompactification vX can be identified with a subspace of the Stone–Čech compactification βX (see [2]).

For any T_0-space X, the set (X, R) forms a conditionally complete lattice $L_R(X)$ (the lattice of real-valued lower semi-continuous functions). It is shown in [5] that this lattice has a certain structure space Ω (a topologized set of equivalence classes of closed prime ideals) which is homeomorphic to the pc-space πX. The points of Ω are the sets $I(A) = \{I(r, A): r \in R\}$ where $I(r, A) = \{f \in L_R(X): f(x) \leqslant r \text{ for all } x \in A\}$ and where A varies through all the irreducible closed subsets of X. It is clear from Lemma 4.1 that the set Φ of all $I(A) \in \Omega$ which correspond to irreducible A with the FCI-property are characterized by the fact that every $f \in L_R(X)$ belongs to some ideal $I(r, A)$ in the equivalence class $I(A)$: in other words, the subspace Φ of Ω is also an invariant of the lattice $L_R(X)$. We now show that Φ is an fc-space. Note first that each point $a \in \pi X$ corresponds to an irreducible set $A \subset X$. In fact, in view of 3.1 and the construction of πX (qv. [1] or [5]) it is possible to write $\mathrm{cl}_{\pi X} A = \mathrm{cl}_{\pi X} a$ and therefore $A = X \cap \mathrm{cl}_{\pi X} a$ (where we have identified X with a dense subspace of πX). The subspace Y of πX consisting of all $a \in \pi X$ which correspond to an irreducible closed set A with the FCI-property is clearly the smallest fc-space such that $X \subset Y \subset \pi X$ holds and the homeomorphism $a \to I(A)$ $(A = X \cap \mathrm{cl}\, a)$ of πX onto Ω (again see [5]) carries Y onto Φ. It is not difficult to verify that each $f \in (X, R)$ has a unique extension to a function in (Y, R) and hence that Φ is homeomorphic to the space φX.

As a result of the above discussion we have:

4.5. THEOREM. *For any T_0-space X, the lattices $L_R(X)$ and $L_R(\varphi X)$*

are isomorphic. Two fc-*spaces* X *and* Y *are homeomorphic iff the lattices* $L_R(X)$ *and* $L_R(Y)$ *are isomorphic.*

Remark. The pc- and fc-spaces play roles in the theory of T_0-spaces analogous to the roles of compact and realcompact spaces in the theory of Tychonoff spaces. It is interesting to note that it is possible to define a concept which is analogous to pseudocompactness. A T_0-space X is said to be a pseudo-pc-space if each element of (X, R) is bounded above on every irreducible closed subset of X.

It is clear from Lemma 4.1 that a T_0-space X is a pseudo-pc-space iff every irreducible closed subset of X has the FCI-property and also that a T_0-space is a pc-space iff it is both an fc- and a pseudo-pc-space.

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On the position of the set of monotone mappings in function spaces

by

R. Pol (Warszawa)

K. Kuratowski and R. C. Lacher have shown in [5] that if X and Y are compact topological spaces and Y is locally connected, then the set of all monotone mappings of X onto Y is closed in Y^X (endowed with the compact-open topology). In an earlier paper [4] K. Kuratowski showed that if the space X is compact and metric and Y an arbitrary metric space, then the monotone mappings of X into Y form a G_δ-set in Y^X.

In this connection the question arises whether the above theorems can be generalized by dropping the assumption of the compactness of X and restricting the considerations to perfect mappings. More generally, in the space Y^X can consider subset $\Phi \subset \Psi \subset Y^X$ (we shall be interested in closed or perfect monotone mappings), and, under certain assumptions on X and Y, one can prove that Φ is closed (or that is a G_δ-set) in Ψ. Below we shall prove a few facts of this type and give examples illustrating role of the assumptions which have been made.

We adopt the terminology and notation of [2] and [3]. All the spaces considered below are Hausdorff spaces. The space Y^X of mappings of X into Y will be considered with the compact-open topology. The symbol $\mathrm{M}(A, B)$, where $A \subset X$, $B \subset Y$, will denote the set $\{f \in Y^X |\ f(A) \subset B\}$.

Lemma 1. *Let X be an arbitrary space, Y a locally connected space and Φ the set $\{f\colon X \to Y |\ f^{-1}(S) \subset X$ is connected for all open and connected $S \subset Y\}$.*

If the mapping $f\colon X \xrightarrow[\text{onto}]{} Y$ satisfies the conditions

(i) *the boundary $\mathrm{Fr} f^{-1}(y)$ is compact for every $y \in Y$,*

(ii) *if $y \in Y$ and U is a neighbourhood of the set $f^{-1}(y)$, then there exists an open set $V \subset X$ such that $f^{-1}(y) \subset V \subset U$ and the boundary $\mathrm{Fr} V$ is compact,*

(iii) *$f \in \overline{\Phi}$,*

then f is a monotone, closed mapping.