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Cardinal properties of lattice ordered groups

by

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Pierce [6], [7] defined a cardinal property of complete Boolean algebras as a rule that assigns to any complete Boolean algebra B a cardinal fB such that $fB_1 = fB_2$ whenever B_1 and B_2 are isomorphic. He proved that if f is increasing, then each complete Boolean algebra B can be decomposed into a complete direct product of Boolean algebras B_i that are homogeneous with regard to f. The aim of this note is to prove some analogical results for lattice ordered groups. In § 1 there are studied "increasing" cardinal properties. In § 2 we prove that a complete and laterally complete l-group G is a complete direct product of l-groups G_i such that either any two non-trivial intervals of G_i are finite or they have the same length; in § 3 an analogical theorem concerning the powers of intervals is proved.

§ 0. Preliminaries. We shall use the standard notations for lattices and lattice ordered groups (cf. [1], [2]). Let G be an l-group; the group operation is denoted by $+$ and the lattice operations by $\wedge$, $\vee$. If $x, y \in G$ and $x \wedge y = 0$, then x and y are said to be *disjoint* (this fact is denoted by $x\,\delta\, y$). A subset $X \subset G$ is *disjoint* if $x > 0$ for any $x \in X$ and any two distinct elements of X are disjoint. $Y \delta x$ ($Y \delta X$) means that the element x (each element of X) is *disjoint* with each element y of the set Y. Let $G^+ = \{x \in G\colon x \geqslant 0\}$ and for any $X \subset G^+$ write $X^\delta = \{y \in G^+\colon X \delta y\}$. G is *laterally complete* if for any disjoint subset $\{x_\alpha\} \subset G$ there exists $\bigvee x_\alpha \in G$. Let L be a lattice, $a, b \in L$, $a \leqslant b$. The *interval* $[a, b]$ is the set $\{x \in L\colon a \leqslant x \leqslant b\}$; $[a, b]$ is a non-trivial interval, if $a \neq b$. $[a, b]$ is a *prime interval* when $\operatorname{card}[a, b] = 2$. L is a *bounded lattice* if it is an interval. A subset $X \subset L$ is *convex* if $[a, b] \subset X$ whenever a and b belong to X. A set $Y \subset L$ is a *closed sublattice* of L if $\{y_\alpha\} \subset L$, $\bigvee y_\alpha = y$ implies $y \in Y$, and dually.

Let $I \neq \emptyset$ be a set and for any $i \in I$ let H_i be an l-group. The *complete direct product* $H = \Pi^* H_i$ $(i \in I)$ is the system of all vectors $h = (\ldots, h_i, \ldots)_{i \in I}$, $h_i \in H_i$, with operations $+$, $\wedge$, $\vee$ that are performed componentwise; then H is an l-group. Instead of h_i we write also $h(i)$.

The l-subgroup K of H consisting of all elements $k \in H$ such that the set $I(k) = \{i \in I\colon k(i) \neq 0\}$ is finite is the (discrete) direct product of l-groups H_i. An l-subgroup G of H is called a *complete subdirect product* of l-groups H_i if for any $i_0 \in I$ and any $h^{i_0} \in H_{i_0}$ there is an element $g \in G$ satisfying $g(i_0) = h^{i_0}$, $g(i) = 0$ for any $i \in I$, $i \neq i_0$. Let A, B be l-ideals of an l-group G. If $A \cap B = \{0\}$, $A+B = G$, then G is *isomorphic* to the direct product of l-groups A and B; in such a case we write $G = A \times B$. The *additive linearly ordered group* of all integers (all reals) is denoted by $N(R_0)$.

Let $\mathcal{B}$ be the class of all bounded lattices containing more than one element and let $\mathcal{K}$ be the class of all cardinals. Let f be a mapping of the class $\mathcal{B}$ into $\mathcal{K}$ such that $fL_1 = fL_2$ whenever L_1 is isomorphic to L_2 and $L_1, L_2 \in \mathcal{B}$. The mapping f is said to be a *cardinal property* defined on $\mathcal{B}$. A lattice L is called *f-homogeneous* if $fL_1 = fL_2$ for any two convex sublattices L_1, L_2 of L such that $L_1, L_2 \in \mathcal{B}$. If card $L = 1$, then no sublattice of L belongs to $\mathcal{B}$ and hence L is f-homogeneous for any cardinal property f. A cardinal property f is *increasing* if $fL_1 \leqslant fL_2$ for any pair of lattices $L_1, L_2 \in \mathcal{B}$ such that L_1 is a convex sublattice of L_2.

§ 1. Increasing cardinal properties. Let $G \neq \{0\}$ and let f be an increasing cardinal property on the class $\mathcal{B}$. We shall consider the following conditions on f:

(c_1) If $t_i \in G$, $0 < t_i$ $(i = 1, 2)$, $f[0, t_1] = f[0, t_2]$ and if $[0, t_1]$ and $[0, t_2]$ are f-homogeneous, then $f[0, t_1+t_2] = f[0, t_1]$.

(c_2) If $t_i \in G$, $0 < t_1 \leqslant t_2 \leqslant \ldots$, $f[0, t_1] = f[0, t_i]$, $\bigvee t_i = t$ and if the intervals $[0, t_i]$ are f-homogeneous $(i = 1, 2, \ldots)$, then $f[0, t] = f[0, t_1]$.

Let $\mathcal{A}$ be the set of all cardinals α such that $f[a, b] = \alpha$ for some non-trivial interval $[a, b]$ of G and for any $\alpha \in \mathcal{A}$ write

$$X_\alpha = \{x \in G\colon x > 0, f[0, x] \leqslant \alpha\} \cup \{0\},$$

$$Y_\alpha = \{y \in G\colon y > 0, f[0, y] < \alpha\} \cup \{0\},$$

$$Z_\alpha = (Y_\alpha)^\delta, \quad A_\alpha = X_\alpha \cap Z_\alpha.$$

1.1. *Assume that* (c_1) *is valid. Let* $\alpha \in \mathcal{A}$. *Then*

(i) *the set A_α is an ideal of the lattice G^+ and a subsemigroup of G^+,*

(ii) *$f[a, b] = \alpha$ for any non-trivial interval $[a, b]$ of A_α,*

(iii) *$A_\alpha \,\delta\, A_\beta$ for any $\beta \in \mathcal{A}$, $\beta \neq \alpha$.*

Proof. Let $t \in A_\alpha$, $t > 0$. Then $t \in X_\alpha$, whence $f[0, t] \leqslant \alpha$. If $f[0, t] < \alpha$, the element t belongs to Y_α and since $t \in Z_\alpha$, we have $t \,\delta\, t$, a contradiction; this implies that $f[0, t] = \alpha$. Let $t_1 \in A_\alpha$, $0 < t_1 \leqslant t$. Since f is increasing, $f[0, t_1] \leqslant \alpha$, whence $t_1 \in X_\alpha$. From $Y_\alpha \,\delta\, t$ if follows that $Y_\alpha \,\delta\, t_1$; thus $t_1 \in Z_\alpha$ and $t_1 \in A_\alpha$. If $0 < t_1 < t$, then the interval $[t_1, t]$ is isomorphic to $[0, t-t_1]$ and $0 < t-t_1 < t$; therefore $f[t_1, t] = \alpha$ and (ii) holds. Let $t_1, t_2 \in A_\alpha$, $t_1 > 0$, $t_2 > 0$. Clearly, $t_1 \wedge t_2 \in A_\alpha$, too. According to (ii) and (c_1) $f[0, t_1+t_2] = \alpha$ and hence $t_1+t_2 \in X_\alpha$. Moreover, from $Y_\alpha \,\delta\, t_1$, $Y_\alpha \,\delta\, t_2$ we get $Y_\alpha \,\delta\, (t_1+t_2)$; thus $t_1+t_2 \in Z_\alpha$ and $t_1+t_2 \in A_\alpha$. Since $0 < t_1 \vee t_2 \leqslant t_1+t_2$, the element $t_1 \vee t_2$ belongs to A_α as well; so the assertion (i) is proved. Let $\beta \in \mathcal{A}$, $\beta \neq \alpha$. If $A_\alpha \,\delta\, A_\beta$ does not hold, then by (i) we have $t \in A_\alpha \cap A_\beta$, $t > 0$ and according to (ii) $f[0, t] = \alpha$, $f[0, t] = \beta$, a contradiction.

1.2. *If* (c_1) *is fulfilled, $t \in G$, $t > 0$, $f[0, t] = \alpha$, and the interval $[0, t]$ is f-homogeneous, then $t \in A_\alpha$.*

Proof. Clearly $t \in X_\alpha$. Let $y \in Y_\alpha$, $t \wedge y = u$. If $u > 0$, then $f[0, u] \leqslant f[0, y] < \alpha$ and, at the same time, $[0, u] \subset [0, t]$, whence $f[0, u] = \alpha$, a contradiction. Therefore $Y_\alpha \,\delta\, t$ and thus $t \in A_\alpha$.

From 1.1 and 1.2 we obtain the following:

1.3. *Assume that* (c_1) *holds and let $\alpha \in \mathcal{A}$. Let F_α be the family of all convex sublattices L_1 of the lattice G^+ such that $0 \in L_1$ and $f[t_1, t_2] = \alpha$ for any non-trivial interval of L_1. Then A_α is the greatest element of the family F_α (ordered by set-inclusion).*

1.4. *Let* (c_1) *be valid and for any $\alpha \in \mathcal{A}$ let $B_\alpha = \{t \in G\colon$ there exist $t_1, t_2 \in A_\alpha$ such that $-t_1 \leqslant t \leqslant t_2\}$. Then* (i) *$B_\alpha$ is an l-ideal of the l-group G;* (ii) *$f[a, b] = \alpha$ for any non-trivial interval $[a, b]$ of B_α;* (iii) *$B_\alpha \cap B_\beta = \{0\}$ for each $\beta \in \mathcal{A}$, $\beta \neq \alpha$.*

Proof. If $t \in B_\alpha$, then clearly $-t \in B_\alpha$ and from 1.1 (i) it follows that B_α is a subsemigroup of G; hence B_α is a subgroup of G. From this and from the convexity of A_α we infer that B_α is a convex subset of G and therefore by the definition of B_α the element $t \vee 0$ belongs to B_α for any $t \in B_\alpha$; thus B_α is a convex l-subgroup of G. For proving that B_α is normal it suffices to verify that $A_\alpha = B_\alpha^+$ is a normal subset of G. Let $t \in A_\alpha$, $t > 0$, $x \in G$ and write $t' = -x+t+x$. Since the intervals $[0, t]$ and $[0, t']$ are isomorphic, it follows from 1.1 (ii) and 1.2 that $t' \in A_\alpha$; thus (i) holds. Let $[a, b]$ be a non-trivial interval of B_α; then $[a, b]$ is isomorphic to $[0, b-a]$ and $[0, b-a] \subset A_\alpha$, whence $f[a, b] = \alpha$. If $x \in B_\alpha \cap B_\beta$, $\alpha \neq \beta$, $x \neq 0$, then $0 \neq |x| \in A_\alpha \cap A_\beta$, a contradiction.

1.5. *Let* (c_1) *be valid and let G_α be the family of all convex sublattices L of G such that $0 \in L$ and $f[t_1, t_2] = \alpha$ for any non-trivial interval of L. Then B_α is the greatest element of the family G_α.*

Proof. According to 1.4, B_α belongs to the family G_α. Assume that $L \in G_\alpha$, $t \in L$, $t \neq 0$. If $[0, 0 \vee t]$ is a non-trivial interval, then it is f-homogeneous and $f[0, 0 \vee t] = \alpha$, whence by 1.2 $0 \vee t \in A_\alpha$. If $[0 \wedge t, 0]$ is a non-trivial interval, then it is f-homogeneous and isomorphic to $[0, -(0 \wedge t)]$; thus $-(0 \wedge t) \in A_\alpha$. This implies that $t \in B_\alpha$ and hence $L \subset B_\alpha$.

1.6. Theorem. *Let f be increasing and assume that* (c_1) *is valid. For any $\alpha \in \mathcal{A}$ and $g \in G$ let $G_\alpha(g)$ be the family of all convex sublattices L of G*

such that $g \in L$ and $f[t_1, t_2] = a$ for each non-trivial interval of L. Let $G_a(g)$ be partially ordered by set-inclusion. Then (i) *any family $G_a(g)$ contains a greatest element (this will be denoted by $B_a(g)$);* (ii) *$B_a(0)$ is an l-ideal of G and $B_a(g) = B_a(0)+g$;* (iii) *$B_a(g) \cap B_\beta(g) = \{g\}$ for any $\beta \in \mathcal{A}$, $\beta \neq a$.*

Proof. Let $g \in G$. The mapping $\varphi(t) = t+g$ being an automorphism of the lattice G, it follows from 1.5 that $B_a+g = B_a(g)$ is the greatest element of the family $G_a(g)$; (ii) and (iii) are consequences of 1.4.

For any $a \in \mathcal{A}$ let $\bar{A}_a$ be the set of all elements $t \in G$ that can be written in the form $t = \bigvee t_i$, $\{t_i\} \subset A_a$.

1.7. *Let $a \in \mathcal{A}$ and assume that* (c₁) *holds. The set $\bar{A}_a$ is a closed ideal of the lattice G^+ and a subsemigroup of G. If $\beta \in \mathcal{A}$, $\beta \neq a$, then $\bar{A}_a \,\delta\, \bar{A}_\beta$. $\bar{A}_a$ is a normal subset of the group G.*

Proof. Let $t \in \bar{A}_a$, $t = \bigvee t_i$, $\{t_i\} \subset A_a$, $t^* \in G$, $0 \leqslant t^* < t$. Since any lattice ordered group is infinitely distributive ([1]), $t^* = t \wedge t^* = \bigvee (t_i \wedge t^*)$ and $t_i \wedge t^* \in A_a$ by 1.1; therefore $t^* \in \bar{A}_a$. Let $S = \{s_j\}_{j\in J} \subset \bar{A}_a$, $\sup S = s$. For any $s_j \in S$ we have $T_j \subset A_a$ such that $s_j = \sup T_j$. Thus

$$s = \sup_{j\in J}(\sup T_j) = \sup(\bigcup_{j\in J} T_j);$$

since $\bigcup T_j \subset A_a$, we have $s \in \bar{A}_a$. This proves that $\bar{A}_a$ is a closed ideal of the lattice G^+. Let $t = \bigvee_{i\in I} t_i$, $t' = \bigvee_{j\in J} t'_j$, $\{t_i\}, \{t'_j\} \subset A_a$. Then $t+t' = \bigvee_{i\in I}\bigvee_{j\in J}(t_i+t'_j)$, whence $t+t' \in \bar{A}_a$. Further, let $\beta \in \mathcal{A}$, $\beta \neq a$, $t = \bigvee t_i$, $\{t_i\} \subset A_a$, $t' = \bigvee t'_j$, $\{t'_j\} \subset A_\beta$. According to 1.1, $t_i \wedge t'_j = 0$, and thus, by using infinite distributivity, $t \wedge t' = 0$. From the normality of A_a it follows that $\bar{A}_a$ is also normal.

Let us put $\bar{B}_a = \{t \in G$: there exist elements $t_1, t_2 \in \bar{A}_a$ such that $-t_1 \leqslant t \leqslant t_2\}$.

From 1.7 we immediately obtain the following:

1.7.1. *Let $a \in \mathcal{A}$ and assume that* (c₁) *is fulfilled. The set $\bar{B}_a$ is an l-ideal of G. If $\beta \in \mathcal{A}$, $\beta \neq a$, then $\bar{B}_a \cap \bar{B}_\beta = \{0\}$.*

Let us now assume that G is a complete l-group (i.e., that the lattice G is relatively complete), $g \in G^+$ and for any $a \in \mathcal{A}$ write

$$g_a = \sup\{t \in A_a \colon t \leqslant g\}.$$

By the definition of $\bar{A}_a$, $g_a \in \bar{A}_a$.

1.8. *Let G be a complete l-group and suppose that* (c₁) *holds. Then $g = \bigvee g_a$ $(a \in \mathcal{A})$ for any $g \in G^+$. If $g = \bigvee h_a$ $(a \in \mathcal{A})$, $h_a \in \bar{A}_a$ for each $a \in \mathcal{A}$, then $h_a = g_a$.*

Proof. Put $\bigvee g_a = h$. Clearly $h \leqslant g$. Assume that $h < g$ and write $-h+g = k$; further, let

$$\beta = \min\{f[0, b] \colon 0 < b \leqslant k\}. \tag{1}$$

There exists $b_0 \in G$, $0 < b_0 \leqslant k$ such that $f[0, b_0] = \beta$. Then for any $b_1 > 0$, $b_1 \leqslant b_0$ we have $f[0, b_1] \leqslant f[0, b_0]$ and according to (1) $f[0, b_1] \geqslant \beta$, whence the interval $[0, b_0]$ is f-homogeneous. Thus, by 1.2, $b_0 \in A_\beta$. There is a subset $\{t_i\} \subset A_\beta$ such that $g_\beta = \bigvee t_i$. Therefore, we have

$$g_\beta + b_0 = (\bigvee t_i) + b_0 = \bigvee (t_i + b_0)$$

and $t_i + b_0 \in A_\beta$ by 1.1. Moreover, $t_i + b_0 \leqslant g_\beta + b_0 \leqslant h+k = g$, whence (by the definition of g_β) $\bigvee (t_i + b_0) \leqslant g_\beta$; thus $g_\beta + b_0 \leqslant g_\beta$, a contradiction. Therefore $\bigvee g_a = g$. If $\bigvee h_a = g$, $h_a \in \bar{A}_a$, then for any $a_0 \in \mathcal{A}$

$$g_{a_0} = g_{a_0} \wedge g = \bigvee (g_{a_0} \wedge h_a) = g_{a_0} \wedge h_{a_0}$$

by 1.4. Analogously, we obtain $h_{a_0} = g_{a_0} \wedge h_{a_0}$, whence $g_{a_0} = h_{a_0}$.

In 1.9-1.20 we assume that G is a complete l-group and that (c₁) is valid.

1.9. *For any $a \in \mathcal{A}$ and any $g \in G^+$, let $\varphi_a(g) = g_a$. Then φ_a is a homomorphism of the lattice ordered semigroup G^+ onto the lattice ordered semigroup $\bar{A}_a$. For $g \in \bar{A}_a$ we have $\varphi_a(g) = g$ and $\varphi_\beta(g) = 0$ whenever $\beta \in \mathcal{A}$, $\beta \neq a$.*

Proof. Let $g, h \in G^+$. Then $g_{a_1} \,\delta\, g_{a_2}$ for any $a_1, a_2 \in \mathcal{A}$, $a_1 \neq a_2$, and thus, by using infinite distributivity, $g \wedge h = \bigvee (g_a \wedge h_a)$; further, we have $g \vee h = \bigvee (g_a \vee h_a)$. Since by 1.7 $g_a \wedge h_a$ and $g_a \vee h_a$ belong to $\bar{A}_a$, it follows from 1.8 that $(g \wedge h)_a = g_a \wedge h_a$, $(g \vee h)_a = g_a \vee h_a$. Moreover,

$$g+h = \bigvee_{a\in\mathcal{A}} g_a + h = \bigvee_{a\in\mathcal{A}} (g_a + h) = \bigvee_{a\in\mathcal{A}}\bigvee_{\beta\in\mathcal{A}} (g_a + h_\beta).$$

If $a \neq \beta$, then $g_a \,\delta\, h_\beta$, whence (cf. [1]) $g_a + h_\beta = g_a \vee h_\beta \leqslant (g_a + h_a) \vee (g_\beta + h_\beta)$; thus $g+h = \bigvee_{a\in\mathcal{A}} (g_a + h_a)$; therefore, according to 1.7 and 1.8, $(g+h)_a = g_a + h_a$. Hence φ_a is an homomorphism. From the definition of g_a it follows immediately that for $g \in \bar{A}_a$ we have $g_a = g$; moreover, from $\bar{A}_a \,\delta\, \bar{A}_\beta$ we obtain $g_\beta = 0$ for any $\beta \in \mathcal{A}$, $\beta \neq a$.

1.10. *Let $g, h, g', h' \in G^+$, $g-h = g'-h'$, $a \in \mathcal{A}$. Then $g_a - h_a = g'_a - h'_a$.*

Proof. Since G is a complete l-group, G is commutative. Hence $g+h' = g'+h$ and thus, by 1.9, $g_a + h'_a = g'_a + h_a$.

For any $k \in G$ there exist elements $g, h \in G^+$ such that $k = g-h$; put $k_a = g_a - h_a$. According to 1.10 k_a is uniquely determined by k. From 1.9 and 1.10 we obtain the following:

1.11. *If $g \in \bar{B}_a$, $\beta \in \mathcal{A}$, $\beta \neq a$, then $g_a = g$, $g_\beta = 0$. The mapping $g \to g_a$ is a homomorphism of the group G onto the group $\bar{B}_a$.*

1.12. *The mapping $g \to g_a$ is a homomorphism of the lattice G onto the lattice $\bar{B}_a$.*

Proof. Let $g, h \in G$. There exists $k \in G$ such that $k \leqslant g$, $k \leqslant h$. Then $g-k$, $h-k \in G^+$ and thus, according to 1.9, and 1.11

$$(g \vee h)_a - k_a = [(g\vee h)-k]_a = [(g-k)\vee(h-k)]_a$$
$$= (g-k)_a \vee (h-k)_a = (g_a \vee h_a) - k_a;$$

therefore $(g\vee h)_a = g_a \vee h_a$. The proof for the operation $\wedge$ is analogous.

Write $H = \Pi^* \bar{B}_a$ $(a \in \mathcal{A})$ and consider the mapping φ: $G \to H$ defined by $\varphi(g) = (\ldots, g_a, \ldots)$. Let $\varphi(G) = H_0$.

1.13. *The mapping φ is an isomorphism of G onto H_0.*

Proof. According to 1.11 and 1.12, φ is a homomorphism, whence it suffices to prove that from $\varphi(g_1) = \varphi(g_2)$ follows $g_1 = g_2$. Let $\varphi(g_1) = \varphi(g_2)$ and write $g = g_1 \vee g_2 - g_1 \wedge g_2$. Then $\varphi(g) = 0$, whence $g_a = 0$ for each $a \in \mathcal{A}$; moreover, $g \in G^+$, whence by 1.8 $g = \bigvee g_a = 0$ and thus $g_1 = g_2$.

According to 1.11, for any $a_0 \in \mathcal{A}$ and any $g^{a_0} \in \bar{B}_{a_0}$ there is an element $h \in H^0$ (namely $h = \varphi(g^{a_0})$) such that $h(a_0) = g^{a_0}$, $h(a) = 0$ for any $a \in \mathcal{A}$, $a \neq a_0$. Thus H_0 is a complete subdirect product of l-groups $\bar{B}_a$ $(a \in \mathcal{A})$.

1.14. *Let P_i $(i \in I)$ be l-groups, $P = \Pi^* P_i$ $(i \in I)$. Assume that an l-subgroup Q of P is a complete subdirect product of l-groups P_i and that Q is laterally complete. Then $Q = P$.*

Proof. Let $p = (\ldots, p_i, \ldots) \in P$. To any p_i there correspond elements $u_i, v_i \in P_i^+$ such that $p_i = u_i - v_i$. Write $u = (\ldots, u_i, \ldots)$, $v = (\ldots, v_i, \ldots)$. Since Q is a complete subdirect product of l-groups P_i, for any $i \in I$ there are elements $u^i, v^i \in Q$ such that $u^i(i) = u_i$, $v^i(i) = v_i$, $u^i(j) = v^i(j) = 0$ whenever $j \in I$, $j \neq i$. The system $\{u^i: i \in I\}$ is disjoint, whence $u = \bigvee u^i \in Q$; similarly, $v = \bigvee v^i \in Q$, whence $p \in Q$ and thus $P = Q$.

By summarizing, we have the following assertion:

1.15. Theorem. *Let G be a complete l-group. Assume that f is increasing and that (c_1) is fulfilled. For any $a \in \mathcal{A}$ let $\bar{B}_a$ be the system of all $b \in G$ such that there are subsets $\{t_i\}, \{t'_j\} \subset A_a$ satisfying $-(\bigvee t_i) \leqslant b \leqslant \bigvee t'_j$. Then $\bar{B}_a$ are convex l-subgroups of G and G is isomorphic to a complete subdirect product of l-groups $\bar{B}_a$ $(a \in \mathcal{A})$. If G is laterally complete, G is isomorphic to a complete direct product of l-groups $\bar{B}_a$ $(a \in A)$.*

The lattices $\bar{B}_a$ need not, in general, be f-homogeneous, and this is the reason for searching for a "better" complete subdirect decomposition of G. (Example. For any non-trivial interval $[a, b] \subset G$ put $f[a, b] = \max\{\mathrm{card}[a, b], \aleph_0\}$. Then f is increasing and satisfies (c_1) (cf. 3.1). Let I be an infinite set, $C_i = E$ for each $i \in I$, $G = \Pi^* C_i$ $(i \in I)$ and let $a = \aleph_0$. Denote by H the discrete direct product of l-groups C_i $(i \in I)$. We have $A_a = H^+$, $B_a = H$. Since each element $0 < g \in G$ is the supremum of some subset of H^+, we get $\bar{B}_a = G$. Let $g \in G$, $g(i) = 1$ for each $i \in I$. Clearly, $f[0, g] > \aleph_0$ and $f[0, h] = \aleph_0$ for any $0 < h \in H$. Therefore $\bar{B}_a$ is not f-homogeneous.) Under the same assumptions as in 1.15 let $a \in \mathcal{A}$ be fixed, $\bar{B}_a \neq \{0\}$. Choose any maximal disjoint subset $\{a_i\}_{i\in I}$ of the l-group $\bar{B}_a$. Hence $a_i \in A_a$ for each $i \in I$. Let $b \in \bar{B}_a$, $b > 0$. Then there is a subset $\{t_j\} \subset A_a$, $t_j > 0$, $\bigvee t_j = b$. For any t_j there exists an a_i such that $t_j \wedge a_i > 0$; thus $b \wedge a_i > 0$, and therefore $\{a_i\}_{i\in I}$ is a maximal disjoint subset of the l-group $\bar{B}_a$. For any $i \in I$ write $C_i = \{b \in \bar{B}_a: |b| \wedge a_j = 0$ for each $j \in I, j \neq i\}$. It is known that C_i is a closed convex l-subgroup of $\bar{B}_a$ (cf. [2], p. 119, Proposition 12).

1.16. *$C_i \cap C_j = \{0\}$ for any $i, j \in I$, $i \neq j$.*

Proof. Let $x \in C_i \cap C_j$, $i \neq j$. Then $|x| \in C_i$, whence $|x|\, \delta\, a_k$ for any $k \in I$, $k \neq i$; moreover, from $x \in C_j$ we obtain $|x|\, \delta\, a_i$. If $x \neq 0$, then $|x| \notin \{a_m\}_{m\in I}$ and $\{a_m\}_{m \in I} \cup \{|x|\}$ is a disjoint set, a contradiction.

For any $0 \leqslant g \in \bar{B}_a$ and any $i \in I$ write $g_i = \sup\{t \in C_i: t \leqslant g\}$. Since C_i is a closed sublattice of $\bar{B}_a$ and $\bar{B}_a$ is a closed sublattice of G, we have $g_i \in C_i$.

1.17. *$g = \bigvee g_i$ for any $g \in \bar{B}_a, g \geqslant 0$.*

Proof. Clearly, $g_i \leqslant g$ for each g_i; let $\bigvee g_i = h$ and assume $h < g$; let $g - h = k$. Then there exists an $i_0 \in I$ such that $k \wedge a_{i_0} = a > 0$. Thus $a \in C_{i_0}$, $g_{i_0} + a \in C_{i_0}$ and $g_{i_0} < g_{i_0} + a \leqslant h + k = g$; hence g_{i_0} is not the greatest element of the set $\{t \in C_{i_0}: t \leqslant g\}$, which is a contradiction.

Now the same method that was used in 1.9–1.15 yields (by applying 1.16 and 1.17) the following:

1.18. *The l-group $\bar{B}_a$ is isomorphic to a complete subdirect product of l-groups C_i $(i \in I)$.*

An element e of an l-group H is called a *weak unit of H* if $h \wedge e > 0$ whenever $h \in H$, $h > 0$.

1.19. *Let e be a weak unit of a complete l-group H, $h \in H$, $h \geqslant 0$. Then*

$$\bigvee_{n=1}^{\infty} (ne \wedge h) = h\,.$$

This assertion is proved in [8], p. 97, for the case where H is a complete vector lattice ("K-space"), but the proof remains valid also for complete l-groups.

Let us remark that for any $i \in I$ the element a_i is a weak unit of A_i (otherwise there would exist a positive element $d \in A_i$ such that $a_i\, \delta\, d$ and then, according to 1.16, we would have $a_j\, \delta\, d$ for each $j \in I$, whence $\{a_i\}_{i \in I} \cup \{d\}$ would be a disjoint set, a contradiction).

1.20. *If (c_2) holds, then $f[a, b] = a$ for any non-trivial interval $[a, b]$ of C_i.*

Proof. Since $[a, b]$ is isomorphic to $[0, b-a]$, it suffices to prove that $f[0, t] = a$ for any $t \in C_i$, $t > 0$. From $a_i \in A_a$ it follows that $na_i \in A_a$ for any positive integer n, and since a_i is a weak unit of $\bar{B}_a$, $0 < na_i \wedge t \in A_a$, we have $f[0, na_i \wedge t] = \alpha$ and all these intervals are f-homogeneous. By 1.19

$$\bigvee_{n=1}^{\infty} (na_i \wedge t) = t ,$$

and thus according to (c_2) $f[0, t] = \alpha$.

From 1.15, 1.18 and 1.20 we obtain:

1.21. Theorem. *Let G be a complete l-group and let f be an increasing cardinal property satisfying* (c_1) *and* (c_2). *Then G is isomorphic to a complete subdirect product of f-homogeneous l-groups. If G is also laterally complete, then it is isomorphic to a complete direct product of f-homogeneous l-groups.*

Under the same assumptions as in Theorem 1.21 let $\alpha \in \mathcal{A}$ be fixed, $\bar{B}_\alpha \neq \{0\}$ and let $A_0 = \{a_i\}_{i \in I_0}$ be the system of all atoms of the lattice $\bar{B}_\alpha^+$. There exists a maximal disjoint subset $A_0 = \{a_i\}_{i \in I}$ such that $I_0 \subset I$. Let $i_0 \in I_0$. Since $[0, a_{i_0}]$ is a prime interval, it is a chain and thus (cf. [5], Thm. 1') there exists a direct decomposition

$$\bar{B}_\alpha = R_{i_0} \times Q_{i_0}$$

such that R_{i_0} is linearly ordered and $a_{i_0} \in R_{i_0}$. Moreover, R_{i_0} is complete and a_{i_0} is an atom of $R_{i_0}^+$, whence R_{i_0} is isomorphic to the l-group N consisting of all integers (cf. [1]). Obviously $A' = A_0 \backslash \{a_{i_0}\}$ is a subset of Q_{i_0} and A' is a maximal disjoint subset of Q_{i_0}; therefore, $R_{i_0} = C_{i_0}$. Now let $i \in I \backslash I_0$ and assume that C_i contains a prime interval $[u, v]$. Then $v - u = a_{i_0}$ is an atom of the lattice $\bar{B}_\alpha^+$, $a_{i_0} \in C_{i_0} \cap C_i$; according to 1.16, $C_{i_0} \cap C_i = \{0\}$, a contradiction. Thus for $i \in I \backslash I_0$ each non-trivial interval of the l-group C_i is infinite. Hence from 1.21 follows:

1.22. Theorem. *Let $G \neq \{0\}$ be a complete l-group and let f be an increasing cardinal property satisfying* (c_1) *and* (c_2). *Then there exists a complete subdirect decomposition of G with factors C_k $(k \in K)$ such that* (i) *each factor C_k is f-homogeneous and* (ii) *for any $k \in K$ either C_k is isomorphic to N or each non-trivial interval of C_k is infinite.*

1.23. *According to the constructions of subdirect decompositions of G* (Thm. 1.15) *and of B_α* (cf. 1.17 and 1.18), *we may assume that the factors C_k in* 1.22 *are l-ideals of G and that, for any $g \in G^+$, $g = \bigvee g^k$ $(k \in K)$, where g^k is the k-th component of g with regard to the subdirect decomposition of G described in* 1.22.

§ 2. Lengths of intervals of a lattice ordered group. Let $[a, b]$ be a non-trivial interval of a lattice L and let $\mathfrak{R}[a, b]$ be the system of all maximal chains of the interval $[a, b]$. We define the length $s[a, b]$ of $[a, b]$ by

$$s[a, b] = \min\{\operatorname{card} R\colon\ R \in \mathfrak{R}[a, b]\} .$$

Write $f_1[a, b] = \max\{s[a, b], \aleph_0\}$.

2.1. *Let L be a complete infinitely distributive lattice, $R \in \mathfrak{R}L$. Let 0 be the least element of L, $a \in L$, $a > 0$. Then $R_1 = \{r \wedge a\colon\ r \in R\}$ belongs to $\mathfrak{R}[0, a]$.*

Proof. Clearly R_1 is a chain, $R_1 \subset [0, a]$; assume that $R_1 \notin \mathfrak{R}[0, a]$. Then there exists $b \in [0, a] \backslash R_1$ such that $R_1 \cup \{b\}$ is a chain. Let $R_u(R_v)$ be the set of all $r \in R$ such that $r \wedge a < b$ $(r \wedge a > b)$. Since L is complete, there exists an $r_0 \in R$ such that $r_0 = \bigwedge r_i$ $(r_i \in R_v)$. Then $r_0 \wedge a = \bigwedge (r_i \wedge a) > b$ ($r_0 \wedge a = b$ cannot hold, since $r_0 \wedge a \in R_1$, $b \notin R_1$). Write $r_1 = \bigvee r$ $(r_j \in R_u)$. Clearly, $r_0 \geqslant r_1$; if $r_0 = r_1$, then $r_0 \wedge a = \bigvee (r_j \wedge a) \leqslant b$, a contradiction. If $r_0 > r_1$, then $[r_1, r_0]$ is a prime interval, whence the set $L_1 = \{r_0, r_1, a \wedge r_0, b, a \wedge r_1\}$ is a non-modular sublattice of L; a contradiction.

The assertion dual to 2.1 can be proved similarly.

2.2. *Let L be a complete infinitely distributive lattice, $[u, v] \subset L$, $u < v$, $R \in \mathfrak{R}L$. Then there is an $R_1 \in \mathfrak{R}[u, v]$ such that $\operatorname{card} R_1 \leqslant \operatorname{card} R$.*

Proof. Let 0 be the least element of L. According to 2.1 $R' = \{r \wedge v\colon\ r \in R\}$ belongs to $\mathfrak{R}[0, v]$ and hence, by the assertion dual to 2.1, $R_1 = \{r' \vee u\colon\ r' \in R'\} \in \mathfrak{R}[u, v]$. Obviously $\operatorname{card} R_1 \leqslant \operatorname{card} R' \leqslant \operatorname{card} R$.

Let G be a complete l-group. Since G is infinitely distributive, it follows from 2.2 that f_1 is increasing.

2.3. *Let G be a complete l-group. Then f_1 satisfies* (c_1).

Proof. Let $t_i \in G$, $t_i > 0$ $(i = 1, 2)$, $f_1[0, t_1] = f_2[0, t_2] = \alpha$, $f_1[0, t_1 + t_2] = \beta$. Since f_1 is increasing, $\alpha \leqslant \beta$. The lattices $[0, t_2]$ and $[t_1, t_1 + t_2]$ are isomorphic, and thus $f_1[t_1, t_1 + t_2] = \alpha$. There are chains $R_1 \in \mathfrak{R}[0, t_1]$, $R_2 \in \mathfrak{R}[t_1, t_1 + t_2]$ such that $\operatorname{card} R_1 \leqslant \alpha$, $\operatorname{card} R_2 \leqslant \alpha$; the set $R_1 \cup R_2$ belongs to $\mathfrak{R}[0, t_1 + t_2]$ and $\operatorname{card} R \leqslant \alpha$; hence $\beta = \alpha$.

Let $\mathcal{A}_1 = \{f_1[a, b]\colon\ [a, b] \subset G, a < b\}$. From 2.3 and 1.6 follows:

2.4. Theorem. *Let $G \neq \{0\}$ be a complete l-group. Let $\alpha \in \mathcal{A}_1$, $\alpha > \aleph_0$ $(\alpha = \aleph_0)$. For any $g \in G$ let $G_\alpha^1(g)$ be the family of all convex sublattices L of G such that $g \in L$ and the length of each non-trivial interval of L equals α (equals or is less than α). Then* (i) *any family $G_\alpha^1(g)$ has a greatest element $B_\alpha(g)$,* (ii) *$B_\alpha(0)$ is an l-ideal of G and $B_\alpha(g) = B_\alpha(0) + g$.*

Let us remark that for a non-complete l-group G f_1 need not be increasing. Example: Let $A(B)$ be the additive group of all rational (real) numbers with the natural order, $G = A \times B$, $t_0 = (0, 0)$, $t_1 = (0, 1)$, $t_2 = (1, 1)$. Then $f_1[t_0, t_1] = c$ (the power of the continuum). Let $R = \{(r, r)\colon\ 0 \leqslant r \leqslant 1, r \in A\}$. R is a maximal chain of the lattice $[t_0, t_2]$ and $\operatorname{card} R = \aleph_0$; hence $f_1[t_0, t_2] = \aleph_0$.

2.5. *Let G be a complete l-group. Then f_1 fulfils* (c_2).

Proof. Let $0 < t_i \in G$, $f_1[0, t_i] = \alpha$ and let $[0, t_i]$ be f_1-homogeneous $(i = 1, 2, \ldots)$, $t_1 \leqslant t_2 \leqslant \ldots$, $\bigvee t_i = t$. Since f_1 is increasing, $f_1[0, t] \geqslant \alpha$. Let S be the system of all intervals $[t_i, t_{i+1}]$ $(t_0 = 0,\ i = 1, 2, \ldots)$ that are non-trivial. According to the f_1-homogenity of $[0, t_{i+1}]$, $f_1[t_i, t_{i+1}] = \alpha$ whenever $[t_i, t_{i+1}]$ is non-trivial. For each $L_i \in S$ there exists a maximal chain $R_i \in \mathfrak{R}L_i$ such that card $R_i \leqslant \alpha$; let R be the union of all these R_i. Then $R \in \mathfrak{R}[0, t]$ and card $R \leqslant \alpha$, whence $f_1[0, t] = \alpha$.

From 2.3, 2.5 and 1.22 we obtain:

2.6. Theorem. *Let G be a complete l-group. Then G is isomorphic to a complete subdirect product of l-groups C_k $(k \in K)$ such that for each $k \in K$ either* (i) *every interval of C_k is finite, or* (ii) *any two non-trivial intervals of C_k have the same length $\alpha_k \geqslant \aleph_0$. If G is laterally complete, then G is isomorphic to a complete direct product of l-groups C_k.*

Now we may ask whether we could obtain analogical results if we define the length of a bounded lattice L $(\mathrm{card} L > 1)$ by the rule

$$s'L = \sup\{\mathrm{card} R\colon\ R \in \mathfrak{R}L\}\,.$$

Put $f_2L = \max\{s'L, \aleph_0\}$. Clearly, f_2 is increasing. Let $G \neq \{0\}$ be an l-group, $\mathcal{A}_2 = \{f_2[a, b]\colon [a, b] \subset G, a < b\}$.

2.7. *f_2 fulfils* (c_1).

Proof. Let $0 < t_i \in G$, $f_2[0, t_i] = \alpha$ $(i = 1, 2)$. Then $f_2[t_1, t_1+t_2] = \alpha$ and, since f_2 is increasing, $f_2[0, t_1+t_2] \geqslant \alpha$. Let $R \in \mathfrak{R}[0, t_1+t_2]$ and write $R_1 = \{r_1\colon\ r_1 = r \wedge t_1, r \in R\}$, $R_2 = \{r_2\colon\ r_2 = r \vee t_1, r \in R\}$. The set $R_1(R_2)$ is a chain in $[0, t_1]([t_1, t_1+t_2])$, whence $\mathrm{card} R_1 \leqslant \alpha$, $\mathrm{card} R_2 \leqslant \alpha$. Since G is distributive and r is the relative complement of the element t_1 in the interval $[r_1, r_2]$, the pair of elements (r_1, r_2) uniquely determines r. Thus $\mathrm{card} R \leqslant \mathrm{card}(R_1 \times R_2) \leqslant \alpha$. This proves that $f_2[0, t_1+t_2] = \alpha$.

From 2.7 and 1.6 we obtain the following:

2.8. *Let $G \neq \{0\}$ be an l-group, $\alpha \in \mathcal{A}_2$, $\alpha > \aleph_0$ $(\alpha = \aleph_0)$. For any $g \in G$ let $G'_\alpha(g)$ be the family of all convex sublattices L of G such that $g \in L$ and for any non-trivial interval L_1 of L $s'L_1 = \alpha$ $(s'L_1 \leqslant \aleph_0)$. Then* (i) *each family $G'_\alpha(g)$ has a greatest element $B'_\alpha(g)$,* (ii) *$B'_\alpha(0)$ is an l-ideal of G and $B'_\alpha(g) = B'_\alpha(0)+g$.*

There exist complete l-groups G such that f_2 fails to satisfy (c_2). Example: Let $I = \{1, 2, \ldots\}$, $G_i = N$ for each $i \in I$, $G = \Pi^* G_i$. For any $i \in I$ define t_i by

$$t_i(j) = 1 \quad \text{for} \quad j \in I,\ j \leqslant i\,, \quad \text{and} \quad t_i(j) = 0 \ \text{otherwise.}$$

Further, let $\bar{0}, \bar{1} \in G$ such that $\bar{0}(j) = 0$, $\bar{1}(j) = 1$ for each $j \in I$. Clearly, $s'[\bar{0}, t_i] = i+1$, whence $f_2[\bar{0}, t_i] = \aleph_0$ and all intervals $[\bar{0}, t_i]$ are f_2-homogeneous. We have $\bar{0} < t_1 < t_2 < \ldots$, $\bigvee t_i = \bar{1}$ and the interval $[\bar{0}, \bar{1}]$ is isomorphic to the Boolean algebra B consisting of all subsets of the set I. There is a chain R in B such that $\mathrm{card} R = c$ (cf. [4]). Thus $f_2[\bar{0}, \bar{1}] \neq \aleph_0$.

Let L be a lattice, $L_1 \subset L$. The set L_1 is dense in L, if $L_1 \cap [a, b] \neq \emptyset$ for any non-trivial interval $[a, b] \subset L$. We define the reduced length s^*L of a bounded lattice by $s^*L = \min\{\alpha \in \mathcal{K}\colon$ there exists an $R \in \mathfrak{R}L$ and a dense subset L_1 of R such that $\mathrm{card} L_1 = \alpha\}$. By the same method as in 2.1–2.6 analogical results for the reduced length can be proved.

§ 3. The powers of intervals of an l-group. Let G be an l-group, $G \neq \{0\}$. For any non-trivial interval $[a, b] \subset G$ we write $f_3[a, b] = \max\{\mathrm{card}[a, b], \aleph_0\}$. Obviously, f_3 is increasing.

3.1. *f_3 satisfies* (c_1).

Proof. Let $0 < t_i \in G$, $f_3[0, t_i] = \alpha$ $(i = 1, 2)$, $f_3[0, t_1+t_2] = \beta$. Then $f_3[t_1, t_1+t_2] = \alpha \leqslant \beta$ and each element $t \in [0, t_1+t_2]$ is uniquely determined by the pair $(t \wedge t_1, t \vee t_1)$. Since $t \wedge t_1 \in [0, t_1]$, $t \vee t_1 \in [t_1, t_1+t_2]$, we have $\mathrm{card}[0, t_1+t_2] \leqslant \mathrm{card}[0, t_1]\, \mathrm{card}[t_1, t_1+t_2] \leqslant \alpha$. Thus $f_3[0, t_1+t_2] = \alpha$.

Let $\mathcal{A}_3 = \{f_3[a, b]\colon [a, b] \subset G, a < b\}$. From 3.1 and 1.6 we obtain:

3.2. Theorem. *Let G be an l-group, $\alpha \in \mathcal{A}_3$, $\alpha > \aleph_0$ $(\alpha = \aleph_0)$. To any $g \in G$ there exists a greatest convex sublattice $B^3_\alpha(g)$ of G containing g such that each non-trivial interval of $B^3_\alpha(g)$ has the power α (the power $\leqslant \aleph_0$). The set $B^3_\alpha(0)$ is an l-ideal of G and $B^3_\alpha(g) = B^3_\alpha(0)+g$.*

3.3. *Let G be a complete l-group. Then there exists a decomposition $G = A \times B$ such that* (i) *A is a complete subdirect product of linearly ordered groups isomorphic to N, and* (ii) *B does not contain any prime interval.*

Proof. Let f be an increasing cardinal property satisfying (c_1) and (c_2) (for example, $f = f_1$). Consider the complete subdirect decomposition with factors C_k $(k \in K)$ treated in 1.22 and 1.23. Let K_0 be the system of all C_k isomorphic to N. We denote by $A(B)$ the set of all $g \in G$ such that $g_k = 0$ for each $k \in K \setminus K_0$ $(k \in K_0)$. Then clearly $G = A \times B$ and $A(B)$ is isomorphic to a complete subdirect product of l-groups C_k, $k \in K_0$ $(k \in K \setminus K_0)$. Let $[t_1, t_2]$ be a non-trivial interval of B. Then $[t_1, t_2]$ is isomorphic to $[0, t]$, $t \in B$, $t_2 - t_1 = t > 0$, whence $t = \bigvee t_k (k \in K \setminus K_0)$. There exists $k_1 \in K \setminus K_0$ such that $t_{k_1} > 0$ and, since C_{k_1} does not contain any prime interval, we have $t' \in C_{k'_1}$, $0 < t' < t_{k_1}$. Therefore the intervals $[0, t]$ and $[t_1, t_2]$ are not prime.

3.4. *Let G be a complete l-group, $a \in G$, $a > 0$, and assume that any disjoint subset of G is finite. Then the lattice $[0, a]$ is isomorphic to a direct product of a finite number of chains.*

Proof. At first we shall prove that for each $b \in [0, a]$, $b > 0$ there is an element b_1, $0 < b_1 \leqslant b$, such that $[0, b_1]$ is a chain. For otherwise there would exist $b_1, b_2 \in [0, b]$, $b_1 > 0$, $b_2 > 0$, $b_1 \,\delta\, b_2$. Further, there would exist positive disjoint elements $b_{21}, b_{22} \in [0, b_2]$. In this way we could construct an infinite disjoint subset $\{b_1, b_{21}, b_{221}, \ldots\} \subset [0, a]$, which is a contradiction. Hence there exists a maximal disjoint subset $B = \{b_1, b_2, \ldots, b_n\}$ of $[0, a]$ such that each interval $[0, b_i]$ is a chain. Since G is Archimedean, for each b_i there exists an integer $n_i \geqslant 1$ such that

$n_i b_i \not\leqslant a$. Put $a_i = a \wedge n_i b_i$. The interval $[0, nb_i]$ is a chain for any integer n (cf. [5], Lemma 17.2), whence $a \wedge nb_i = a_i$ for each $n \geqslant n_i$. Let $\bigvee a_i = a'$, $a - a' = k$. Clearly $k \geqslant 0$; assume that $k > 0$. Then there exists $b_{i_0} \in B$ such that $k_{i_0} = k \wedge b_{i_0} > 0$. Thus $a_{i_0} + k_{i_0} \leqslant (n_{i_0}+1) b_{i_0}$, $a_{i_0} + k_{i_0} \leqslant a' + k = a$, whence $a_{i_0} + k_{i_0} \leqslant a \wedge (n_{i_0}+1) b_{i_0} = a_{i_0}$, a contradiction. Therefore $k = 0$ and $a = \bigvee a_i$. Each interval $[0, a_i] \subset [0, n_i b_i]$ is a chain and the mapping $x \to \{x \wedge a_i\}$ $(i = 1, \ldots, n)$ is an isomorphism of the lattice $[0, a]$ onto the direct product $\Pi[0, a_i]$ $(i = 1, 2, \ldots, n)$.

3.5. *Let G be a complete l-group and let B have the same meaning as in* 3.3. *Let $[0, b] \subset B$ be a non-trivial f_3-homogeneous interval, $f_3[0, b] = \alpha$. Then $\alpha^{\aleph_0} = \alpha$.*

Proof. At first assume that each disjoint subset of $[0, b]$ is finite. Then by 3.4 there exist elements $b_1, \ldots, b_n \in [0, b]$, $b_i > 0$, such that each interval $[0, b_i]$ is a chain and $[0, b]$ is isomorphic to the direct product of intervals $[0, b_i]$. According to [5, Thm. 1'] there exist l-ideals B_i of G such that B_i are linearly ordered and $b_i \in B_i$. Moreover, B_i are complete and since $B_i \subset B$ does not contain any prime interval, each B_i is isomorphic to the additive l-group R_0 of all reals (cf. [1]); thus $\operatorname{card}[0, b_i] = c$ and $\operatorname{card}[0, b] = c = c^{\aleph_0}$. Now let us suppose that there exists an infinite disjoint subset of the interval $[0, b]$; then there exists a disjoint subset $\{b_1, b_2, \ldots\}$ of $[0, b]$. Since any non-trivial interval of B is infinite, $\operatorname{card}[0, b] = \alpha \geqslant \aleph_0$ and, according to the f_3-homogenity of $[0, b]$, $\operatorname{card}[0, b_i] = \alpha$ for $i = 1, 2, \ldots$ Write $b' = \bigvee b_i$ $(i = 1, 2, \ldots)$ and consider the mapping φ: $x \to \{x \wedge b_i\}$ of the lattice $[0, b']$ into $\Pi^*[0, b_i]$. By using the infinite distributivity of $[0, b']$ it is easy to verify that φ is an isomorphism. Hence $\alpha = \operatorname{card}[0, b'] = \operatorname{card}\Pi^*[0, b_i] = \alpha^{\aleph_0}$.

3.6. *Let G be a complete l-group and let B be as in* 3.3. *Then f_3 satisfies* (c_2) *with regard to B.*

Proof. Let $t_i \in B$, $f_3[0, t_i] = \alpha$ $(i = 1, 2, \ldots)$, $0 < t_1 \leqslant t_2 \leqslant \ldots$, $\bigvee t_i = t$ and assume that all intervals $[0, t_i]$ are f_3-homogeneous. Since B does not contain prime intervals, $\operatorname{card}[0, t_i] = \alpha$ for $i = 1, 2, \ldots$ For any $x \in [0, t]$ we have $x = \bigvee (x \wedge t_i)$, whence the mapping $x \to \{x \wedge t_i\}$ $(i = 1, 2, \ldots)$ is a monomorphism of the set $[0, t]$ into the complete direct product $\Pi^*[0, t_i]$; from this we obtain $f_3[0, t] = \operatorname{card}[0, t] \leqslant \operatorname{card}\Pi^*[0, t_i] = \alpha^{\aleph_0}$ and $\alpha^{\aleph_0} = \alpha$ according to 3.5. Therefore (since f_3 is increasing) $f_3[0, t] = \alpha$.

According to 3.1 and 3.6, we may apply Th. 1.21 to the l-group B; since A is isomorphic to a complete subdirect product of linearly ordered groups C_k $(k \in K_0)$ such that any interval of C_k is finite, we have the following result:

3.7. THEOREM. *Let G be a complete l-group. Then G is isomorphic to a complete subdirect product of l-groups C_k $(k \in K)$ such that, for each C_k, one of the following conditions holds*: (i) *any interval of C_k is finite and C_k is linearly ordered, or* (ii) *any non-trivial interval of C_k has the same cardinality α_k and $\alpha_k^{\aleph_0} = \alpha_k$.*

Let α be a cardinal, $\alpha^{\aleph_0} = \alpha$. Then there is a lattice ordered group $G_\alpha \neq \{0\}$ such that $\operatorname{card}[a, b] = \alpha$ for each non-trivial interval of G_α. We construct G_α as follows:

Since $\alpha^{\aleph_0} = \alpha$, there exists a Boolean algebra $B_\alpha \neq \{0\}$ such that $\operatorname{card}[b_1, b_2] = \alpha$ for any non-trivial interval of B_α (cf. Pierce [6]). Let E be the vector lattice consisting of all elementary Carathéodory functions on B_α (cf. Goffman [3]); i.e., E is the set consisting of all forms

$$f = a_1 b_1 + \ldots + a_n b_n \tag{2}$$

(where $a_i \neq 0$ are reals and $b_i \in B_\alpha$, $b_i > 0$, $b_{i_1} \wedge b_{i_2} = 0$ for any $i_1, i_2 \in \{1, \ldots, n\}$, $i_1 \neq i_2$) and of the "empty form"; if g is another such form,

$$g = a_1 b_1' + \ldots + a_m' b_m', \tag{3}$$

then f, g are considered as equal if $\bigvee_{i=1}^{n} b_i = \bigvee_{j=1}^{m} b_j'$ and if $a_i = a_j'$ whenever $b_i \wedge b_j' \neq 0$. For any $b, b' \in B_\alpha$ let $b - b'$ be the relative complement of $b \wedge b'$ in the interval $[0, b]$. The operation $+$ in E is defined by

$$f + g = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + a_j')(b_i \wedge b_j') + \sum_{i=1}^{n} a_i \Big(b_i - \bigvee_{j=1}^{m} b_j'\Big) + \sum_{j=1}^{m} a_j' \Big(b_j' - \bigvee_{i=1}^{n} b_i\Big),$$

where in the summations only those terms are taken into account in which $a_i + a_j' \neq 0$ and the elements $b_i \wedge b_j'$, $b_i - \bigvee_{j=1}^{m} b_j'$, $b_j' - \bigvee_{i=1}^{n} b_i$ are non-zero. The multiplication by a real $a \neq 0$ is defined by $af = (aa_1) b_1 + \ldots + (aa_n) b_n$; $0f$ is the empty form. The form (2) is positive, if $a_i > 0$ for $i = 1, \ldots, n$. Let G_α be the subset of E consisting of the empty form f_0 and of all forms (2) such that $a_i \neq 0$ are integers $(i = 1, 2, \ldots, n)$. Then G_α is an l-subgroup of the l-group E and $\operatorname{card} G_\alpha = \alpha$. For proving that $\operatorname{card}[f, g] = \alpha$ for any non-trivial interval $[f, g]$ of G_α it suffices to examine the intervals $[f_0, f]$, $f > f_0$. Let $f \in G_\alpha$ be the form (2) with $a_i \geqslant 1$ $(i = 1, \ldots, n)$. Let

$$Y = \{b \in B_\alpha\colon\ 0 < b \leqslant b_1\}, \qquad \bar{Y} = \{g \in G_\alpha\colon\ g = 1b,\ b \in Y\}.$$

Since $\operatorname{card}[0, b_1] = \alpha$, we have $\operatorname{card}\bar{Y} = \alpha$ and, because $\bar{Y} \subset [f_0, f] \subset G_\alpha$, $\operatorname{card}[f_0, f] = \alpha$.

It remains as an open question whether for any cardinal α satisfying $\alpha^{\aleph_0} = \alpha$ there exists a complete l-group G such that $\operatorname{card} L = \alpha$ for any non-trivial interval of G.

Analogously as in § 2 we may define the reduced power $\operatorname{card}^* L$ of a bounded lattice L to be the least cardinal α such that there exists a dense subset L_1 of L, $\operatorname{card} L_1 = \alpha$. Write $f_4 L = \max\{\operatorname{card}^* L, \aleph_0\}$. Obviously f_4 is increasing, but f_4 fails to satisfy the condition (c_1). Example: Let $G = R_0 \times R_0$, $g_0 = (0, 0)$, $g_1 = (1, 0)$, $g_2 = (0, 1)$. Clearly, $f_4[g_0, g_1] = f_4[g_0, g_2] = \aleph_0$ and the intervals $[g_0, g_1]$, $[g_0, g_2]$ are f_4-homogeneous. Let L_1 be a dense subset of $[g_0, g]$, $g = g_1 + g_2 = (1, 1)$. Let $r \in [0, 1]$, $h_1 = (0, r)$, $h_2 = (1, r)$. Then there exists $g_r \in L_1 \cap [g_1, g_2]$ and $g_r = (x_r, r)$, $x_r \in [0, 1]$. Thus $g_{r_1} \neq g_{r_2}$ whenever $r_1 \neq r_2$ and therefore $\operatorname{card} L_1 = c = f_4[0, g_1 + g_2] \neq f_4[0, g_1]$.

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Algèbre du calcul propositionnel trivalent de Heyting

par

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1. Introduction. Nous nous proposons dans cette note de déterminer le nombre d'éléments de l'algèbre H_3 avec un nombre fini de générateurs libres (¹).

1.1. DÉFINITION. Une algèbre de Heyting (²) A sera dite une *algèbre* H_3 si l'égalité suivante est vérifiée:

(T) $$\big((a \to c) \to b\big) \to \Big(\big((b \to a) \to b\big) \to b\Big) = 1$$

quels que soient les éléments a, b et c de A.

Ces algèbres jouent dans l'étude du calcul propositionnel trivalent de Heyting (A. Heyting [5], J. Łukasiewicz [6], I. Thomas [16]) un rôle analogue à celui des algèbres de Boole dans le calcul propositionnel classique.

Il est évident que toute algèbre de Boole, est une algèbre H_3, car dans les algèbres de Boole est valable l'égalité $(b \to a) \to b = b$, qui implique (T).

Indiquons l'exemple le plus simple d'une algèbre H_3, qui n'est pas une algèbre de Boole: Soit $T = \{0, a, 1\}$ l'ensemble formé par trois éléments distincts sur lequel on définit les opérations $\wedge$, $\vee$ et $\to$ au moyen des tables suivantes (auxquelles nous ajoutons la table de l'opération de négation $\neg$ définie par $\neg x = x \to 0$).

$\wedge$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

$\vee$	0	a	1
0	0	a	1
a	a	a	1
1	1	1	1

$\to$	0	a	1	$\neg x$
0	1	1	1	1
a	0	1	1	0
1	0	a	1	0

Cette algèbre a été considérée pour la première fois par A. Heyting (1930).

L'algèbre de Boole $B = \{0, 1\}$ est une sous-algèbre de T que nous aurons à utiliser par la suite.

(¹) Un résumé de cette note a été publié dans Notas de Lógica Matemática № 19 (1964).

(²) Voir: T. Skolem [14], G. Birkhoff [1], p. 459, [3], p. 147, M. Ward [17] et A. Monteiro [8]. Nous avons adopté la terminologie de H. Rasiowa et R. Sikorski [11].